Concerning irreducible cuttings of continua [1].

By

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A subset K of a continuum M will be called a *cutting* of M, or will be said to *cut* M, provided that the set of points $M-K$ is not connected, i. e., is the sum of two non-vacuous mutually separated points sets; K will be called a *cutting of M between two points A and B* of M, or will be said to *cut M between A and B*, provided that $M-K$ is the sum of two mutually separated point sets M_a and M_b containing A and B respectively. A subset K of a continuum M will be called an *irreducible cutting of M* provided K cuts M but no proper subset of K cuts M; K will be called an *irreducible cutting of M between the points A and B of M* provided K cuts M between these two points but contains no proper subset which does.

These notions are related to the notions of *„coupure du plan“* and *„coupure irréductible du plan“* as used by Kuratowski in his memoir *Sur les coupures irréductibles du plan* [2]. For the case where M is the entire plane, or indeed where M is any continuous curve, the above definitions are equivalent [3]) to those of Kuratowski, or to this definition extended in an obvious way to continuous curves. However, for a continuum in general, such is not the case. If M is not a continuous curve, then a *coupure* of M in the sense of Kuratowski is not [3]) necessarily a cutting of M in the sense above defined. I shall quite frequently have occasion to refer to the above mentioned paper of Kuratowski. In this paper, among other results, I shall show that a large number of the theorems in Kuratowski's paper concerning the „coupures du plan“ subsist for cuttings of any continuous curve.

It will be shown in § 2 that every cutting of a continuous curve M between two points A and B of M contains an irreducible cutting of M between these two points. This theorem does not remain true for continua M in general. Indeed, as shown below, if A and B are points of *any indecomposable continuum M whatever*, then every cutting of M between these two points is reducible. As shown by Kuratowski (loc. cit.), not every cutting of M, even if M is in the plane, contains an irreducible cutting of M. However, I shall show, in § 3, that if every subcontinuum of a plane continuous curve M is a continuous curve then every cutting of M contains an irreducible cutting of M.

In § 1 I shall show that if K is an irreducible cutting of a bounded plane continuum M such that $M-K$ has at least three components, then K itself has at most two components. Hence, only two kinds of irreducible cuttings of a bounded plane continuum M exist which cut M into more than two components; these are continua and point sets which are the sum of two continua (either of which may reduce to a single point).

The point sets considered are assumed to lie in a Euclidean space. Theorems 4, 5, 12, 13, 14 and 15 hold only in 2-dimensions, whereas all the remaining ones are true in n-dimensions.

Definitions and notations. The term continuous curve will be used to designate any connected im kleinen continuum (bounded or not). By a *component* of a point set M is meant a connected subset of M which is not a proper subset of any other connected subset of M. A subset R of a point set M is an open subset of M provided $M-R$ is closed (in M). If R is an open subset of a set M, $F_M(R)$ will be used to denote the boundary of R with respect to M, i. e., the set of all those points of $M-R$ which are limit points of R; $F(R)$ will denote the boundary of R with respect to the whole space. A cutting K of a continuum M will be said to be a *componentwise irreducible* cutting of M provided that every subset of K which cuts M contains a point in each component of K. Obviously every irreducible cutting of a continuum is also componentwise irreducible. A point P is said to be acces-

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[1]) Presented to the American Mathematical Society May 7, 1927. See also papers presented Dec. 28, 1927 and Feb. 26, 1928.

[2]) Fund. Math., vol. 6 (1924) pp. 130—145.

[3]) Cf. R. L. Moore, Math. Zeit. vol. 15 (1922) pp. 254—260, Theorem 1.

sible from a point set R provided that if A is any point of R there exists a simple continuous arc AP such that $AP - P \subset R$.

1. *Irreducible cuttings of continua in general.*

Theorem 1. *If K is an irreducible cutting of a continuum M between the points A and B of M, then K is a closed set of points.*

Proof. It has been shown by R. L. Moore[1]) with the aid of a theorem due to Knaster and Kuratowski[1]) that every cutting of a continuum M between two of its points A and B contains a closed subset which cuts M between A and B. Hence, if K is an irreducible cutting of M between A and B, K must be closed.

Corollary 1 a. *Every irreducible cutting of a continuum is a closed set of points.*

Theorem 2. *Let K be an irreducible cutting of a continuum M between the points A and B of M, and let M_a and M_b be any two mutually separated point sets such that $M_a \supset A$, $M_b \supset B$, and $M_a + M_b = M - K$. Then $M_a + K$ and $M_b + K$ are continua.*

Proof. It follows with the aid of Theorem 1 that $M_a + K$ and $M_b + K$ are closed sets. Suppose, contrary to this theorem, that one of them, say $M_a + K$, is not connected. Then $M_a + K = N_1 + N_2$, where N_1 and N_2 are mutually separated sets and N_1 contains A. Since M is connected, $K \cdot N_1 \neq 0$ and $K \cdot N_2 \neq 0$. Hence $K \cdot N_1$ is a proper subset of K. But $M - K \cdot N_1$ is the sum of the two mutually separated point sets N_1 and $M_b + N_2$ which contain A and B respectively, contrary to the fact that K is an irreducible cutting of M between A and B. Therefore $M_a + K$ and $M_b + K$ are continua.

Corollary 2 a. *If K is any irreducible cutting of a continuum M, and M_1 and M_2 are any two mutually separated point sets whose sum is $M - K$, then $M_1 + K$ and $M_2 + K$ are continua.*

Theorem 3. *If A and B are any two points of an indecomposable continuum M, then every cutting of M between A and B is reducible.*

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[1]) R. L. Moore, *Concerning upper semi-continuous collections of continua which do not separate a given continuum.* Proc. Nat. Ac. Sc., vol. 10 (1924) pp. 356–360 Lemma 1; Knaster and Kuratowski, *Sur les ensembles connexes*, Fund. Math., vol. 2, (1921) pp. 206–255. Theorem 37. Moore's lemma is stated and proved only for the case of M bounded. However, by the method of inversion it is not difficult, with the aid of Knaster and Kuratowski's theorem, to show that the lemma holds without the restriction of boundedness on M.

Proof. Suppose, on the contrary, that there exists an irreducible cutting K of M between some two points A and B of M. By definition there exist two mutually separated point sets M_a and M_b such that $M_a \supset A$, $M_b \supset B$ and $M_a + M_b = M - K$. But by Theorem 2, $M_a + K$ and $M_b + K$ are continua, and since $(M_a + K) + (M_b + K) = M$, this contradicts the fact M is indecomposable.

Corollary 3. *Every cutting of an indecomposable continuum is reducible.*

In the light of Theorem 3, it might be supposed that some basic relation exists between the indecomposability of a continuum M and the fact that not every cutting of M between two points of M contains an irreducible cutting of M between these two points. In connection with such questions, the following example is of interest.

Example. *There exists a plane bounded continuum M every subcontinuum of which is decomposable which contains two points A and B such that every cutting of M between A and B is reducible.*

Let T be a non-dense perfect set on the interval $(0, 1)$ of the X-axis, and let A be the point $(\frac{1}{2}, 1)$. For each point X of T, let L_x be the straight line interval from A to X. Let $M = \sum_{X \subset T} L_x$. Then M is a bounded continuum, every subcontinuum of M is decomposable, and if B is any point of T, it is easily seen with the aid of Theorem 2 that every cutting of M between A and B is reducible.

This example, however, still leaves open the following questions. (1) *If a continuum M has the property that for every two points A and B of M it is true that no irreducible cutting of M between A and B exists, then is it true that M is indecomposable or that it contains an indecomposable continuum?* (2) *If every cutting of a continuum M is reducible, is M necessarily indecomposable?*

Theorem 4. *If the closed componentwise irreducible cutting K of a plane bounded continuum M has more than two components, then $M - K$ has just two components. Hence, $M - K$ is the sum of two mutually separated and connected point sets.*

Proof. Since by hypothesis K has more than one component and is a componentwise irreducible cutting of M, therefore if H is any component of K, $M - H$ is connected and hence lies wholly in one complementary domain of H; and for not more than one component H of K is it true that $M-H$ lies in a bounded complementary domain of H. Hence, by the principle of inversion, we may assume, without loss of generality, that for each component H of K, $M-H$ is a subset of the unbounded complementary domain of H. For each component H of K let H_0 denote the continuum H plus all of its bounded complementary domains, and let G denote the collection of all the continua $[H_0]$ thus obtained. Then since K is closed, it follows with the aid of a theorem of R. L. Moore's [1]) that G is an upper semi-continuous collection of bounded continua. Clearly no one of these continua separates the plane and no one has any point in common with $M-K$. Hence, if S' denotes the space whose elements are the continua of the collection G plus all the points in the plane which belong to no element of G, by a result of R. L. Moore's [2]) S' is homeomorphic with the ordinary euclidean plane, and axioms $1-8$ of R. L. Moore's paper „*On the foundations of plane analysis situs*" [3]) are satisfied. Then since K is a subset of a totally disconnected set of elements in S', and $M-K$ is not connected, it follows by a theorem of R. G. Lubben's [4]), that there exists a simple closed curve J of elements of S' such that $J.M\equiv K$ and both the interior and the exterior of J contain points of $M-K$. Let I and E denote the interior and exterior respectively of J. Then I and E are ordinary point sets, and if M_1 and M_2 denote the point sets $I.M$ and $E.M$ respectively, M_1 and M_2 are mutually separated. I shall show that each of them is connected. Suppose M_1 is not connected. Then $M_1+E+J-K$ is not connected, and by application of R. G. Lubben's theorem quoted above, there exists a simple closed curve J_1 of elements of S' such that $J_1.(M_1+E+J-K)=0$, and both the interior and exterior of J_1 contain points of M_1. Hence J_1 must lie, except for those elements which belong to G, wholly in I. It follows that there exist two elements („points") A and B of G and an „arc" AXB of elements of J_1 such that (1) $AXB-(A+B)\subset I$, (2) each of the two domains R_1 and R_2 into which AXB divides

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[1]) *Concerning upper semi-continuous collections of continua*, Trans. Amer. Math. Soc., vol. 26 (1925), pp. 416—428.

[2]) Loc. cit.

[3]) Trans. Amer. Math. Soc., vol. 17 (1916), pp. 131—164.

[4]) *The separation of mutually separated subsets of a continuum by curves*, Bull. Amer. Math. Soc., vol. 32 (1926), p. 114 (abstract).

I contains points of M_1. Let AWB and AZB denote the two „arcs" of J from A to B. Since by hypothesis K has at least three components, therefore one of the segments $AWB-(A+B)$ and $AZB-(A+B)$, say $AWB-(A+B)$, must contain a component C of K. Let J_3 denote the „simple closed curve" $AZBXA$ and suppose R_1 is its interior. Then since J_3 does not contain C, $J_3.K$ is a proper subset of K which contains no point of the component C of K. But $0\neq R_1.M\neq M-J_3.K$, and clearly $R_1.M$ and $M-(J_3.K+R_1.M)$ are mutually separated point sets whose sum is $M-J_3.K$. This contradicts the hypothesis that K is a componentwise irreducible cutting of M, and therefore M_1 is connected. That M_2 is connected follows by a similar argument, after performing an inversion of the plane. The truth of Theorem 4 is therefore established.

Corollary 4a. *If K is a closed componentwise irreducible cutting of a bounded plane continuum M such that $M-K$ has at least three distinct components, then K is either a continuum or the sum of two continua (either of which may reduce to a single point).*

Corollary 4b. *Let H, L, and N be bounded continua in the plane such that (1) $H.L=H.N=L.N=K$, (2) $H-K$, $L-K$, and $N-K$ are connected point sets. Then K is either a continuum or the sum of two continua.*

Proof. Let M denote the continuum $H+L+N$. Then since each component of K contains a limit point of each of the connected sets $H-K$, $L-K$, and $N-K$, it follows that K is a closed componentwise irreducible cutting of M. And since $M-K$ has the three distinct components $H-K$, $L-K$, and $N-K$, then by corollary 4a, K is either a continuum or the sum of two continua.

Since every irreducible cutting of a continuum is also componentwise irreducible, and since, by Corollary 1a, every irreducible cutting of a continuum is a closed set of points, we have the following theorem.

Theorem 5. *If the irreducible cutting K of a bounded plane continuum M has more than two components, then $M-K$ has just two components, and hence $M-K$ is the sum of two mutually separated connected point sets.*

Corollary 5a. *If the irreducible cutting K of a bounded plane continuum is totally disconnected and contains more than two points,* *then $M-K$ is the sum of two mutually separated and connected point sets.*

It is to be noted that for cuttings K of a continuum which are totally disconnected, the properties of being an „irreducible cutting of M" and a „componentwise irreducible cutting of M" are equivalent.

Theorem 6. *Let K be an irreducible cutting of a continuum M between the points A and B of M, let P be any point of K and R any bounded domain containing P, and let N denote the component of $M.[R+F(R)]$ which contains P. Then if $K.F(R)=0$, $K.N$ is a cutting of N.*

Proof. By hypothesis $M-K=M_a+M_b$, where M_a and M_b are mutually separated and contain A and B respectively. By Theorem 2, M_a+K and M_b+K are continua. Then since $N.K$ and $F(R)$ are bounded and $(N.K).F(R)=0$, it follows with the aid of two theorems of Miss Mullikin's[1]) that $M_a.N\neq 0$ and $M_b.N\neq 0$. But $N-N.K=M_a.N=M_b.N$, and clearly $M_a.N$ and $M_b.N$ are mutually separated point sets. Hence $N.K$ is a cutting of N.

Corollary 6a. *Let P be any point of an irreducible cutting K of a continuum M, let R be any bounded domain containtng P, and let N be the component of $M.[R+F(R)]$ containing P. Then if $K.F(R)=0$, $K.N$ is a cutting of N.*

Examples are easily constructed showing that Theorem 6 is not true in the absence of the condition that $K.F(R)=0$.

2. *Irreducible cuttings of continuous curves.* **Theorem 7.** *In order that a closed cutting K of a continuous curve M between the points A and B of M should be an irreducible cutting of M between A and B it is necessary and sufficient that if R_a and R_b denote the components of $M-K$ containing A and B respectively then $F_M(R_a)=F_M(R_b)=K$*[2]).

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[1]) *Certain theorems relating to plane connected point sets*, Trans. Amer. Math. Soc., vol. 24 (1922), pp. 144—162, Theorems 2 and 1. These theorems hold in n-dimensions.

[2]) For the case where M is the entire euclidean space, this theorem is equivalent to a theorem of Kuratowski's (Cf. *Sur les coupures irréductibles du plan*, loc. cit.). The same is true of Theorems 8 and 9 in this section. The methods of proof and lemmas used are similar to those used by Kuratowski to prove his theorems.

In proving Theorem 7, use will be made of the following easily established lemma.

Lemma 7a. *If R is any open subset of a continuous curve M, and $M-\overline{R}\neq 0$, then $F_M(R)$ cuts M between every pair of points belonging to R and $M-\overline{R}$ respectively.*

Proof of Theorem 7. The condition is sufficient. For if H is any proper subset of K, and P is a point of $K-H$, then since $P\subset F_M(R_a)$ and $P\subset F_M(R_b)$, therefore $P+R_a+R_b$ is connected; and hence H does not cut M between A and B. The condition is also necessary. For suppose one of sets $F_M(R_a)$, and $F_M(R_b)$, say $F_M(R_a)$, is $\neq K$. Then since $F_M(R_a)\subset K$, $F_M(R_a)$ is a proper subset of K. But by Lemma 7a, $F_M(R_a)$ cuts M between A and B, contrary to the fact that K is an irreducible cutting of M between A and B.

Corollary 7a. *In order that a closed cutting K of a continuous curve M should be irreducible it is necessary and sufficient that if R is any component of $M-K$, then $F_M(R)=K$.*

Examples are easily constructed to show that the condition of Theorem 7 is not necessary in the absence of the stipulation that the continuum M is a continuous curve.

Theorem 8. *Every cutting K_0 of a continuous curve M between the points A and B of M contains an irreducible cutting of M between A and B.*

Proof. By a lemma of R. L. Moore's[1]), K_0 contains a closed, subset K which cuts M between A and B. Let D denote the component of $M-K$ which contains A. Then by lemma 7a, $F_M(D)$ cuts M between A and B. And if R_b denotes the component of $M-F_M(D)$ which contains B, then by lemma 7a, $F_M(R_b)$ cuts M between A and B. But if R_a denotes the component of $M-F_M(R_b)$ which contains A, then since R_a contains D and $F_M(R_b)\subset F_M(D)$, it follows that $F_M(R_a)=F_M(R_b)$. Hence, by Theorem 7, $F_M(R_a)$ is an irreducible cutting of M between A and B,

Theorem 9. *If the cutting K of a continuous curve M contains no interior point relative to M and is such that $M-K$ has only a finite number of components, then K contains an irreducible cutting of M.*

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[1]) Cf. reference to R. L. Moore in proof of Theorem 1. and note on the unbounded case in the same footnote.

Proof. Let the components of $M-K$ be denoted by $E_1, E_2, \dots, E_n$. By R. L. Moore's lemma just quoted, K contains a closed subset K_1 which cuts M. Then $M-K_1$ has at most n components. For if not, it would have one component R which contains no point of $M-K$, because, for each $i \leqslant n$, E_i lies wholly in some single component of $M-K_1$. Then $R \subset K$. But if P is a point of R, then since M is connected im kleinen it follows that P is not a limit point of $M-K$. Hence P is an interior point of K relative to M, contrary to hypothesis. Hence the closed subset K_1 of K cuts M into just a finite number of components; and with the aid of inductive methods similar to those used by Kuratowski to prove an analogous theorem for a „coupure du plan" (Cf. Kuratowski, loc. cit.). The proof of Theorem 9 is easily completed.

The question as to whether or not Theorem 9 remains true on the omission of the condition that „K contain no interior point relative to M" is very interesting. This raised the following interesting questions: (1) *Does every continuous curve contain an irreducible cutting of itself?* [1] (2) *Does every open subset of a continuous curve M contain an irreducible cutting of M?* [1]. These questions are related to question (2) stated above just before the statement of Theorem 4.

We may extend the notion of „irreducible cutting between two points" to closed sets as follows.

Definitions. If A and B are mutually exclusive closed subsets of a continuum M, the subset K of M is said to be a cutting of M between A and B, or to cut M between A and B, provided $M-K$ is the sum of two mutually separated point sets M_a and M_b containing A and B respectively; K is said to be an irreducible cutting of M between A and B provided K cuts M between A and B but no proper subset of K cuts M between A and B.

Theorem 10. *If A and B are any two mutually exclusive closed subsets of a continuous curve M, and K_0 is any bounded cutting of M between A and B, then K_0 contains an irreducible cutting of M between A and B.*

In proving Theorem 10 use will be made of the following lemmas.

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[1] Since this paper was written, Mr. J. H. Roberts has shown that for plane continuous curves the answer to both of these questions is affirmative.

Lemma 10a. *If K_1 and K_2 are any two closed mutually exclusive subsets of a continuous curve M one of which is bounded, then K_1 is contained wholly in the sum of a finite number of the components of $M-K_2$* [1].

Proof. Suppose, on the contrary, that there exists an infinite collection $R_1, R_2, R_3, \dots$, of components of $M-K_2$ each of which contains a point of K_1. For each $i>0$ let P_i denote a point of $K_1 . R_i$, and let H denote the set of points $P_1+P_2+P_3+\dots$ Then H is bounded. For if K_1 is bounded, H is bounded because H is a subset of K_1; and if K_2 is bounded, there exists a hypersphere S enclosing K_2, and since M is a continuous curve, only a finite number of the components of $M-K_2$ can contain points without S; and therefore H is bounded, since all save possibly a finite number of its points lie on or within S. Hence, in any case, H is bounded; and since it is infinite, it must have at least one limit point P. The point P cannot belong to K_2, for $P \subset H \subset K_1$ and $K_1 . K_2 = 0$. Hence P belongs to some component R of $M-K_2$. But this is impossible, since M is connected im kleinen and R contains at most one point of H. Thus the supposition that lemma 10a is false leads to a contradiction.

Lemma 10b. *Let K be closed cutting of a continuous curve M, let G_1 and G_2 be mutually exclusive subsets of $M-K$ each of which is the sum of a finite number of the components of $M-K$, and let A_1 and A_2 be closed subsets of G_1 and G_2 respectively such that A_i $(i=1, 2)$ contains at least one point in each component of G_i. A necessary and sufficient condition that K be an irreducible cutting of M between A_1 and A_2 is that $F_M(G_1) = F_M(G_2) = K$.*

Lemma 10b is an obvious extension of Theorem 7.

Proof of Theorem 10. With the aid of R. L. Moore's lemma quoted in the proof of Theorem 1, together with the Borel Theorem, it is readily shown that K_0 contains a closed subset K which cuts M between A and B. Since K is bounded, and A and K are closed mutually exclusive subsets of M. then by lemma 10a there exists a finite collection $R_1, R_2, \dots, R_m$ of components of $M-K$ each containing a point of A and such that $A \subset \sum_{n=1}^{m} R_n$. Let K_1

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[1] For the case where M is the entire space, see G. T. Whyburn and W. L. Ayres, *On continuous curves in n-dimensions*, Bull. Amer. Math. Soc. vol. 34 (1928), Theorem 3.

denote $\sum_{n=1}^{m} F_M(R_n)$. Then by lemma 7a, it follows that K_1 cuts M between A and B. Since K_1 and B are closed mutually exclusive subsets of M, and K_1 is bounded, by lemma 10a a finite collection $D_2, D_2, \ldots, D_k$ of components of $M-K$, exists each containing a point of B and such that $B \subset \sum_{i=1}^{k} D_i$. Let K_2 denote the set of points $\sum_{i=1}^{k} F_M(D_i)$. Then K_2 likewise cuts M between A and B; and since $K_2 \subset K_1 = \sum_{i=1}^{k} F_M(R_i)$, it follows readily by lemma 10b that K_2 is an irreducible cutting of M between A and B.

Theorem 11. *Let K be a bounded irreducible cutting of a continuous curve M. Then if K is not connected, the components of $M-K$ are finite in number.*

Proof. Suppose, on the contrary, that the collection G of all the components of $M-K$ is infinite. Now since K is not connected, it is the sum of two mutually exclusive closed and bounded point sets K_1 and K_2. Then by a theorem due to W. L. Ayres and the author [1]) there exists a finite number of subcontinua $L_1, L_2, \ldots, L_n$ of M whose sum separates K_1 and K_2 in M, i. e., $M-(L_1 + L_2 + \ldots + L_n)$ is the sum of two mutually separated point sets containing K_1 and K_2 respectively. But since each of the continua $L_1, L_2, \ldots L_n$ is contained wholly in some single element of G, and G is infinite, it follows that there exists an element g of G which contains no point of $L_1 + L_2 + \ldots + L_n$. But by Theorem 7 and corollary 7a, $F_M(g) = K$; hence $g + K$ is a continuum which contains both K_1 and K_2 but contains no point of $L_1 + L_2 + \ldots + L_n$, and therefore $L_1 + L_2 + \ldots + L_n$ does not separate K_1 and K_2 in M, contrary to supposition. Thus the supposition that Theorem 10 is false leads to a contradiction.

It is obvious from the proof of Theorem 11 that the theorem remains true if in its statement we substitute the words „closed and componentwise irreducible cutting“ for the words „irreducible cutting“. That Theorem 11 is not true, even in case M is the entire euclidean plane, in the absence of the condition that „K is not connected“ has been shown by Knaster [1]). That the theorem does not remain true, in 3-dimensions, in the absence of the condition that „K is bounded“ is shown by the following example. Using cylindrical coordinates ϱ, θ, ζ, in 3-dimensions, let C_n $(n = 2, 3, 4, \ldots)$ be defined by the relations $0 \leqslant \varrho \leqslant 1/n$, $\theta = \pi/n$, $-\infty < \zeta < \infty$. Let N denote the continuum $\sum_{n=2}^{\infty} C_n$, and let N' be the image of N reflected in the plane $\varrho \cos\theta = 3/2$. Let $L_n (n = 2, 3, 4, \ldots)$ be a straight line interval joining the point $(1/n, \pi/n, n)$ to its image in N'. Let M be the continuum $N + N' + \sum_{n=2}^{\infty} L_n$, and let K be the set of points $\varrho = 0$, $\theta = 0$, $-\infty < \zeta < \infty$ (the ζ axis) plus its image in N'. Then K is an irreducible cutting of M, but obviously for each value of n, $[C_n + C'_n$ (image of C_n in N') $+ L_n] - K$ is a component of $M-K$.

It would be interesting to determine whether or not Theorem 11 would remain true in the absence of the condition that K is bounded, for the case where M lies in the plane. Also it would be interesting to know whether or not, under the same conditions, either K must consist of exactly two points or else $M-K$ has exactly *two* components. At present I am unable to answer either one of these questions.

3. *Cuttings of plane continuous curves all of whose subcontinua are continuous curves.*

In this section use will be made of the following theorem.

Theorem A. *If R_1, R_2, and R_3 are mutually exclusive connected point sets in the plane S, and G denotes the set of all points in $S-(R_1 + R_2 + R_3)$ which are accessible from each of the sets R_1, R_2, and R_3, then G contains not more than two points.*

A proof for Theorem A will be found in my paper „*Concerning plane closed point sets which are accessible from certain subsets of their complement*“ [2]). A very interesting proof of Theorem A based on the results of the present paper is as follows: Suppose, contrary to Theorem A, that G contains three points X, Y and Z. Let A, B and C be points in R_1, R_2 and R_3 respectively. By hypothesis there exist arcs AX, AY, AZ, BX, BY, BZ, CX, CY, and CZ such that $AX + AY + AZ \subset R_1 + X + Y + Z$, $BX + BY + BZ \subset R_2 + X + Y + Z$, and

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[1]) G. T. Whyburn and W. L. Ayres, loc. cit., Theorem 4.

[1]) B. Knaster, *Quelques coupures singulières du plan*, Fund. Math., vol. 7 (1925), pp. 264—289.

[2]) Offered to Proc. Nat. Acad. of Sciences.

$CX+CY+CZ \subset R_3+X+Y+Z$. Let H, L and N denote the continua $AX+AY+AZ$, $BX+BY+BZ$, and $CX+CY+CZ$ respectively, and K the set of points $X+Y+Z$. Then $H.L=L.N=H.N=K$ and, clearly $H-K$, $L-K$ and $N-K$ are connected point sets. Hence by corollary 4b to Theorem 4, K is either a continuum or the sum of two continua — a conclusion which is absurd, since K consists of exactly three points.

Theorem 12. *Every cutting of a plane continuous curve M every subcontinuum of which is a continuous curve contains an irreducible cutting of M.*

Proof. Let K be any cutting of M. Then K cuts M between some two points A and B of M; and by Theorem 8, K contains an irreducible cutting K_1 of M between A and B. Let R_a and R_b denote the components of $M-K$ containing A and B respectively; and let R_c be any other component of $M-K_1$. (If R_c does not exist, then obviously K_1 is an irreducible cutting of M). By Theorem 7, $F_M(R_a)=F_M(R_b)=K_1$; and since $F_M(R_c) \subset K_1$ and, by a theorem of the author's [1], the boundary with respect to M of every connected open subset of M is accessible from that open subset, it follows that every point of K_1 is accessible from each of the three sets R_a, R_b and R_c. Therefore, by Theorem A, K_1 contains at most two points. By lemma 5a, K_1 cuts M. Hence either K_1 is an irreducible cutting of M or it contains a *cut point* of M, which is necessarily an irreducible cutting of M. Therefore K contains an irreducible cutting of M.

The following example shows that Theorem 12 is not true in 3-dimensions.

Example. *There exists, in 3-dimensions, a continuous curve M every subcontinuum of which is a continuous curve and a cutting K of M which contains no irreducible cutting of M.*

Using cylindrical coordinates, ϱ, θ, ζ, let K denote the interval of the ζ-axis from 0 to 1. For each positive integer n, let K be subdivided into a set I_n of n equal intervals by inserting $n-1$ points of division on K (i. e., we effect a rational subdivision of K). For each $n>0$ let us construct, in the plane $\theta=\pi/n$, an equilateral triangle on each interval of the collection I_n as a base, and with each of these triangles as a basis construct a Sierpiński regular curve [1]; let C_n be the sum of all such regular curves thus constructed.

Let M denote the continuum

$$K+\sum_{n=1}^{\infty} C_n.$$

Then every subcontinuum of M is a continuous curve, and K is a cutting of M. But K contains no irreducible cutting of M. For suppose, on the contrary, that K contains an irreducible cutting H of M. Clearly H must contain at least one point P which is an interior point of the interval K [i. e., a point different from either (0, 0, 0) or (0, 0, 1)]. There exists an integer n_1 and an interval J_1 of the collection of intervals I_{n_1} such that P is an interior point of J_1. Let N_1 be the Sierpiński regular curve which was constructed in the equilateral triangle with base J_1. Since N_1-J_1 is connected and $N_1-J_1 \subset M-K$, therefore N_1-J_1 lies wholly in some component R_1 of $M-H$. And if R_2 is any other component of $M-H$, $(R_2 \neq R_1)$, then since, by Theorem 7, $F_M(R_2)=H \supset P$, it easily follows that every neighborhood of P contains a point of R_2 which does not belong to K. Hence there exists an integer n_2 and an interval J_2 of the collection of intervals I_{n_2} such that (1) $J_2 . J_1$ contains an interval I, (2) if N_2 denotes the Sierpiński regular curve which was constructed in the equilateral triangle whose base is J_2, (see above), then $N_2-J_2 \subset R_2$. And since $F_{N_1}(N_1-J_1)=J_1$ an $F_{N_2}(N_2-J_2)=J_2$, it follows that $F_M(R_1) \supset I$ and $F_M(R_2) \supset I$. Hence $H \supset I$. But there exists an integer n_3 and an interval J_3 of the collection of intervals I_{n_3} such that J_3 is a proper subset of I; and if N_3 denotes the Sierpiński regular curve constructed in the equilateral triangle whose base is J_3 (see above), then N_3-J_3 is an open subset of M and, by lemma 7a, $F_M(N_3-J_3)$ is a cutting of M; but $F_M(N_3-J_3)=J_3$ is a proper subset of H, and H, by supposition is an irreducible cutting of M. Thus the supposition that K contains an irreducible cutting of M, leads to a contradiction.

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[1] G. T. Whyburn, *Concerning the open subsets of plane continuous curves*, Proc. Nt. Acad. Sc., vol. 13 (1927), pp. 650—657, Theorem 3.

[1] Cf. W. Sierpiński, Prace Matematyczno-Fizyczne, vol. 27 (1915); Comptes Rendus, vol. 160 (1915), p. 302. See also Knaster and Kuratowski, Bull. Amer. Math. Soc. vol. 33 (1927) pp. 106, 107.

Theorem 13. *If K is an irreducible cutting of a plane continuous curve M every subcontinuum of which is a continuous curve such that $M-K$ has more than two components, then K contains at most two points.*

Proof. By hypothesis $M-K$ has at least three components R_a, R_b, and R_c. By Corollary 7 a, $F_M(R_a)=F_M(R_b)=F_M(R_c)=K$. Hence, by the above quoted theorem of the author's, each point of K is accessible from each of the sets R_a, R_b, and R_c. Therefore, by Theorem A. K contains at most two points.

Theorem 14. *Let M be any plane continuous curve, let K be any irreducible cutting of M between the points A and B of M, let R_a and R_b be the components of $M-K$ containing A and B respectively; and suppose that every point of K is accessible from each of the sets R_a and R_b. Then either K contains two points whose sum cuts M (and hence K contains an irreducible cutting of M) or $M-K=R_a+R_b$ (and hence K itself is an irreducible cutting of M).*

Proof. Suppose R_a+R_b is not identical with $M-K$. Then a component R_c of $M-(K+R_a+R_b)$ exists. Now $F_M(R_1)$ contains not more than two points; for if it did, it would contain at least three points X, Y, and Z each accessible from R_c. But $F_M(R_c)\subset K$, and hence, by hypothesis, each of the points X, Y, and Z is accessible from R_a and also from R_b. But this is contrary to Theorem A. Hence $F_M(R_c)$ contains at most two points; and since, by lemma 5a, $F_M(R_c)$ cuts M, it follows that K contains two points whose sum cuts M. This completes the proof.

Corollary 14a. *Let K be an irreducible cutting of a plane continuous curve M such that $M-K$ has more than two components and such that there exist two of these R_1 and R_2 such that every point of K is accessible from both R_1 and R_2; then K contains at most two points.*

Theorem 15. *If every point of the irreducible cutting K of a plane continuous curve M is accessible from at least two components of $M-K$ then either $M-K$ has just two components or K contains not more than two points.*

Proof. Suppose $M-K$ has at least three components. Then K must be countable. For if K is uncountable, it readily follows that there exist components R_1 and R_2 of $M-K$ and an uncountable subset E of K such that every point of E is accessible from both R_1 and R_2. Now by hypothesis $M-K$ has at least one component R_3 different from R_1 and from R_2. And since, by Corollary 7a, $F_M(R_3)=K$, then every point of E is accessible from R_1 and R_2 and is a limit point of R_3. But this is contrary to a theorem of the author's 1). Therefore K must be countable. Now by Theorem 4 it follows that K has at most two components. And therefore, since K is countable, it must contain at most two points.

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1) G. T. Whyburn, *Concerning plane closed point sets which are accessible from certain subsets of their complements*, loc. cit., see remark following proof of Theorem 1. The theorem here used is an extension of Theorem A (above).

The University of Texas, March 2, 1928.