Posons

$$U = U_1 + U_2 + U_3 + \dots \tag{8}$$

Nous avons $U_n \subset T_{k_{r_n}} \subset S_{r_n} E_{k_{r_n}} \subset S E_{k_{r_n}}$, et $k_{r_n} < k_{r_{n+1}}$ (pour $n = 1, 2, 3, \dots$): les termes de la somme (6) étant disjoints et ouverts dans S, il résulte de (8) que les termes de la somme (8) sont disjoints et ouverts dans U. Or, U_n, en tant que homéomorphe à R_n, est homéomorphe à E_n: il en résulte donc, de de (1) et (8), que les ensembles E et U sont homéomorphes.

Or, de $U_n \subset S$, pour $n = 1, 2, 3, \dots$, et de (8), résulte que $U \subset S$. Nous avons donc $dE \leqslant dS$, donc (S étant homéomorphe à H): $dE \leqslant dH$, contrairement à (4).

Notre assertion est ainsi démontrée.

Or, il est à remarquer qu'on pourrait sans peine construire des espaces, dont les types de dimensions sont intermédiaires entre dE. et $d\omega$.

D'autre part, on pourrait démontrer l'existence des ensembles H dont les types de dimensions ne sont pas finis, et tels que $dH < dE$.

En effet, M. Mazurkiewicz a démontré [1]) l'existence, pour tout n naturel, d'un ensemble G_δ, soit M_n, situé dans R_n, punctiforme et non homéomorphe à aucun sous-ensemble de R_{n-1}. Or, R_n étant superposable avec E_n, il existe un sous-ensemble de N_n de E_n, superposable avec M_n, et on voit sans peine que l'ensemble $H = N_1 + N_2 + N_3 + \dots$ jouit des propriétés désirées.

[1]) *Fund. Math* t. X, p. 311.

Remark on the generalised Bernstein's theorem.

By

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Bernstein's theorem, that from $2\mathfrak{m} = 2\mathfrak{n}$ follows $\mathfrak{m} = \mathfrak{n}$, for every pair of cardinal numbers was recently generalised in the following way [1]):

Let a set E be twice decomposed into two equivalent parts:

$$E = M + N = P + Q, \quad MN = 0 = PQ$$

and let φ and ψ be one-one correspondances between M and N and P and Q respectively, then the sets M and Q can be decomposed into *four* disjunctive subsets:

$$M = M_1 + M_2 + M_3 + M_4$$

$$Q = Q_1 + Q_2 + Q_3 + Q_4$$

such that:

$$Q_1 = \alpha_1(M_1), \quad Q_2 = \alpha_2(M_2), \quad Q_3 = \alpha_3(M_3), \quad Q_4 = \alpha_4(M_4),$$

where α_i $(i = 1 \dots 4)$ are four functions belonging to the group P created by combining the functions φ and ψ. (The first of them are:

$$0 = \text{Identity}, \quad \varphi, \ \psi, \ \varphi\psi, \ \psi\varphi, \ \varphi\psi\varphi, \ \psi\varphi\psi, \text{ etc.})$$

I shall prove, that in this theorem the number *four* cannot be diminished [2]). I shall define indeed a set E. two subsets M and Q and two one-one correspondances between M and $E - M$ and Q

[1]) See C. Kuratowski, *Fund. Math.* VI. p. 240—243, D. König *Math. Ann.* 77, and *Fund. Math.* VIII. See also Banach and Tarski, *Fund. Math.* VI, where use is made of this theorem.

[2]) This problem was raised in the Seminary of Prof. C. Kuratowski.

and $E-Q$ respectively, so that there does not exist any decomposition of the sets M ane Q into *three* disjunctive subsets united by some functions belonging to the group Γ.

For the sake of better understanding I shall at first give an example proving the impossibility of a decomposition into *two* parts possessing the above stated properties.

The set E consists of 8 points (see fig. 1).

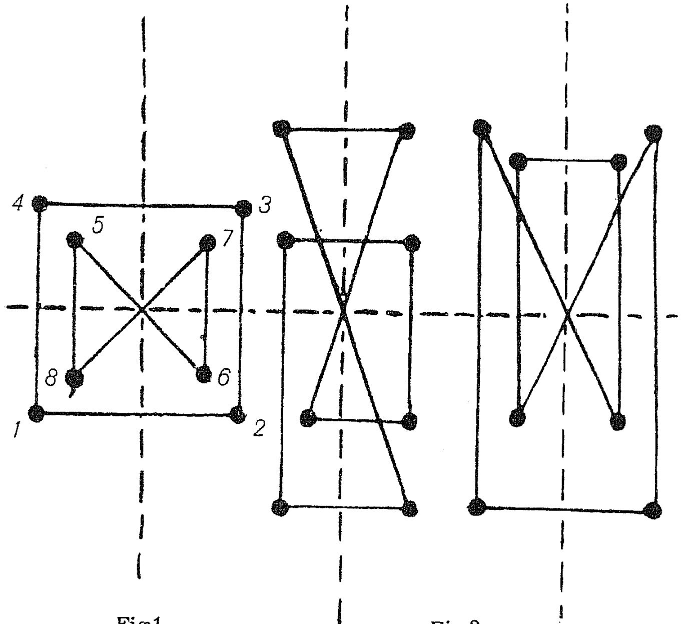

Fig 1 Fig 2

M is the part of E on the left of the vertical line, Q below the horizontal one.

$$\varphi(1)=2,\quad \varphi(4)=3,\quad \varphi(5)=6,\quad \varphi(8)=7;$$

$$\psi(1)=4,\quad \psi(2)=3,\quad \psi(8)=5,\quad \psi(6)=7:$$

and inversely:

$$\varphi(2)=1.\quad \varphi(3)=4,\ldots,\quad \psi(7)=6.$$

Two corresponding points are connected by a segment. In our case the group Γ reduces itself evidently to 4 functions:

$$0,\quad \varphi,\quad \psi\quad \varphi\psi(=\psi\varphi)$$

Let us now suppose, there are existing two functions α_1 and α_2 of Γ and a decomposition of both M and Q into two disjunctive parts, such that:

$$M = M_1 + M_2,\quad Q = Q_1 + Q_2,$$

$$Q_1 = \alpha_1(M_1),\quad Q_2 = \alpha_2(M_2).$$

It follows that

1°: The elements 1 and 4 correspond to 1 and 2.

2°: The elements 5 and 8 correspond to 8 and 6.

From 1° it follows that there are only two possibilities: one of the transformations α_1 and α_2 is identic with 0 and the other one with $\varphi\psi$, or one with φ and the other one with ψ; i. e. the only two possibilities are: $(0, \varphi\psi)$ and (φ, ψ).

On the other hand we conclude from 2° that the only two possibilities are there $0, \varphi$ and $\psi, \varphi\psi$.

Thus we get a contradiction. if we suppose that there exists a decomposition of M and Q into *two* parts possessing the above mentioned properties. However, there exists in the case of the fig. 1 a decomposition into *three* parts.

In the example of fig. 2 even such a decomposition into *three* parts is impossible.

(For typographic reasons the set E is drawn in two parts. We have to imagine the drawings one *on* the other).

The set E is composed of 16 elements. the sets M and Q are as in the former example on the left of the vertical and below the horizontal line, resp.

The proof may be carried out in a quite analogous way as before.

The group Γ reduces here itself actually to 8 functions:

$$0,\quad \varphi,\quad \psi,\quad \psi\varphi,\quad \varphi\psi,\quad \varphi\psi\varphi,\quad \psi\varphi\psi,\quad \varphi\psi\varphi\psi(=\psi\varphi\psi\varphi).$$