Concerning irreducible cuttings of continua 1).

By

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A subset $K$ of a continuum $M$ will be called a cutting of $M$, or will be said to cut $M$, provided that the set of points $M - K$ is not connected, i.e., is the sum of two non-vacuous mutually separated point sets; $K$ will be called a cutting of $M$ between two points $A$ and $B$ of $M$, or will be said to cut $M$ between $A$ and $B$, provided that $M - K$ is the sum of two mutually separated point sets $M_1$ and $M_2$ containing $A$ and $B$ respectively. A subset $K$ of a continuum $M$ will be called an irreducible cutting of $M$ provided $K$ cuts $M$ but no proper subset of $K$ cuts $M$; $K$ will be called an irreducible cutting of $M$ between the points $A$ and $B$ of $M$ provided $K$ cuts $M$ between these two points but contains no proper subset which does.

These notions are related to the notions of "coupage du plan" and "coupage irréductible du plan" as used by Kuratowski in his memoir *Sur les coupures irréductibles du plan* 2). For the case where $M$ is the entire plane, or indeed where $M$ is any continuous curve, the above definitions are equivalent 3) to those of Kuratowski, or to this definition extended in an obvious way to continuous curves. However, for a continuum in general, such is not the case. If $M$ is not a continuous curve, then a coupure of $M$ in the sense of Kuratowski is not necessarily a cutting of $M$ in the sense above defined. I shall quite frequently have occasion to refer to the above mentioned paper of Kuratowski. In this paper, among other results, I shall show that a large number of the theorems in Kuratowski's paper concerning the "coupages du plan" subsist for cuttings of any continuous curve.

It will be shown in § 2 that every cutting of a continuous curve $M$ between two points $A$ and $B$ of $M$ contains an irreducible cutting of $M$ between these two points. This theorem does not remain true for continua $M$ in general. Indeed, as shown below, if $A$ and $B$ are points of any indecomposable continuum $M$ whatever, then every cutting of $M$ between these two points is reducible. As shown by Kuratowski (loc. cit.), not every cutting of $M$, even if $M$ is in the plane, contains an irreducible cutting of $M$. However, I shall show, in § 3, that if every subcontinuum of a plane continuous curve $M$ is an irreducible curve, then every cutting of $M$ contains an irreducible cutting of $M$.

In § 1 I shall show that if $K$ is an irreducible cutting of a bounded plane continuum $M$ such that $M - K$ has at least three components, then $K$ itself has at most two components. Hence, only two kinds of irreducible cuttings of a bounded plane continuum $M$ exist which cut $M$ into more than two components; these are continua and point sets which are the sum of two continua (either of which may reduce to a single point).

The point sets considered are assumed to lie in a Euclidean space. Theorems 4, 5, 12, 13, 14 and 15 hold only in 2-dimensions, whereas all the remaining ones are true in $n$-dimensions.

*Definitions and notations.* The term continuous curve will be used to designate any connected im kleinen continuum (bounded or not). By a component of a point set $M$ is meant a connected subset of $M$ which is not a proper subset of any other connected subset of $M$. A subset $R$ of a point set $M$ is an open subset of $M$ provided $M - R$ is closed (in $M$). A subset $R$ of a set $M$, $F(R)$ will be used to denote the boundary of $R$ with respect to $M$, i.e., the set of all those points of $M - R$ which are limit points of $R$; $F(R)'$ will denote the boundary of $R$ with respect to the whole space. A cutting $K$ of a continuum $M$ will be said to be a componentwise irreducible cutting of $M$ provided that every subset of $K$ which cuts $M$ contains a point in each component of $K$. Obviously every irreducible cutting of a continuum is also componentwise irreducible. A point $P$ is said to be acces-

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2) Fund. Math. vol. 6 (1924) pp. 130—145.
Irreducible cuttings of continua.

Proof. Suppose, on the contrary, that there exists an irreducible cutting \( K \) of \( M \) between some two points \( A \) and \( B \) of \( M \). By definition there exist two mutually separated point sets \( M_1 \) and \( M_2 \), each containing \( M \), such that \( M_1 \cap A \), \( M_2 \cap B \), and \( M_1 \cup M_2 = M - K \). But by Theorem 2, \( M_1 + K \) and \( M_2 + K \) are continua, and since \((M_1 + K) + (M_2 + K) = M\) this contradicts the fact that \( M \) is indecomposable.

Corollary 2. Every cutting of an indecomposable continuum is reducible.

In the light of Theorem 2, it might be supposed that some basic relation exists between the indecomposability of a continuum \( M \) and the fact that not every cutting of \( M \) between two points of \( M \) contains an irreducible cutting of \( M \) between these two points.

Example. There exists a plane bounded continuum \( M \) every subcontinuum of which is decomposable which contains two points \( A \) and \( B \) such that every cutting of \( M \) between \( A \) and \( B \) is reducible.

Let \( T \) be a non-dense perfect set on the interval \((0, 1)\) of the \( X \)-axis, and let \( A \) be the point \((1, 1)\). For each point \( X \) of \( T \), let \( L_x \) be the straight line interval from \( A \) to \( X \). Let \( M \equiv \sum L_x \). Then \( M \) is a bounded continuum, every subcontinuum of \( M \) is decomposable, and, if \( B \) is any point of \( T \), it is easily seen with the aid of Theorem 2 that every cutting of \( M \) between \( A \) and \( B \) is reducible.

This example, however, still leaves open the following questions.

(1) If a continuum \( M \) has the property that for every two points \( A \) and \( B \) of \( M \) it is true that no irreducible cutting of \( M \) between \( A \) and \( B \) exists, then is it true that \( M \) is indecomposable or that it contains an indecomposable continuum? (2) If every cutting of a continuum \( M \) is reducible, is \( M \) necessarily indecomposable?

Theorem 3. If \( A \) and \( B \) are any two points of an indecomposable continuum \( M \), then every cutting of \( M \) between \( A \) and \( B \) is reducible.

\section*{Irreducible cuttings of continua.}

Proof. Suppose, on the contrary, that there exists an irreducible cutting \( K \) of \( M \) between some two points \( A \) and \( B \) of \( M \). By definition there exist two mutually separated point sets \( M_1 \) and \( M_2 \), each containing \( M \), such that \( M_1 \cap A \), \( M_2 \cap B \), and \( M_1 \cup M_2 = M - K \). But by Theorem 2, \( M_1 + K \) and \( M_2 + K \) are continua, and since \((M_1 + K) + (M_2 + K) = M\) this contradicts the fact that \( M \) is indecomposable.

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Theorem 4. If the closed componentwise irreducible cutting \( K \) of a plane bounded continuum \( M \) has more than two components, then \( M - K \) has just two components. Hence, \( M - K \) is the sum of two mutually separated and connected point sets.

Proof. Since by hypothesis \( K \) has more than one component and is a componentwise irreducible cutting of \( M \), therefore if \( H \) is any component of \( K \), \( M - H \) is connected and hence lies wholly in one complementary domain of \( H \); and for not more than one
component $H$ of $K$ is such that $M - H$ lies in a bounded complementary domain of $H$. Hence, by the principle of inversion, we may assume, without loss of generality, that for each component $H$ of $K$, $M - H$ is a subset of the unbounded complementary domain of $H$. For each component $H$ of $K$ let $H_0$ denote the continuum $H$ plus all of its bounded complementary domains, and let $G$ denote the collection of all the continua $[H_0]$ thus obtained. Then since $K$ is closed, it follows with the aid of a theorem of R. L. Moore's that $G$ is an upper semi-continuous collection of bounded continua. Clearly no one of these continua separates the plane and no one has any point in common with $M - K$. Hence, if $S'$ denotes the space whose elements are the continua of the collection $G$ plus all the points in the plane which belong to no element of $G$, by a result of R. L. Moore's $S'$ is homeomorphic with the ordinary euclidean plane, and axioms 1 - 8 of R. L. Moore's paper "On the foundations of plane analysis situs" are satisfied. Then since $K$ is a subset of a totally disconnected set of continua, then $M - K$ is not connected, it follows by a theorem of R. G. L. L. h. b. b. n's that there exists a simple curve $J$ of elements of $S'$ such that $J$ and $M - K$ is not connected, and the exterior of $J$ contain points of $M - K$. Let $I$ and $E$ denote the interior and exterior respectively of $J$. Then $I$ and $E$ are ordinary point sets, and if $M_1$ and $M_2$ denote the point sets $I$, $M$ and $E$, $M$ respectively, $M_1$ and $M_2$ are mutually separated. I shall show that each of them is connected. Suppose $M_2$ is not connected. Then $M_1 + E + J - K$ is not connected, and by application of R. G. L. L. h. b. b. n's theorem quoted above, there exists a simple curve $J_1$ of elements of $S'$ such that $J_1$, $(M_1 + E + J - K)$ is not connected, and the exterior of $J_1$ contain points of $M_1$. Hence $J_1$ must lie, except for those elements which belong to $G$, wholly in $I$. It follows that there exist two elements (or points $A$ and $B$ of $G$ and $A + B$ of elements of $J_1$ such that (1) $AXB - (A + B) \subseteq I$, (2) each of the two domains $R_1$ and $R_2$ into which $AXB$ divides $I$ contains points of $M_1$. Let $AWB$ and $AZB$ denote the two "areas" of $J$ from $A$ to $B$. Since by hypothesis $K$ has at least three components, therefore one of the segments $AWB - (A + B)$ and $AZB - (A + B)$, say $AWB - (A + B)$, must contain a component $C$ of $K$. Let $J_2$ denote the "simple closed curve" $AZBX$, and suppose $R_2$ is its interior. Then since $J_2$ does not contain $C$, $J_2 - K$ is a proper subset of $K$ which contains no point of the component $C$ of $K$. But $0 \notin R_1, M - M - J_2, M$, and clearly $R_1, M$ and $M - (J_2, K + R_1)$ are mutually separated point sets whose sum is $M - J_2 - K$. This contradicts the hypothesis that $K$ is a componentwise irreducible cutting of $M$, and therefore $M_1$ is connected. That $M_2$ is connected follows by a similar argument, after performing an inversion of the plane. The truth of Theorem 4 is therefore established.

**Corollary 4a.** If $K$ is a closed componentwise irreducible cutting of a bounded plane continuum $M$ such that $M - K$ has at least three distinct components, then $K$ is either a continuum or the sum of two continua (either of which may reduce to a single point).

**Corollary 4b.** Let $H$, $L$, and $N$ be bounded continua in the plane such that (1) $H \cup L = H \cup N = L$, $H = L$, and $L \cup K$, and $N \cup K$ are connected point sets. Then $K$ is either a continuum or the sum of two continua.

**Proof.** Let $M$ denote the continuum $H + L + N$. Then since each component of $K$ contains a limit point of each of the connected sets $H - K$, $L - K$, and $N - K$, it follows that $K$ is a closed componentwise irreducible cutting of $M$. And since $M - K$ has the three distinct components $H - K$, $L - K$, and $N - K$, then by corollary 4a, $K$ is either a continuum or the sum of two continua.

Since every irreducible cutting of a continuum is also componentwise irreducible, and since, by Corollary 1a, every irreducible cutting of a continuum is a closed set of points, we have the following theorem.

**Theorem 5.** If the irreducible cutting $K$ of a bounded plane continuum $M$ has more than two components, then $M - K$ has just two components, and hence $M - K$ is the sum of two mutually separated connected point sets.

**Corollary 5a.** If the irreducible cutting $K$ of a bounded plane continuum is totally disconnected and contains more than two points,
then $M - K$ is the sum of two mutually separated and connected point sets.

It is to be noted that for cuttings $K$ of a continuum which are totally disconnected, the properties of being an "irreducible cutting of $M^a$" and a "componentwise irreducible cutting of $M^a$" are equivalent.

**Theorem 6.** Let $K$ be an irreducible cutting of a continuum $M$ between the points $A$ and $B$ of $M$, let $P$ be any point of $K$ and $R$ any bounded domain containing $P$, and let $N$ denote the component of $M - [R + F(R)]$ which contains $P$. Then if $K.F(R) = 0$, $K.N$ is a cutting of $N$.

**Proof.** By hypothesis $M - K = M_1 + M_2$, where $M_1$ and $M_2$ are mutually separated and contain $A$ and $B$ respectively. By Theorem 2, $M_1 + K$ and $M_2 + K$ are continua. Then since $N.K$ and $F(R)$ are bounded and $(N.K).F(R) = 0$, it follows with the aid of two theorems of Miss Mullikin's \(^1\) that $M_1.N = 0$ and $M_2.N = 0$. But $N = N.K = M_1.N = M_2.N$, and clearly $M_1.N$ and $M_2.N$ are mutually separated point sets. Hence $K.N$ is a cutting of $N$.

**Corollary 6 a.** Let $P$ be any point of an irreducible cutting $K$ of a continuum $M$, let $R$ be any bounded domain containing $P$, and let $N$ be the component of $M - [R + F(R)]$ containing $P$. Then if $K.F(R) = 0$, $K.N$ is a cutting of $N$.

Examples are easily constructed showing that Theorem 6 is not true in the absence of the condition that $K.F(R) = 0$.

2. **Irreducible cuttings of continuous curves.**

**Theorem 7.** In order that a closed cutting $K$ of a continuous curve $M$ between the points $A$ and $B$ of $M$ should be an irreducible cutting of $M$ between $A$ and $B$ it is necessary and sufficient that if $R_a$ and $R_b$ denote the component of $M - K$ containing $A$ and $B$ respectively then $F_a(R_a) = F_b(R_b) = K^s$.

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\(^1\) Certain theorems relating to plane connected point sets, Trans. Amer. Math. Soc., vol. 24 (1922), pp. 144-162, Theorems 2 and 1. These theorems hold in n-dimensions.

\(^2\) For the case where $M$ is the entire Euclidean space, this theorem is equivalent to a theorem of Kuratowski's (Cf. Sur les coupures irreductibles du plan, loc. cit.). The same is true of Theorems 8 and 9 in this section. The methods of proof and lemmas used are similar to those used by Kuratowski to prove his theorems.

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**Irreducible cuttings of continua.**

In proving Theorem 7, use will be made of the following easily established lemma.

**Lemma 7 a.** If $R$ is any open subset of a continuous curve $M$, and $M - R = 0$, then $F_a(R)$ cuts $M$ between every pair of points belonging to $R$ and $M - R$ respectively.

**Proof of Theorem 7.** The condition is sufficient. For if $H$ is any proper subset of $K$, and $P$ is a point of $K - H$, then since $P \subseteq F_a(R_a)$ and $P \subseteq F_b(R_b)$, therefore $P + R_a + R_b$ is connected; and hence $H$ does not cut $M$ between $A$ and $B$. The condition is also necessary. For suppose one of sets $F_a(R_a)$ and $F_b(R_b)$, say $F_a(R_a)$, is $0$. Then since $F_a(R_a) \subseteq K$, $F_a(R_a)$ is a proper subset of $K$. But by Lemma 7 a, $F_a(R_a)$ cuts $M$ between $A$ and $B$, contrary to the fact that $K$ is an irreducible cutting of $M$ between $A$ and $B$.

**Corollary 7 a.** In order that a closed cutting $K$ of a continuous curve $M$ should be irreducible it is necessary and sufficient that if $R$ is any component of $M - K$, then $F_a(R) = K$.

Examples are easily constructed to show that the condition of Theorem 7 is not necessary in the absence of the stipulation that the continuum $M$ is a continuous curve.

**Theorem 8.** Every cutting $K_a$ of a continuous curve $M$ between the points $A$ and $B$ of $M$ contains an irreducible cutting of $M$ between $A$ and $B$.

**Proof.** By a lemma of R. L. Moore's \(^1\), $K_a$ contains a closed subset $K$ which cuts $M$ between $A$ and $B$. Let $D$ denote the component of $M - K$ which contains $A$. Then by lemma 7 a, $F_a(D)$ cuts $M$ between $A$ and $B$. And if $R_a$ denotes the component of $M - F_a(D)$ which contains $B$, then by lemma 7 a, $F_a(R_a) = B$. But if $R_b$ denotes the component of $M - F_b(R_b)$ which contains $A$, then since $R_b$ contains $D$ and $F_a(R_b) \subseteq F_a(D)$, it follows that $F_a(R_b) = K_a(R_b)$. Hence, by Theorem 7, $F_a(R_a)$ is an irreducible cutting of $M$ between $A$ and $B$.

**Theorem 9.** If the cutting $K$ of a continuous curve $M$ contains no interior point relative to $M$ and is such that $M - K$ has only a finite number of components, then $K$ contains an irreducible cutting of $M$.

\(^1\) Cf. reference to R. L. Moore in proof of Theorem 1. and note on the unbounded case in the same footnote.

Fundamenta Mathematicae. T. XIII.
Proof. Let the components of $M - K$ be denoted by $E_1, E_2, \ldots, E_n$. By R. L. Moore's lemma just quoted, $K$ contains a closed subset $K_1$ which cuts $M$. Then $M - K_1$ has at most $n$ components. For if not, it would have one component $K$ which contains no point of $M - K$, because, for each $i \leq n$, $E_i$ lies wholly in some single component of $M - K_i$. Then $K \subset K_i$. But if $P$ is a point of $K$, then since $M$ is connected in the kleinian it follows that $P$ is not a limit point of $M - K$. Hence $P$ is an interior point of $K$ relative to $M$, contrary to hypothesis. Hence the closed subset $K_1$ of $K$ cuts $M$ into just a finite number of components; and with the aid of inductive methods similar to those used by Kuratowski to prove an analogous theorem for a "coupe du plan" (cf. Kuratowski, loc. cit.). The proof of Theorem 9 is easily completed.

The question as to whether or not Theorem 9 remains true on the omission of the condition that $K$ contain no interior point relative to $M^2$ is very interesting. This raised the following interesting questions: (1) Does every continuous curve contain an irreducible cutting of itself? (2) Does every open subset of a continuous curve $M$ contain an irreducible cutting of $M^2$? These questions are related to the question (2) stated above just before the statement of Theorem 4.

We may extend the notion of "irreducible cutting between two points" to closed sets as follows.

Definitions. If $A$ and $B$ are mutually exclusive closed subsets of a continuum $M$, the subset $K$ of $M$ is said to be a cutting of $M$ between $A$ and $B$, or to cut $M$ between $A$ and $B$, provided $M - K$ is the sum of two mutually separated point sets $M_1$ and $M_2$, containing $A$ and $B$ respectively; $K$ is said to be an irreducible cutting of $M$ between $A$ and $B$ provided $K$ cuts $M$ between $A$ and $B$ but no proper subset of $K$ cuts $M$ between $A$ and $B$.

Theorem 10. If $A$ and $B$ are any two mutually exclusive closed subsets of a continuous curve $M$, and $K_1$ is any bounded cutting of $M$ between $A$ and $B$, then $K_1$ contains an irreducible cutting of $M$ between $A$ and $B$.

In proving Theorem 10 use will be made of the following lemmas.

Lemma 10a. If $K_1$ and $K_2$ are any two closed mutually exclusive subsets of a continuous curve $M$ one of which is bounded, then $K_1$ is contained wholly in the sum of a finite number of the components of $M - K_2$.

Proof. Suppose, on the contrary, that there exists an infinite collection $K_1, K_2, K_3, \ldots$, of components of $M - K_1$ each of which contains a point of $K_2$. For each $i > 0$ let $P_i$ denote a point of $K_i$. Let $H$ denote the set of points $P_1 + P_2 + \ldots$ Then $H$ is bounded. For if $K_1$ is bounded, $H$ is bounded because $H$ is a subset of $K_1$; and if $K_2$ is bounded, there exists a hypersphere $S$ enclosing $K_2$, and since $M$ is a continuous curve, only a finite number of the components of $M - K_2$ can contain points without $S$; and therefore $H$ is bounded, since all save possibly a finite number of its points lie on or within $S$. Hence, in any case, $H$ is bounded; and since it is infinite, it must have at least one limit point $P$. The point $P$ cannot belong to $K_2$, for $P \subset H \subset K_1$ and $K_1 \cdot K_2 = 0$. Hence $P$ belongs to some component $R$ of $M - K_2$. But this is impossible, since $M$ is connected in the kleinian and $R$ contains at most one point of $H$. Thus the supposition that lemma 10a is false leads to a contradiction.

Lemma 10b. Let $K$ be a closed cutting of a continuous curve $M$, let $G_1$ and $G_2$ be mutually exclusive subsets of $M - K$ each of which is the sum of a finite number of the components of $M - K$, and let $A_1$ and $A_2$ be closed subsets of $G_1$ and $G_2$ respectively such that $A_i$ $(i = 1, 2)$ contains at least one point in each component of $G_i$. A necessary and sufficient condition that $K$ be an irreducible cutting of $M$ between $A_1$ and $A_2$ is that $F_n(G_1) = F_n(G_2) = K$.

Lemma 10b is an obvious extension of Theorem 7.

Proof of Theorem 10. With the aid of R. L. Moore's lemma quoted in the proof of Theorem 1, together with the Borel Theorem, it is readily shown that $K_1$ contains a closed subset $K$ which cuts $M$ between $A$ and $B$. Since $K$ is bounded, and $A$ and $B$ are closed mutually exclusive subsets of $M$. Then by lemma 10a there exists a finite collection $R_1, R_2, \ldots, R_n$ of components of $M - K$, each containing a point of $A$ and such that $A \subset \bigcup_{i=1}^{n} R_i$. Let $K_1$
denote $\sum_{i=1}^{n} F_i(R_i)$. Then by lemma 7a, it follows that $K_i$ cuts $M$ between $A$ and $B$. Since $K_i$ and $B$ are closed mutually exclusive subsets of $M$, and $K_i$ is bounded, by lemma 10a a finite collection $D_1, D_2, \ldots, D_k$ of components of $M - K_i$ exists each containing a point of $B$ and such that $B \subseteq \sum_{i=1}^{k} D_i$. Let $K_i$ denote the set of points $\sum_{i=1}^{k} F_i(D_i)$. Then $K_i$ likewise cuts $M$ between $A$ and $B$; and since $K_i \subseteq K_i = \sum_{i=1}^{k} F_i(R_i)$, it follows readily by lemma 10b that $K_i$ is an irreducible cutting of $M$ between $A$ and $B$.

**Theorem 11.** Let $K$ be a bounded irreducible cutting of a continuous curve $M$. Then if $K$ is not connected, the components of $M - K$ are finite in number.

Proof. Suppose, on the contrary, that the collection $G$ of all the components of $M - K$ is infinite. Now since $K$ is not connected, it is the sum of two mutually exclusive closed and bounded point sets $K_1$ and $K_2$. Then by a theorem due to W. L. Ayres and the author $^1$ there exists a finite number of subcontinua $L_1, L_2, \ldots, L_n$ of $M$ whose sum separates $K_1$ and $K_2$ in $M$, i.e., $M - (L_1 + L_2 + \ldots + L_n)$ is the sum of two mutually separated point sets containing $K_1$ and $K_2$ respectively. But since each of the continua $L_1, L_2, \ldots, L_n$ is contained wholly in some single element of $G$, and $G$ is infinite, it follows that there exists an element of $G$ which contains no point of $L_1 + L_2 + \ldots + L_n$. But by Theorem 7 and corollary 7a, $F_\infty(g) = K_1$; hence $g + K$ is a continuum which contains both $K_1$ and $K_2$ but contains no point of $L_1 + L_2 + \ldots + L_n$, and therefore $L_1 + L_2 + \ldots + L_n$ does not separate $K_1$ and $K_2$ in $M$, contrary to supposition. Thus the supposition that Theorem 10 is false leads to a contradiction.

It is obvious from the proof of Theorem 11 that the theorem remains true if in its statement we substitute the words „closed and componentwise irreducible cutting“ for the words „irreducible cutting“. That Theorem 11 is not true, even in case $M$ is the entire euclidean plane, in the absence of the condition that $g$ is not connected$^2$ has been shown by Knauster$^3$. That the theorem does not remain true, in 3-dimensions, in the absence of the condition that $g$ is bounded$^4$ is shown by the following example. Using cylindrical coordinates $\phi, \theta, \xi$, in 3-dimensions, let $C_n (n = 2, 3, 4, \ldots)$ be defined by the relations $0 \leq \phi \leq \frac{\pi}{16}, \theta = \frac{\pi}{n}, -\infty < \xi < \infty$.

Let $N$ denote the continuum $\sum_{n=1}^{\infty} C_n$, and let $N'$ be the image of $N$ reflected in the plane $\phi \cos \theta = \frac{3}{2}$. Let $L_n (n = 2, 3, 4, \ldots)$ be a straight line interval joining the point $(\frac{1}{n}, \frac{\pi}{n}, n)$ to its image in $N'$. Let $M$ be the continuum $N + N' + \sum_{n=1}^{\infty} L_n$, and let $K$ be the set of points $\phi = 0, \theta = 0, -\infty < \xi < \infty$ (the $\xi$ axis) plus its image in $N'$. Then $K$ is an irreducible cutting of $M$, but obviously for each value of $n$, $[C_n + C_n$ (image of $C_n$ in $N') + L_n] - K$ is a component of $M - K$.

It would be interesting to determine whether or not Theorem 11 would remain true in the absence of the condition that $g$ is bounded, for the case where $M$ lies in the plane. Also it would be interesting to know whether or not, under the same conditions, either $F_\infty$ must consist of exactly two points or else $M - K$ has exactly two components. At present I am unable to answer either one of these questions.

3. **Cuttings of plane continuous curves all of whose subcontinua are continuous curves.** In this section we will be made of the following theorem.

**Theorem A.** If $R_1, R_2$, and $R_3$ are mutually exclusive connected point sets in the plane $S$, and $G$ denotes the set of all points in $S - (R_1 + R_2 + R_3)$ which are accessible from each of the sets $R_1, R_2$, and $R_3$, then $G$ contains not more than two points.

A proof for Theorem A will be found in my paper „Concerning plane closed point sets which are accessible from certain subsets of their complement“ $^5$.

A very interesting proof of Theorem A based on the results of the present paper is as follows: Suppose, contrary to Theorem A, that $G$ contains three points $X, Y$, and $Z$. Let $A, B$, and $C$ be points in $R_1, R_2$, and $R_3$ respectively. By hypotheses there exist arcs $AX, AY, AZ, BX, BY, BZ, CX, CY$, and $CZ$ such that $AX + AY + AZ \subseteq R_1 + X + Y + Z, BX + BY + BZ \subseteq R_2 + X + Y + Z$, and

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$^2$ W. L. Ayres, loc. cit., Theorem 4.


$^4$ Offered to Proc. Nat. Acad. of Sciences.
Irreducible cuttings of continua.

Let $M$ denote the continuum

$$K = \sum_{n=1}^{\infty} C_n.$$ 

Then every subcontinuum of $M$ is a continuous curve, and $K$ is a cutting of $M$. For suppose, on the contrary, that $K$ contains an irreducible cutting $H$ of $M$. Clearly $H$ must contain at least one point $P$ which is an interior point of the interval $K$ [i.e., a point different from either $(0, 0, 0)$ or $(0, 0, 1)$]. There exists an integer $n_1$ and an interval $J_1$ of the collection of intervals $I_n$ such that $P$ is an interior point of $J_1$. Let $N_1$ be the Sierpiński regular curve which was constructed in the equilateral triangle with base $J_1$. Since $N_1 - J_1$ is connected and $N_1 - J_1 \subseteq M - K$, therefore $N_1 - J_1$ lies wholly in some component $R_1$ of $M - H$. And if $R_2$ is any other component of $M - H$, $(R_2 \not\subseteq R_1)$, then, by lemma 5, $F_n(R_2) \cap H$ contains a point of $P$ which does not belong to $K$. Hence there exists an integer $n_2$ and an interval $J_2$ of the collection of intervals $I_n$ such that (1) $J_2 \subseteq J_1$ contains an interval $I$, (2) if $N_2$ denotes the Sierpiński regular curve which was constructed in the equilateral triangle whose base is $J_2$, (see above), then $N_2 - J_2 \subseteq H$. And since $F_n(N_2 - J_2) = J_2$ and $F_{n+1}(N_2 - J_2) = J_2$, it follows that $F_n(R_2) \cap I$ and $F_{n+1}(R_2) \cap I$. Hence $H \subseteq I$. But there exists an integer $n_3$ and an interval $J_3$ of the collection of intervals $I_n$ such that $J_3$ is a proper subset of $I$; and if $N_3$ denotes the Sierpiński regular curve constructed in the equilateral triangle whose base is $J_3$ (see above), then $N_3 - J_3$ is an open subset of $M$ and, by lemma 7, $F_n(N_3 - J_3)$ is a cutting of $M$; but $F_n(N_3 - J_3) = J_3$ is a proper subset of $H$, and $H$, by supposition, is an irreducible cutting of $M$. Thus the supposition that $K$ contains an irreducible cutting of $M$ leads to a contradiction.

Theorem 13. If $K$ is an irreducible cutting of a plane continuous curve $M$ every subcontinuum of which is a continuous curve such that $M - K$ has more than two components, then $K$ contains at most two points.

Proof. By hypothesis $M - K$ has at least three components $R_1, R_2,$ and $R_3$. By Corollary 7a, $F_{a}(R_1) = F_{a}(R_2) = F_{a}(R_3) = K$. Hence, by the above quoted theorem of the author's, each point of $K$ is accessible from each of the sets $R_1, R_2,$ and $R_3$. Therefore, by Theorem A, $K$ contains at most two points.

Theorem 14. Let $M$ be any plane continuous curve, let $K$ be any irreducible cutting of $M$ between the points $A$ and $B$ of $M$, let $R_1$ and $R_2$ be the components of $M - K$ containing $A$ and $B$ respectively, and suppose that every point of $K$ is accessible from each of the sets $R_1$ and $R_2$. Then either $K$ contains two points whose sum cuts $M$ (and hence $K$ contains an irreducible cutting of $M$) or $M - K = R_1 + R_2$ (and hence $K$ itself is an irreducible cutting of $M$).

Proof. Suppose $R_1 + R_2$ is not identical with $M - K$. Then a component $R_3$ of $M - (R_1 + R_2)$ exists. Now $F_{a}(R_3)$ contains not more than two points; if it did, it would contain at least three points $X, Y,$ and $Z$ each accessible from $R_3$. But $F_{a}(R_3) \subseteq K$, and hence, by hypothesis, each of the points $X, Y,$ and $Z$ is accessible from $R_1$ and also from $R_2$. But this is contrary to Theorem A. Hence $F_{a}(R_3)$ contains at most two points; and since, by lemma 5a, $F_{a}(R_3)$ cuts $M$, it follows that $K$ contains two points whose sum cuts $M$. This completes the proof.

Corollary 14a. Let $K$ be an irreducible cutting of a plane continuous curve $M$ such that $M - K$ has more than two components and such that there exist two of these $R_1$ and $R_2$ such that every point of $K$ is accessible from both $R_1$ and $R_2$; then $K$ contains at most two points.

Theorem 15. If every point of the irreducible cutting $K$ of a plane continuous curve $M$ is accessible from at least two components of $M - K$ then either $M - K$ has just two components or $K$ contains not more than two points.

Proof. Suppose $M - K$ has at least three components. Then $K$ must be countable. For if $K$ is uncountable, it readily follows that there exist components $R_i$ and $R_j$ of $M - K$ and an uncountable subset $E$ of $K$ such that every point of $E$ is accessible from both $R_i$ and $R_j$. Now by hypothesis $M - K$ has at least one component $R_4$ different from $R_i$ and from $R_j$. And since, by Corollary 7a, $F_{a}(R_4) = K$, then every point of $E$ is accessible from $R_i$ and $R_j$ and is a limit point of $R_4$. But this is contrary to a theorem of the author's. Therefore $K$ must be countable. Now by Theorem 4 it follows that $K$ has at most two components. And therefore, since $K$ is countable, it must contain at most two points.

1) G. T. Whyburn, Concerning plane closed point sets which are accessible from certain subsets of their complements, loc. cit., see remark following proof of Theorem 1. The theorem here used is an extension of Theorem A (above).

The University of Texas, March 2, 1928.