# The notion of a directed point in $n$ dimensions.

By

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The appearance, in analysis, of the symbol $x+0$, $x-0$, also written $x^+$, $x^-$ or $x_+$, $x_-$, carries with it a new idea.

It is true that this idea seems to have received no explicit formulation up to the present, and that the use of the symbols has been purely incidental and notational, an abbreviation of language in the course of reasoning or explanation. But from their mere use and its gradual extension [1]), possible interpretations necessarily arise in the background of our minds.

We have still to work out these interpretations, to see by precisely what means the symbols $x+0$ and $x-0$ may be consistently invested with an existence independant of their incidental use, and in particular to examine what there may be of truth or untruth underlying the verbal convention sometimes made [2]), simultaneously with the symbolical convention, to speak of there as actual points, on a footing with the point $x$.

[1]) In its narrowest form, which seems to have been current as early as Riemann's time, it comprises solely the inclusion of the symbols within the functional brackets, to denote the right-hand and left-hand limits of the function:

$$f(x+0), \quad f(x-0).$$

Thence it is extended to the symbolical expression of the fact of a sequence of points converging to the point $x$ from the right or left:

$$x' \to x+0, \quad x'' \to x-0,$$

and finally to that of other than functional limits involving this mode of approach to $x$, as for instance right-hand and left-hand derivation of one-variabled functions.

[2]) Cf. L. C. Young, „The Theory of Integration" — Cambridge Tracts in Mathematics and Mathematical Physics, N° 21 (1927).



In a paper [1]) dealing with functions of bounded variation in $n$-space, I have briefly worked out the interpretations in the one-dimensional case. Il appears that the two symbols $x+0$ and $x-0$ should not be considered as on a footing with the point $x$, but rather as dividing between them the properties of that point. Or else they may be regarded more logically still perhaps as entities of a superior order, *represented* e. g. by monotone sequences of real points, as real numbers are *represented* by sequences of the original fractional numbers.

In the present paper, I propose to develop the theory and its interpretations systematically and in direct $n$-dimensional form. I shall illustrate it by the consideration of continuous functions of the new „point", and indicate the special significance of these functions in the theory of functions of position.

1. A set of $n$ real numbers $x_1, x_2, \dots, x_n$, in a given order, defines a *real point* $P$ in space of $n$ dimensions; the number $x_k$ is the $k^{\text{th}}$ *coordinate* of the *real point*. In one dimension, a real point is a real number.

The set of all real points whose $n$ coordinates satisfy a given linear relation

$$a_1 x_1 + a_2 x_2 + \dots + a_n x_n = c$$

is called a *plane*. The plane

$$x_k = 0$$

is called the $k^{\text{th}}$ *coordinate plane*. Planes whose defining relations only differ in the constant term are parallel (to one another). Division by a constant throughout the relation of course does not alter the plane it defines.

The *distance* between two real points is the square root of the sum of the squares of the differences between their $k^{\text{th}}$ coordinates, for $k = 1, 2, \dots, n$. This is independent of the order in which the two points are taken, and the magnitudes, only, not the signs, of the differences are involved.

A set $E$ of real points is said to be *bounded* if the distance

[1]) „Les fonctions additives d'ensemble, les fonctions de point à variation bornée et la généralisation de la notion d'espace à $n$ dimensions" Enseignement mathématique, 1927, Nos. 1—2—3, p. 63.

between any two of its points is bounded, and the upper bound of this distance is called the *span* of the set. A bounded set can always be exhibited as the sum of a finite number of sets each of span less than a given positive $\delta$.

A set of $n$ *signs* $\theta_1, \theta_2, \ldots, \theta_n$, $(\theta_n \equiv +, -, \text{ or } 0)$, is a given order, defines a *sense* $\Theta$ in space of $n$ dimensions; the sign $\theta_k$ is the $k^{\text{th}}$ *coordinate* of the *sense*. A sense is *definite* if none of its coordinates are 0. There are $3^n$ senses in space of $n$ dimensions, $2^n$ of which are definite.

Two senses, like two points, are only said to *coincide* if their $k^{\text{th}}$ coordinates, for each $k$, are identical. Two senses are said to be *opposite* (to one another) if their $k^{\text{th}}$ coordinates, for each $k$, are either opposite signs (the one $+$, the other $-$), or both 0. The sense opposite to a given sense $\Theta$ is thus unique; we may denote it by $-\Theta$.

Let us agree to define the *sum of two signs* as follows:

the sum of two equal signs is again the same sign;

the sum of any sign $\theta$ and of 0 is again $\theta$;

the sum of $+$ and $-$ is 0.

The *sum of two senses* $\Theta_1$ and $\Theta_2$ is then defined as the sense having for its $k^{\text{th}}$ coordinate the sum of the $k^{\text{th}}$ coordinates of the two given senses, and is written $\Theta_1 + \Theta_2$. The sum of $\Theta_1$ and $-\Theta_2$ is also written $\Theta_1 - \Theta_2$ and is called the *difference* between the two senses $\Theta_2$ and $\Theta_1$, in that order.

A *pair of real points* in a given order, $P_1$, $P_2$, say, is denoted by $\overrightarrow{P_1P_2}$. It has for *span* the distance between its two points and it has a *sense*, whose $k^{\text{th}}$ coordinate is the sign of the difference between the $k^{\text{th}}$ coordinates of $P_1$ and $P_2$, in that order (excess of the latter over the former). The absolute value of this difference is called the $k^{\text{th}}$ *component* of the span.

The two pairs $\overrightarrow{P_1P_2}$ and $\overrightarrow{P_2P_3}$ are said to have the *resultant* $\overrightarrow{P_1P_3}$. If $\overrightarrow{P_1P_2}$ has a sense $\Theta_1$ and $\overrightarrow{P_2P_3}$ a sense $\Theta_2$, then $\overrightarrow{P_1P_3}$ has a sense $\Theta$ whose $k^{\text{th}}$ coordinate is the sum of the $k^{\text{th}}$ coordinates of $\Theta_1$ and $\Theta_2$, unless these coordinates are opposite signs, when that of $\Theta$ may be $+$, $-$ or 0 according to the relative positions of $P_1$ and $P_2$.



If $\Theta_1$ and $\Theta_2$ only differ in places where the coordinate of one is 0, then we have therefore

$$\Theta = \Theta_1 + \Theta_2.$$

Another useful rule is that when the $k^{\text{th}}$ components of the spans of $\overrightarrow{P_1P_2}$ and $\overrightarrow{P_2P_3}$ do not coincide, the $k^{\text{th}}$ coordinates of the senses of $\overrightarrow{P_1P_3}$ and of the pair $\overrightarrow{P_1P_2}$ or $\overrightarrow{P_3P_3}$ which has the largest $k^{\text{th}}$ component, coincide, unless the latter coordinate were 0.

A succession (i. e. any countable, more than finite set, in countable order)

$$P_1, P_2, \ldots, P_i, \ldots$$

of real points is said to be a *monotone sequence* if $\overrightarrow{P_iP_{i+1}}$ has an invariable sense for all $i$. This sense is called *the sense of the sequence*. The adverb *strictly* is added to the word *monotone* if this sense is *definite*. In a monotone sequence, if two of the points coincide, then all coincide. The repetition of a single point constitutes an *improper* monotone sequence. In a *proper* monotone sequence no two points coincide. *In one dimension every proper monotone sequence is definite.* A monotone sequence of real points $P_i$ in $n$ dimensions may be described in terms of one-dimensional monotone sequences by saying that *the $k^{\text{th}}$ coordinate of $P_i$, for each $i$, describes a monotone sequence of real numbers*, whose sense is then the $k^{\text{th}}$ coordinate of the sense of the given sequence.

*A* succession

$$P_1, P_2, \ldots, P_i, \ldots$$

of real points is said to have *the unique limit* $P_0$ if the span of $\overrightarrow{P_0P_i}$ tends to zero with $1/i$. Each of its sub-successions then has the same unique limit $P_0$. In particular this is the case with every monotone sub-sequence of the succession. Since *every succession of real points contains monotone sub-sequences*, it follows that, if every monotone sub-sequence of a succession has the same unique limit $P_0$, the given succession has the unique limit $P_0$. — Thus a *necessary and sufficient condition for a given succession of real points to have the unique limit $P_0$ is that each of its monotone subsequences should have the unique limit $P_0$.*

A *bounded* monotone sequence of real points $P_i$ always has a unique limit $P_0$ which is called *the limit of the sequence.* The sense of $\overrightarrow{P_0 P_i}$ is the same for all $i$, and is opposite to the sense of the sequence. *If a succession of real points $P_i$, of unique limit $P_0$, without being itself necessarily monotone, satisfies the condition that $\overrightarrow{P_0 P_i}$ has an invariable sense, then every monotone subsequence of the succession has the same sense, opposite to that of $\overrightarrow{P_0 P_i}$.*

From a set $E$ of real points [1]), not merely finite in number, we can always form successions and hence monotone sequences of its points. The limits of bounded proper monotone subsequences of a set $E$ are called the *limiting points* of the set. A limiting point of $E$ may, or may not, belong to $E$. A bounded set which contains all its limiting points is a *closed set*. Points of a set which are not limiting points of the set are *isolated* points of the set. A closed set without isolated points is a *perfect set.*

The set of the $k^{\text{th}}$ coordinates of the points of a set is called the $k^{\text{th}}$ *projection* of the set.

We also recall the definition of the limit of a *succession of sets.* A succession of sets $E_n$ of real points is said to have the *unique limit* $E$ if every point of $E$ belongs to all but a finite number of the sets $E_n$, and every point belonging to more than a finite number of the sets $E_n$ is a point of $E$.

The $k^{\text{th}}$ projection of $E_n$ then describes a succession of linear sets whose unique limit is the $k^{\text{th}}$ projection of $E$.

If $E_n$ is the set of *all* points whose $k^{\text{th}}$ projection for each $k$ belongs to a given set $e_n^{(k)}$, and the $k^{\text{th}}$ projection $e_n^{(k)}$ of $E_n$ describes for each $k$ a succession having a unique limit $e^{(k)}$, then $E_n$ describes a succession having as unique limit the set of all points whose $k^{\text{th}}$ projection belongs to $e^k$, $(k = 1, 2, \ldots, n)$.

2. A real point $P$ and a *definite* sense $\Theta$, in space of $n$ dimensions, together constitute a **directed point** $\mathscr{P}$ in that space. The real point $P$ is the *position* of the directed point $\mathscr{P}$, the definite sense $\Theta$ is the *sense* of the directed point. The $k^{\text{th}}$ coordinate of the position of $\mathscr{P}$, together with the $k^{\text{th}}$ coordinate ($+$ or $-$) of its sense, constitute the $k^{\text{th}}$ *coordinate* of $\mathscr{P}$; it is a directed point in space of one dimension, or *a directed number*. Two directed points only coincide if they have the same position and the same sense, i. e. if their $k^{\text{th}}$ coordinates, for each $k$, are identical in position and sense. As there are $2^n$ definite senses in space of $n$ dimensions, *every real point in n-space is the position of $2^n$ directed points.*

[1]) In an unordered set, repetitions of elements are indistinguishable and therefore not admitted.



To *represent* a directed point $\mathscr{P}$ of position $P$ and sense $\Theta$, in the space of real points, we have in principle two chief means:

a) to take the set of all real points $P'$ for which $\overrightarrow{PP'}$ has the definite sense $\Theta$. This is called *the open quadrant of index $\Theta$* at $P$.

b) to take any monotone sequence of real points whose limit is $P$, *i. e.* any monotone sequence of real points of sense $-\Theta$ and limit $P$.

In one dimension these two modes of representation correspond to the two main schemes for the representation of a real number by rational numbers:

a′) as a Dedekind section of the realm of rational numbers, with the modification that, if the threshold between the two sides of the section is formed by a rational number, this is excluded from both sides.

b′) as a pair of strictly monotone sequences of rational numbers, of opposite senses, whose term-by-term difference tends to zero (nest of intervals).

The difference is now that the two sides of the Dedekind section (and the two sequences forming the nest) are regarded as representing two different directed numbers.

A third method, a modification of a), resembles still more closely the original Dedekind definition of real number:

A *closed quadrant* at $P$ of index $\Theta$, is defined as the open quadrant at $P$ of index $\Theta$ together with its limiting points; the latter are the real points $P'$ for which the sense of $\overrightarrow{PP'}$ has its $k^{\text{th}}$ coordinate, of each $k$, either equal to that of $\Theta$, or zero.

If we omit from the closed quadrant at $P$, of index $\Theta$, those of these limiting points $P'$ for which the sense of $\overrightarrow{PP'}$ has its $k_1^{\text{th}}, k_2^{\text{th}}, \ldots, k_q^{\text{th}}$ coordinates (where the $k_i$ are arbitrary fixed integers less than $n$, and $q \geqq 1$) all zero, we obtain, a *half-open* quadrant at $P$ of the same index $\Theta$.

In one dimension, there are no half-open quadrants, every quadrant is either open or closed. The whole of the one-dimensional space can be expressed as the sum of two quadrants, at any given point $x_0$, having no common point. If the one quadrant, of index $\theta_0$, say, is open, the other, of index $-\theta_0$, is closed. The two quadrants are said to form a *Dedekind section at $x_0$, of sense $\theta_0$*, in the one-dimensional realm of *real numbers,* and the open quadrant of the two, whose index gives the sense of the section, is called the *principal quadrant* of the section.

At every real point-number, there are two Dedekind sections, corresponding to the two definite senses $\theta = +$ and $\theta = -$, and these may be taken to represent the two directed numbers, having for position that real number and for senses those of the respective sections.

In $n$-dimensions, a quadrant of index $\Theta$ at $P$ (of coordinates $\theta_k$ and $x_k$) is the set of points whose $k^{\text{th}}$ projection, for each $k$, is a quadrant of index $\theta_k$ at $x_k$.

In $n$ dimensions, the whole of the space of real points can again be expressed as the sum of quadrants, at a given point $P$, no two of which have a common point. This subdivision of space will be called a *Dedekind section of sense $\Theta$ at $P$* (of coordinates $\theta_k$ and $x_k$) in the $n$-dimensional realm of real points, *if each of its quadrants has for $k^{\text{th}}$ projection a quadrant of the Dedekind section of index $\theta_k$ at $x_k$.* Thus the quadrant of index $\Theta'$, (of $k^{\text{th}}$ coordinate $\theta_k'$), of the Dedekind section at $P$, of index $\Theta$, is the set of points whose $k^{\text{th}}$ projection is a quadrant of index $\theta'_k$ at $x_k$, open if $\theta'_k$ and $\theta_k$ coincide, closed if they are opposite. The quadrant of index $\Theta$ of the section is open and is called the *principal* quadrant of the section; the quadrant of index $-\Theta$ is closed. Every other quadrant of the section is half-open.

The Dedekind section of sense $\Theta$ at $P$ may also be defined as the set of $2^n$ quadrants, each of a different index $\Theta'$, and such that for any point $P'$ of such a quadrant, whenever the $k^{\text{th}}$ coordinates of $\Theta$ and $\Theta'$ coincide, that of the sense of $\overrightarrow{PP'}$ coincides with them. Thus the sense of $\overrightarrow{PP'}$ has the $k^{\text{th}}$ coordinate of $\Theta$, except possibly for values of $k$ for which the $k^{\text{th}}$ coordinate of $\Theta+\Theta_0$ is 0; and for these it may also have the $k^{\text{th}}$ coordinate 0.

The Dedekind section of sense $\Theta$ at $P$ may again adequately represent the directed number of position $P$ and sense $\Theta$. The difference between this and the ordinary Dedekind conception is that, in the latter, the sections of various indices would be axiomatically identified, instead of being distinguished and maintained distinct.

The second and third of these modes of representation afford the readiest illustrations where sets and sequences of directed points are concerned, while the first and second are most convenient for the construction of functions of a directed point.

A *pair of directed points* in a given order, $\mathscr{P}_1$, $\mathscr{P}_2$, say, is denoted by $\overrightarrow{\mathscr{P}_1\mathscr{P}_2}$. If $P_i$ be the position, $\Theta_i$ the sense, of $\mathscr{P}_i$ the *span* of $\overrightarrow{\mathscr{P}_1\mathscr{P}_2}$ coincides with that of $\overrightarrow{P_1P_2}$, *i. e.* is the distance between the positions of the two directed points, and the *sense* of $\overrightarrow{\mathscr{P}_1\mathscr{P}_3}$ has its $k^{\text{th}}$ coordinate coinciding with that of the sense of $\overrightarrow{P_1P_3}$ when not 0, and otherwise with that of $\Theta_2-\Theta_1$.

This is the $n$-dimensional form of the convention that, in space of a single dimension, the sense of a pair of directed points is that of the pair of their positions, unless these coincide; and the sense of a pair of points whose positions coincide, is the difference between their two senses, in the same order.

The sense of the pair $\overrightarrow{\mathscr{P}_1\mathscr{P}_2}$ of directed points in $n$-space, may then be described, in terms of the sense of pairs of directed numbers, as having for its $k^{\text{th}}$ coordinate, for each $k$, the sense of the pair of $k^{\text{th}}$ coordinates of $\mathscr{P}_1$ and of $\mathscr{P}_2$, in the same order.

It follows that, if the $k^{\text{th}}$ coordinates of the senses of $\overrightarrow{\mathscr{P}_1\mathscr{P}_2}$ and $\overrightarrow{\mathscr{P}_2\mathscr{P}_3}$ are the same, say $\theta_k$, or the one is $\theta_k$ and the other 0, then that of the sense of $\overrightarrow{\mathscr{P}_1\mathscr{P}_3}$ is again $\theta_k$. It is only when the two former coordinates are opposite definite signs, that they do not determine the latter absolutely, which may indeed be $+$, $-$, or 0, according to the positions of $\mathscr{P}_1$ and $\mathscr{P}_3$. Thus the same rule applies to the sense of $\overrightarrow{\mathscr{P}_1\mathscr{P}_3}$ when those of $\overrightarrow{\mathscr{P}_1\mathscr{P}_2}$ and $\overrightarrow{\mathscr{P}_2\mathscr{P}_3}$ are given, as



in the case of pairs of *real* points (q. v.). In particular, *if the $k^{\text{th}}$ coordinates of the two given senses, those of $\overrightarrow{\mathscr{P}_1\mathscr{P}_2}$ and $\overrightarrow{\mathscr{P}_2\mathscr{P}_3}$, only differ where the one is* 0, *then the sense of the resultant $\overrightarrow{\mathscr{P}_1\mathscr{P}_3}$ is the sum of the two given senses.*

Given two directed points $\mathscr{P}_1$ and $\mathscr{P}_2$, whose pair has a definite sense $\Theta$, a directed point $\mathscr{P}$ is said to be *strictly between* $\mathscr{P}_1$ and $\mathscr{P}_2$; if $\overrightarrow{\mathscr{P}_1\mathscr{P}}$ and $\overrightarrow{\mathscr{P}\mathscr{P}_2}$ have the same definite sense $\Theta$. (For this the positions of $\mathscr{P}_1$ and $\mathscr{P}_2$ must be distinct). $\mathscr{P}$ is still said to be *between* $\mathscr{P}_1$ and $\mathscr{P}_2$, if the senses of $\overrightarrow{\mathscr{P}_1\mathscr{P}}$ and $\overrightarrow{\mathscr{P}\mathscr{P}_2}$ only differ from that of $\overrightarrow{\mathscr{P}_1\mathscr{P}_2}$ by the substitution of the sign 0 for some of its coordinates, in particular, the points $\mathscr{P}_1$ and $\mathscr{P}_2$ themselves are between $\mathscr{P}_1$ and $\mathscr{P}_2$ in accordance with this definiton, as also the $2^{n-1}$ other points whose $k^{\text{th}}$ coordinates, for each $k$, coincide with the $k^{\text{th}}$ coordinate of $\mathscr{P}_1$ or of $\mathscr{P}_2$.

When $\mathscr{P}$ is between $\mathscr{P}_1$ and $\mathscr{P}_2$, its $k^{\text{th}}$ coordinate is between the $k^{\text{th}}$ coordinates of $\mathscr{P}_1$ and $\mathscr{P}_2$, in accordance with the one-dimensional form of the definition.

The set of directed points between two points $\mathscr{P}_1$ and $\mathscr{P}_2$ whose pair has a definite sense $\Theta$, is called an *interval.*; $\mathscr{P}_1$ and $\mathscr{P}_2$ are *opposite endpoints* of this interval; its other *endpoints* are the $2^{n-1}$ other points whose $k^{\text{th}}$ coordinates, for each $k$, coincide with the $k^{\text{th}}$ coordinate of $\mathscr{P}_1$ or of $\mathscr{P}_2$. Each pair of endpoints whose sense is definite is a pair of opposite endpoints of the interval; each endpoint of the interval belongs to one and only one such pair, whose sense, when the endpoint in question is placed second in the pair, is the *index* of the endpoint. Thus an endpoint $\mathscr{P}_0$ has the index $\Theta$ in the interval, if $\overrightarrow{\mathscr{P}\mathscr{P}_0}$, for each $\mathscr{P}$ of the interval, has the definite sense $\Theta$ or a sense obtained from $\Theta$ by the substitution of 0 for some of its coordinates. The span of a pair of opposite endpoints of an interval does not depend on the sense of the pair; it constitutes the *span of the interval* and is the maximum of the span of any pair of points of the interval.

Every interval, in $n$-dimensional space of directed points, has $2^n$ endpoints; it may have no other points. This will be the case if the endpoints all have the same position, and its span is then zero. In every other case the positions of opposite endpoints are distinct and the span is positive.

A set $\mathscr{E}$ of directed points is said to be *bounded*, if the set $E$ of the positions of its points is bounded; and the *span* of $\mathscr{E}$ is defined as equal to that of $E$. The set of the $k^{\text{th}}$ coordinates of the points of $\mathcal{E}$ is the $k^{\text{th}}$ *projection* of $\mathcal{E}$ and the span of this projection of $\mathcal{E}$ is always $\leqq$ that of $\mathcal{E}$ itself. *A bounded set can always be exhibited as the sum of a finite number of sets of span less than a given* $\delta > 0$.

4. A succession of directed points

$$\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_i, \ldots$$

in space of $n$ dimensions, is said to be *monotone*, or to be a *monotone sequence*, if the pair $\overrightarrow{\mathcal{P}_i\mathcal{P}_{i+1}}$ has an invariable sense for all $i$. This sense is called the *sense of the sequence*. The adverb *strictly* is added to the word *monotone* when this sense is definite. If two of the directed points of the sequence coincide, then all do, and the sequence is called an *improper* monotone sequence; otherwise it is a *proper* monotone sequence.

*In one dimension, every proper monotone sequence is definite.* A monotone sequence of directed points $\mathcal{P}_i$ in $n$ dimensions may be described in terms of one-dimensional monotone sequences, by saying that *the* $k^{\text{th}}$ *coordinate of* $\mathcal{P}_i$, *for each* $k$, *describes a monotone sequence of directed numbers*, the sense of which sequence is the $k^{\text{th}}$ coordinate of the sense of the given sequence.

Any pair of points of a monotone sequence, in increasing order of their indices, has the sense of the sequence. It follows also that all the intermediary points of a monotone sequence, between two given non-consecutive ones, are actually *between* these. In particular, no two non-consecutive points of a proper monotone sequence can have the same position.

In a proper monotone sequence of directed points $\mathcal{P}_i$, the positions $P_i$ of the directed points themselves form a proper monotone sequence of real points, if we omit the repetitions (at most once) which may occur in the positions of consecutive directed points.

The succession of positions of the directed points in a bounded monotone sequence thus certainly has a unique limit $P_0$. The **limit** $\mathcal{P}_0$ of the bounded monotone sequence of directed points $\mathcal{P}_i$ is then defined as the directed point having the *position* $P_0$, and the *sense* whose $k^{\text{th}}$ coordinate is opposite to that of the sense of the sequence when this is $\neq 0$, but coincides, when this coordinate is 0, with the corresponding coordinate of the sense of $\mathcal{P}_i$, which is the same for all $i$. An unbounded monotone sequence has no limit.

When $\mathcal{P}_i$ describes a strictly monotone sequence of limit $\mathcal{P}_0$, every point $\mathcal{P}_i'$ having the position of $\mathcal{P}_i$ describes a strictly monotone sequence with the same limit $\mathcal{P}_0$. And in general when $\mathcal{P}_i$ describes a monotone sequence of sense $\Theta$ and limit $\mathcal{P}_0$, so does any point $\mathcal{P}_i'$ having the position of $\mathcal{P}_i$ and a sense differing from the sense of $\mathcal{P}_i$ at most in coordinates corresponding to definite coordinates of $\Theta$.



With this definition of the limit $\mathcal{P}_0$ of a bounded monotone sequence of directed points $\mathcal{P}_i$, we ensure that, *while the span of* $\overrightarrow{\mathcal{P}_i\mathcal{P}_0}$ *tends to zero, its sense remains invariable and coincides with the sense of the sequence.*

In fact, for values of $k$ for which the $k^{\text{th}}$ coordinate of the sense of the sequence (*i. e.* of $\overrightarrow{\mathcal{P}_i\mathcal{P}_{i+1}}$) is $\neq 0$, it coincides with that of the sense of $\overrightarrow{P_iP_{i+2}}$, hence of $\overrightarrow{P_iP_0}$ and therefore, since it is $\neq 0$, with that of $\overrightarrow{\mathcal{P}_i\mathcal{P}_0}$.

For values of $k$ for which the $k^{\text{th}}$ coordinate of the sense of the sequence is 0, that of the sense of $\mathcal{P}_0$ coincides with that of sense of $\mathcal{P}_i$, by definition, and so that of the sense of $\overrightarrow{\mathcal{P}_0\mathcal{P}_i}$ is also 0.

A succession of directed points

$$\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_i, \ldots$$

is said to have the *unique limit* $\mathcal{P}_0$ if $\overrightarrow{\mathcal{P}_0\mathcal{P}_i}$ has, for all $i > N$, a sense whose $k^{\text{th}}$ coordinate coincides with that of $\mathcal{P}_0$, or is 0, while the span of $\overrightarrow{\mathcal{P}_0\mathcal{P}_i}$ tends to zero. Every subsuccession then has the same unique limit. It is clear that no two points can satisfy the conditions simultaneously; in fact, the succession of the positions $P_i$ of the points $\mathcal{P}_i$ has as unique limit the position $P_0$ of $\mathcal{P}_0$, which is uniquely determined; while the sense of $\mathcal{P}_0$ must have its $k^{\text{th}}$ coordinate coinciding with that of the sense of $\overrightarrow{P_0P_i}$ for every $i$ for which this is $\neq 0$, and otherwise with that of the sense of $\mathcal{P}_i$, and is therefore also determinate.

The points $\mathcal{P}_i$ are said to *tend* to the limit $\mathcal{P}_0$, and we write

$$\mathcal{P}_i \to \mathcal{P}_0.$$

The $k^{\text{th}}$ coordinate of $\mathcal{P}_i$ then tends to that of $\mathcal{P}_0$ in accordance with the one-dimensional form of the definition.

A bounded monotone sequence always has a unique limit which coincides with the *limit* already defined. Since *every succession of directed points has monotone subsequences*, it follows that *a necessary* *and sufficient condition for a given succession of directed points to have the unique limit $\mathcal{P}_0$ is that each of its monotone subsequences should have the limit $\mathcal{P}_0$.*

The second mode of representing a directed point furnishes a good interpretation of the notion of limit. We have indeed the theorem:

*A necessary and sufficient condition for a succession of directed points $\mathcal{P}_i$, of position $P_i$ and sense $\Theta_i$, to have the unique limit $\mathcal{P}_0$, of position $P_0$ and sense $\Theta_0$, is that, if $P'_i$ be any real point of any strictly monotone sequence „representing“ $\mathcal{P}_i$, for which $\overrightarrow{P_iP'_i}$ has span less than a certain positive $\delta_i$, then every monotone subsequence of the real points $P'_i$ „represents“ $\mathcal{P}_0$.*

We may, of course, omit from consideration the points which coincide in position with $\mathcal{P}_0$.

Let $\delta_i$ be chosen equal to the smallest among the $k^{\text{th}}$ components of the span of $\overrightarrow{P_iP_0}$, for that $i$, which are not zero; $\delta_i$ is positive and tends to zero with $1/_i$.

From any given sequence representing $\mathcal{P}_i$, choose $P'_i$ so that

$$\text{span } P_iP'_i < \delta_i.$$

Then *a fortiori* each component of this span is less than $\delta_i$ and therefore less than the corresponding component of the span of $\overrightarrow{P_0P_i}$, excepting when this component is zero. Therefore (cf. pp. 247 and 243) the $k^{\text{th}}$ coordinate of the sense of $\overrightarrow{P_0P'_i}$, resultant of $\overrightarrow{P_0P_i}$ and $\overrightarrow{P_iP'_i}$, concides with that of the sense of $\overrightarrow{P_0P_i}$ when this is $\neq 0$, and if either is known to coincide with that of the sense of $\mathcal{P}_0$, so is the other.

On the other hand, for values of $k$ and $i$ for which the $k^{\text{th}}$ component of the span of $\overrightarrow{P_0P_i}$ is zero, the $k^{\text{th}}$ components of the senses of $\overrightarrow{P_0P'_i}$ and of $\overrightarrow{P_iP'_i}$ — which latter is that of $\mathcal{P}_i$ — coincide, and, if either is known to coincide with that of the sense of $\mathcal{P}_0$, then so is the other.

Thus there is complete equivalence between the two hypotheses:

1) The definite coordinates of the sense of $\overrightarrow{P_0P_i}$ coincide with the corresponding ones of $\mathcal{P}_0$; for those of its coordinates which are 0, the corresponding ones of $\mathcal{P}_0$ and $\mathcal{P}_i$ coincide.

2) The sense of $\overrightarrow{P_0P'_i}$ coincides with that of $\mathcal{P}_0$.

And these represent precisely the two hypotheses:

1) Every monotone subsequence of the directed points $\mathcal{P}_i$ has the limit $\mathcal{P}_0$.

2) Every monotone subsequence of the real points $P'_i$ represents $\mathcal{P}_0$.

Thus the theorem is true.

Another simple illustration is provided by the third mode of representing a directed point, in virtue of the following theorem:

*A necessary and sufficient condition for the succession of directed points $\mathcal{P}_i$, of positions $P_i$ and senses $\Theta_i$, to have the unique limit $\mathcal{P}_0$, of position $P_0$ and sense $\Theta_0$, is that the quadrant of any given index $\Theta'$ in the Dedekind section representing $\mathcal{P}_i$ should describe a sequence of sets (of real points) having as unique limit the quadrant of the same index $\Theta'$ in the Dedekind section representing $\mathcal{P}_0$.*



We first note that, if for a special $\Theta'_0$, the quadrant of index $\Theta'_0$ in the Dedekind section representing $\mathcal{P}_i$ describes such a sequence, with a unique limit coinciding with the quadrant of the same sense in the Dedekind section representing $\mathcal{P}_0$, then the same occurs for every other $\Theta'$. In fact, since, in the Dedekind section representing $\mathcal{P}_i$, we have

$$k^{\text{th}} \text{ proj of quadrant of index } \Theta' = k^{\text{th}} \text{ proj. of quadrant of index } \Theta'_0$$
$$\text{or} = \text{its complementary},$$

according as the $k^{\text{th}}$ coordinate of $\Theta'$ is equal or opposite to that of $\Theta'_0$, it follows that the same relation holds at the limit, when $i \to \infty$. Each projection of the quadrant of index $\Theta'$ in the Dedekind section representing $\mathcal{P}_i$ tends to a unique limit, the corresponding projection of the quadrant of index $\Theta'_0$ in the Dedekind section representing $\mathcal{P}_0$, or the complementary set, according as the corresponding coordinate of $\Theta'_0$ is equal or opposite to that of $\Theta'$. This limit is precisely the projection of the quadrant of index $\Theta'$ in the Dedekind section representing $\mathcal{P}_0$. This is the necessary and sufficient condition (cf. p. 244) for the quadrant of index $\Theta'$ in the Dedekind section representing $\mathcal{P}_i$, to describe a successsion whose unique limit is the quadrant of the same index $\Theta'$ in the Dedekind section representing $\mathcal{P}_0$.

Let us choose $\Theta' = \Theta_0$, the sense of $\mathcal{P}_0$. We have to show first that, if $\mathcal{P}_i$ describes a succession with the unique limit $\mathcal{P}_0$, the quadrant $Q_i$ of index $\Theta_0$ in the Dedekind section of sense $\Theta_i$ at $P_i$ describes a succession of sets, having as unique limit the quadrant of index $\Theta_0$, *i. e.* the principal or open quadrant $Q_0$, in the Dedekind section of sense $\Theta_0$ at $P_0$.

Now if $P'$ be any point of the quadrant $Q_i$, the $k^{\text{th}}$ coordinate of the sense of $\overrightarrow{P_iP'}$ is that of $\Theta_0$, or 0, (the latter only when the $k^{\text{th}}$ coordinate of $\Theta_0 + \Theta_i$ is 0). The same is true of the sense of $\overrightarrow{P_0P_i}$, (except that here the coordinate is 0 only when that of $\Theta_0 - \Theta_i$ is 0). The senses of $\overrightarrow{P_0P_i}$ and $\overrightarrow{P_iP'}$ thus both differ from $\Theta_0$, and hence from one another, in the one having possibly 0 as a coordinate, when the corresponding coordinate of the other is definite. The sense of their resultant $\overrightarrow{P_0P'}$ is therefore the sum of their senses, and can only differ from $\Theta_0$ by having a possible 0-coordinate, when the corresponding coordinates of $\overrightarrow{P_0P_i}$ and $\overrightarrow{P_iP'}$ are *both* 0. But this can never happen, since both $\Theta_0 + \Theta_i$ and $\Theta_0 - \Theta_i$ would have to have that coordinate $= 0$, which is impossible. Therefore $\overrightarrow{P_0P'}$ has the sense $\Theta_0$, and $P'$ belongs to $Q_0$.

On the other hand, if $P$ be any point of the quadrant $Q_0$, then $\overrightarrow{P_0P}$ has the definite sense $\Theta_0$, and the components of its span are, in particular, all greater than some positive $\delta$.

For all $i > N_\delta$, the pair $\overrightarrow{P_0P_i}$ has its span, and *a fortiori* the components of its span, less than the positive $\delta$. Hence, for these $i$, the resultant $\overrightarrow{P_iP}$ of $\overrightarrow{P_iP_0}$ and $\overrightarrow{P_0P}$ has the sense of the latter, since this is definite, — *i. e.* the sense $\Theta_0$. $P$ belongs to $Q_i$ for all $i > N_\delta$.

Therefore $\lim Q_i = Q_0$, as asserted.

We have next to show that, *if $Q_i$ describes a sequence of sets with unique limit $Q_0$, $\mathcal{P}_i$ describes a succession of directed points with unique limit $\mathcal{P}_0$.*

The verification of the fact that the span of $\overrightarrow{P_0P_i}$ tends to zero is immediate.

For, if $P_\varepsilon$ be a point of $Q_0$ with span of $\overrightarrow{P_0P_\varepsilon} < \varepsilon$, then, $P_\varepsilon$ belongs to $Q_i$ for all $i > N_\varepsilon$, and $\overrightarrow{P_iP}$ has a sense whose $k^{\text{th}}$ coordinate is that of $\Theta_0$ or 0, whence

$$k^{\text{th}} \text{ component span } \overrightarrow{P_0P_i} \leqq k^{\text{th}} \text{ component span } \overrightarrow{P_0P_\varepsilon}$$

*i. e.*

$$\text{span } \overrightarrow{P_0P_i} \leqq \text{span } \overrightarrow{P_0P_\varepsilon} < \varepsilon, \text{ for all } i > N_\varepsilon.$$

We have next to ascertain that the $k^{\text{th}}$ coordinate of the sense of $\overrightarrow{P_0P}$ is, (for all $i > N$), equal to that of $\Theta_0$ or 0, and that, when it is 0, the corresponding coordinates of $\Theta_0$ and $\Theta_i$ coincide.

Now if the sense of $P_0P_i$ had its $k^{\text{th}}$ coordinate opposite to that of $\Theta_0$ for a sequence of values of $i$, any point $P$ for which the sense of $\overrightarrow{P_0P}$ had that $k^{\text{th}}$ coordinate 0 and the others coinciding with those of $\Theta_0$, would, without being a point of $Q_0$, belong to $Q_i$ for all but a finite number of these values of $i$, (contrary to the hypothesis $\lim Q_i = Q_0$). In fact, we should only have to take $i$ sufficiently large to ensure that the span of $\overrightarrow{P_0P_i}$ was less than the least positive component of the span of $\overrightarrow{P_0P}$: the senses of $\overrightarrow{P_iP}$ and $P_0P$ would then have the same coordinates (those of $\Theta_0$), for all but the particular given $k$ for which the latter has $k^{\text{th}}$ coordinate 0; for this $k$, the coordinates of the senses of $\overrightarrow{P_iP}$ and $\overrightarrow{P_iP_0}$ would coincide (again with that of $\Theta_0$); $\overrightarrow{P_iP}$ would have the sense $\Theta_0$, *i. e.* belong to $Q_i$, for all those $i$.

Finally we may note that, in the last theorem, *The succession described by $\mathcal{P}_i$ is monotone, if, and only if, one (at least) of the quadrants in the Dedekind section representing $\mathcal{P}_i$ describes a monotone increasing sequence of sets.* The index of this quadrant is, of course, necessarily the same for all $i$, and, if properly chosen is that of the principal quadrant in the Dedekind section representing $\mathcal{P}_0$. We have only to take the quadrant whose $k^{\text{th}}$ projection, for each $k$, is either invariable and open, or increases with $i$.

From a set $\mathcal{E}$ of directed points, not merely finite in number, we can always form successions, and hence monotone sequences, of its points. The limits of bounded proper monotone subsequences of a set $\mathcal{E}$ are called the *limiting points* of the set. Their positions are limiting points of the set of positions af the points of $\mathcal{E}$. A limiting point of $\mathcal{E}$ may, or may not, belong to $\mathcal{E}$. A bounded set which contains all its limiting points is a *closed set*. The set $E$ of the positions of its points is then closed. An unbounded set is not closed. Points of a set which are not limiting points of the set are *isolated* points of the set. A closed set without isolated points is *perfect*. The set of the positions of its points is then also perfect. A set $\mathcal{E}$ is said to be *complete*, if all the points having the position of a point of $\mathcal{E}$ always belong to $\mathcal{E}$.

The set of the $k^{\text{th}}$ coordinates of the points of a set is called the $k^{\text{th}}$ *projection* of the set. The $k^{\text{th}}$ projection of a closed set is closed.



The common part of two closed sets is closed.

An interval is a closed set, and, if its span is positive, it is a perfect set. It is not necessarily complete.

The common part of an interval and a given set $\mathcal{E}$ is called a *portion of $\mathcal{E}$*.

Every portion of a closed set is therefore a closed set. The *span of a set* is the upper bound of the span of any pair of its points. *A bounded set can always be exhibited as the sum* [1]) *of a finite number of portions of the set, each of span less than a given $\delta$.*

*The neighbourhood of span $\delta$* of a directed point $\mathcal{P}$, ($\delta > 0$) is the set of all directed points $\mathcal{P}'$, with

$$\text{span } \overrightarrow{\mathcal{P}\mathcal{P}'} < \delta,$$

belonging to monotone sequences of limit $\mathcal{P}$. This will comprise all those points $\mathcal{P}'$ for which $\overrightarrow{\mathcal{P}\mathcal{P}'}$ has span less than $\delta$ and sense differing from that of $\mathcal{P}$ only in so far as some of the coordinates which are definite in the latter may be 0 in the former. This neighbourhood is thus *fan-shaped*. It contains $\mathcal{P}$ and every neighbourhood of $\mathcal{P}$ of span less than $\delta$. And it contains a neighbourhood of each of its points [2]).

5. A numerical value (finite real number) attached to each directed point $\mathcal{P}$ defines a (*numerical*) *function* of that point, say $f(\mathcal{P})$. In general only the values of the function at points of a given set are considered. A function $f(\mathcal{P})$ is said to be *bounded* in a set $\mathcal{E}$ of directed points, if the set of its values in $\mathcal{E}$ is bounded. The span of this set of values is called the *oscillation* of $f(\mathcal{P})$ in $\mathcal{E}$. To say

[1]) Two or more sets $\mathcal{E}_i$ are said to have for *sum* the set of all the points which belong to one *at least* of the set $\mathcal{E}_i$.

[2]) If we wished to introduce a parallel to the *open set* of the modern theory of sets of real points, we should define it as a set which contains a neighbourhood of each of its points. But this would involve the complication that a set could be both open in accordance with this definition, and closed; thus, for instance, an interval each of whose endpoints had its index opposite to its sense. This shows that the terminology „open set“ is not a happy one in the present connexion.

that $f(\mathscr{P})$ is bounded in $\mathscr{E}$ is equivalent to saying that is oscillation in $\mathscr{E}$ is finite.

A function $f(\mathscr{P})$ is said to be *continuous in $\mathscr{E}$ at a point $\mathscr{P}_0$*, if, for every (proper or improper) monotone sequence of points $\mathscr{P}$ of $\mathscr{E}$ with limit $\mathscr{P}_0$, $f(\mathscr{P}_n)$ tends to the same unique limit $f(\mathscr{P}_0)$. Thus $f(\mathscr{P})$ is „continuous in $\mathscr{E}$" at every isolated point of $\mathscr{E}$, and at a limiting point $\mathscr{P}_0$ of $\mathscr{E}$ it is continuous in $\mathscr{E}$ if

$$\lim_{\mathscr{P}_i \to \mathscr{P}_0} f(\mathscr{P}_i) = f(\mathscr{P}_0), \qquad (\mathscr{P}_i \text{ in } \mathscr{E})$$

(cf. p. 249).

If $f(\mathscr{P})$ is continuous in $\mathscr{E}$ at every point of $\mathscr{E}$, we say simply that $f(\mathscr{P})$ is *continuous in $\mathscr{E}$*. It is then also continuous in every subset of $\mathscr{E}$. If $\mathscr{E}$ is a fundamental set, which has been specified once for all in a certain connexion, functions of $\mathscr{P}$ which are continuous in $\mathscr{E}$ are spoken of as *continuous functions*, in that connexion.

**Theorem I.**

*If $f(\mathscr{P})$ is continuous in a closed set $\mathscr{E}$, it is bounded in $\mathscr{E}$.*

For it then assumes a closed (hence bounded) set of values in $\mathscr{E}$, since every sequence of these values has a subsequence corresponding to a monotone sequence of points of $\mathscr{E}$, and therefore every limit of such a sequence of values is a value of the function in $\mathscr{E}$.

Let

$$w(t_1, t_2, \dots, t_q) = w_q(T)$$

— where $t_i$ is any real number, and $1 \leqq q \leqq n$, — denote the upper bound of the difference between the values of $f(\mathscr{P})$ at any two points of $\mathscr{E}$ which coincide in all but $q$ of their coordinates, these $q$ coordinates differing, but only in sense, and occupying the assigned positions $t_1, t_2, \dots, t_q$ [1]). When, for a given $T$, such points do not exist in $\mathscr{E}$, we equate $w(T)$ to 0 for that $T$. The quantity $w(t_1, t_2, \dots, t_q)$ is a non-negative function of position in real space of $q$ dimensions, whose generic point $T$ has the $r^{\text{th}}$ coordinate $t_r$. Il is certainly always finite, if $f(\mathscr{P})$ is bounded in $\mathscr{E}$.

[1]) Thus for $q=1$, $w(t)$ is the upper bound of $|f(\mathscr{P}') - f(\mathscr{P}'')|$ for all pairs of points of $\mathscr{E}$ whose positions $P = (x_1, x_2, \dots, x_n)$ coincide, and lie on one of the straight lines $x_i = t$ $(i = 1, 2, \dots, n)$, while their senses $\Theta' = (\theta_1, \theta_2, \dots \theta_i', \dots \theta_n)$ and $\Theta'' = (\theta_1, \theta_2, \dots, \theta_i'', \dots, \theta_n)$ only differ in their $i^{\text{th}}$ coordinates, $\theta_i' \neq \theta_i''$. Similarly for any $q$, when $q$ indices $i_1, i_2, \dots, i_q$ take the place of the one index $i$.



**Lemma.** *If $\mathscr{E}$ is closed and $f(\mathscr{P})$ continuous in $\mathscr{E}$, then $w_q(T_i) \to 0$ whenever $T_i$ describes a strictly monotone sequence of points in real space of $q$ dimensions.*

We shall show that any strictly monotone sequence of points $T_i$ contains a subsequence for which $w_q(T_i) \to 0$. This implies that $w_q(T_i) \to 0$ for every such strictly monotone sequence.

Given any strictly monotone sequence of real points $T_i$, to each $T_i$ for which $w_q(T_i) > 0$, there correspond two points $\mathscr{P}_i'$, $\mathscr{P}_i''$ of $\mathscr{E}$, — differing only in the $q$ coordinates of their sense corresponding to which their position has the $q$ coordinates of $T_i$ — and such that

$$|f(\mathscr{P}_i') - f(\mathscr{P}_i'')| > \tfrac{1}{2} w_q(T_i).$$

For each of the $\binom{m}{q}$ ordered combinations $k_1, k_1, \dots, k_q$ of $q$ of the integers $1, 2, \dots, n$, let us pick out the indices $i$ for which $\mathscr{P}_i'$ and $\mathscr{P}_i''$ have precisely their $k_r^{\text{th}}$ coordinates coinciding in position with the $r^{\text{th}}$ coordinate of $T_i$, but differing in sense. As, to every index $i$, corresponds one, and only one, such combination, at least one of the combinations corresponds to an infinity of the indices $i$.

No two of the points $\mathscr{P}_i'$ having these indices $i$ then have the same $k_r^{\text{th}}$ coordinate, for any $r$ (since no two $T_i$ have the same $r^{\text{th}}$ coordinate); and so the sense of any monotone subsequence of the points $\mathscr{P}_i'$ having these indices $i$ has its $k_r^{\text{th}}$ coordinates $(r = 1, 2, \dots q)$ all $\neq 0$. At the same time the point $\mathscr{P}_i''$ corresponding to $\mathscr{P}_i'$ only differs from it in its $k_r^{\text{th}}$ coordinates, $(r = 1, 2, \dots, q)$; hence (by p. 249); that monotone subsequence of points $\mathscr{P}_i'$ and the corresponding points $\mathscr{P}_i''$ tend to the same limit $\mathscr{P}_0$, a point of $\mathscr{E}$; $f(\mathscr{P}_i')$ and $f(\mathscr{P}_i'')$ both tend to $f(\mathscr{P}_0)$. Thus for some subsequence of the indices $i$, $|f(\mathscr{P}_i') - f(\mathscr{P}_i'')|$ and *a fortiori* $w_q(T_i)$, tends to 0. Q. E. D.

It follows that the set of points $T$ at which $w_q(T) \geqq \varepsilon > 0$ contains no strictly monotone subsequences. It must therefore be distributed over a finite number of planes parallel to the coordinate planes in the space of $T$. For if, given any finite number of points $T_i$ of that set, the planes through these points parallel to the coordinate planes, did not contain the whole of the set, there would be a further point $T$ of the set, such that $\overrightarrow{T_i T}$ had a definite sense for every $i$. Hence we could form by induction a succession of points $T_i$ of the set, each of whose monotone subsequences was strictly monotone, and there would be strictly monotone subsequences of the set. Thus

**Corollary.** *If $\mathcal{E}$ is closed, and $f(\mathcal{P})$ is continuous in $\mathcal{E}$, $w_q(T)$ is $<\varepsilon$ except at most at points $T$ of a finite number of planes parallel to the coordinate planes in the space of $T$.*

For each $q$ there are thus a certain finite number $K_q$ of exceptional values $c_i^{(q)}$ such that, if $w_q(T)\geqq\varepsilon$, one of the coordinates of $T$ is equal to one of these $c_i^{(q)}$.

If $\mathcal{P}'$ and $\mathcal{P}''$ be any two points of $\mathcal{E}$ occupying the same position $P$ (of $k^{\text{th}}$ coordinate $x_k$), and whose senses coincide in all but their $k_1{}^{\text{th}}, k_2{}^{\text{th}},\ldots, k_q{}^{\text{th}}$ coordinates, say, $(1\leqq q\leqq n)$, and $T$ denote the real point in space of $q$ dimensions having the coordinates $x_{k_1}, x_{k_2},\ldots, x_{k_q}$, then by definition,

$$w_q(T)\geqq|f(\mathcal{P}')-f(\mathcal{P}'')|.$$

The latter difference can therefore only be $\geqq\varepsilon$ if one of the coordinates $x_{k_r}$ of the position $P$, corresponding to which the senses of $\mathcal{P}'$ and $\mathcal{P}''$ have opposite $k_r{}^{\text{th}}$ coordinates, coincides with one of the exceptional values $c_i^{(q)}$ Thus

*For two points $\mathcal{P}'$, $\mathcal{P}''$, of $\mathcal{E}$, having the same position $P$, we can only have*

$$|f(\mathcal{P}')-f(\mathcal{P}'')|\geqq\varepsilon$$

*if some (say the $k_0{}^{\text{th}}$), coordinate, of $P$ has one of a certain finite number of exceptional values, while the corresponding (i. e. the $k_0{}^{\text{th}}$) coordinates of the senses of $\mathcal{P}'$ and $\mathcal{P}''$ are opposite.*

Let the exceptional values, in order of magnitude, (without further reference to the index $q$), be

$$c_1<c_2<\ldots<c_{K-1}$$

and let $c_0<c_1$, $c_K>c_{K-1}$ be bounds of the $k^{\text{th}}$ projections of the positions of the points of $\mathcal{E}$, for every $k$.

Let $A$ be any real point in $n$-space whose $k^{\text{th}}$ coordinate, for each $k$, has one of the exceptional values $c_{i_k}$ $(1\leqq i_k<K)$ or the value $c_0$ [1]), and let $B$ be the point whose $k^{\text{th}}$ coordinate, for each $k$, is $c_{i_k+1}$.

[1]) I. e. one of the *nodes*, in the quadrangular network of planes parallel to the coordinate planes:

$$x_i=c_0, c_1, c_2,\ldots \text{ or } c_{K-1}.$$

There are $(K-1)$ of these planes parallel to each of the $n$ coordinate planes, and therefore $(K-1)^n$ nodes.



Let $\mathfrak{A}'$ be the directed point of position $A$ having the sense of $\overrightarrow{AB}$, i. e. with all its coordinates $+$, and let $\mathfrak{B}''$ be the directed point of position $B$ having the opposite sense.

*Between $\mathfrak{A}'$ and $\mathfrak{B}''$ there are no two points $\mathcal{P}'$ and $\mathcal{P}''$ of $\mathcal{E}$ with the same position $P$, for which*

$$|f(\mathcal{P}')-f(\mathcal{P}'')|\geqq\varepsilon.$$

For, if $\mathcal{P}$ is between $\mathfrak{A}'$ and $\mathfrak{B}''$, its $k^{\text{th}}$ coordinate is between $(c_{i_k},+)$ and $(c_{i_k+1},-)$, and the only exceptional values $c_i$ with which the position of this coordinate may coincide are $c_{i_k}$ or $c_{i_k+1}$, and for these the sense of the coordinate is determined and unique.

The point $A$ (and hence $\mathfrak{A}'$ and the corresponding $\mathfrak{B}''$) may be chosen in a finite number $(k-1)^n$ of ways. In the interval of endpoints $\mathfrak{A}'$, $\mathfrak{B}''$ so determined, $|f(\mathcal{P}')-f(\mathcal{P}'')|$ is always $<\varepsilon$, when $\mathcal{P}'$ and $\mathcal{P}''$ coincide in position.

Now every point $\mathcal{P}$ of $\mathcal{E}$ belongs to one, and only one, such interval; for it determines uniquely, for each $k$, a value $c_{i_k}$ such that its $k^{\text{th}}$ coordinate is between $(c_{i_k},+)$ and $c_{i_k+1},-)$.

Thus *$\mathcal{E}$ may be divided into a finite number of portions in each of which, whenever $\mathcal{P}'$ and $\mathcal{P}''$ have the same position,*

$$|f(\mathcal{P}')-f(\mathcal{P}'')|<\varepsilon$$

We thus obtain the following theorem:

**Theorem II.**

*If $f(\mathcal{P})$ is continuous in a closed set $\mathcal{E}$, the latter is the sum of a finite number of closed subsets, (portions of the set) in each of which the maximum difference between values of $f(\mathcal{P})$ at points occupying the same position, is always less than $\varepsilon$, a given positive number* [1]).

Now if $f(\mathcal{P})$ is continuous in a closed set $\mathcal{E}$ and its values for points of $\mathcal{E}$ having the same position always differ by less than $\varepsilon$, the *oscillation* of $f(\mathcal{P})$ cannot be $\geqq\varepsilon$ in subsets of $\mathcal{E}$ of span as small as we please.

[1]) With the interpretation of the notion of function of a directed point given below in section 6, this theorem expresses the generalisation of a well-known theorem in the theory of functions of bounded variation. It is usually proved there by means of the hypothesis of bounded variation, whereas the above shows that it is independent of this hypothesis and depends only on the unicity of the limits of the function in every open quadrant at every point, on which the definition and continuity of the corresponding function of $\mathcal{P}$ depends.

Otherwise there would be two points $\mathcal{P}_i'$ and $\mathcal{P}_i''$ of $\mathcal{E}$, with

$$|f(\mathcal{P}_i') - f(\mathcal{P}_i'')| > \left(1 - \frac{1}{i}\right),$$

whose pair had a span less than $\delta_i$, where $\delta_i \to 0$ with $1/i$.

We could pick out a subsequence of the indices $i$, so that the corresponding $\mathcal{P}_i'$ and $\mathcal{P}_i''$ both formed a monotone sequence. Their limits $\mathcal{P}_0'$ and $\mathcal{P}_0''$ occupy the same position (by p. 248) and are points of $\mathcal{E}$, and

$$|f(\mathcal{P}_0') - f(\mathcal{P}_0'')| \geqq \varepsilon.$$

This is contrary to the hypothesis about $f(\mathcal{P})$. Thus

**Theorem III.**

*If $f(\mathcal{P})$ is continuous in a closed set $\mathcal{E}$ and its values at points of $\mathcal{E}$ having the same position always differ by less than $\varepsilon$, the oscillation of $f(\mathcal{P})$ is less than $\varepsilon$ in every subset of $\mathcal{E}$ of span less than a certain positive $\delta_\varepsilon$.*

Since we can express $\mathcal{E}$ as the sum of a finite number of portions of $\mathcal{E}$, each of span less than $\delta_\varepsilon$, it follows, in particular, that $\mathcal{E}$ is the sum of a finite number of closed sets (portions of the given set) in each of which $f(\mathcal{P})$ has oscillation less than $\varepsilon$. Combining this with Theorem II, we obtain the analogue of the uniform continuity theorem of the theory of functions of position [1]):

**Theorem IV.**

*If $f(\mathcal{P})$ is continuous in a closed set $\mathcal{E}$, the latter is the sum of a finite number of closed subsets (portions of the set) in each of which the oscillation of $f(\mathcal{P})$ is less than $\varepsilon$, a given positive number.*

6. A function $f(\mathcal{P})$ whose values in a set $\mathcal{E}$ are independent of the sense of $\mathcal{P}$ constitutes a *function of position* in the ordinary sense, in the set $E$ of the positions of the points of $\mathcal{E}$.

More generally, given any function $f(\mathcal{P})$, considered in a set $\mathcal{E}$, we can deduce from it a function of position by taking the mean of its values for each group of points of $\mathcal{E}$ having the same position $P$. We shall speak of this as the *associated function of position* of $f(\mathcal{P})$.

The limits of a function of position for strictly monotone sequences of positions, of the same (definite) sense $-\Theta_0$ and limit $P_0$, and belonging to a set $E$, are called its *limits at $P_0$ in the open quadrant of index $\Theta_0$ in $E$*. (When $E$ is a specific fundamental set, which does not change in the course of a discussion, its mention is omitted). Each such limit of the associated function of position of a given $f(\mathcal{P})$, considered in a set $\mathcal{E}$, is the mean between some limits of $f(\mathcal{P})$ for sequences of points of $\mathcal{E}$ of limit $\mathcal{P}_0 = (P_0, \Theta_0)$.

In particular, if $f(\mathcal{P})$ is continuous in $\mathcal{E}$ at $\mathcal{P}_0$, the associated function of position has the unique limit $f(\mathcal{P}_0)$ for every strictly monotone sequence of positions in $E$, of sense $\Theta_0$ and limit $P_0$.

Therefore:

*The associated function of position of a function $f(\mathcal{P})$ considered in a set $\mathcal{E}$ in which it is continuous, has a unique limit in each open quadrant in the set $E$ of the positions of the points of $\mathcal{E}$, and for the quadrant at $P$ of index $\Theta$, this limit is the value of $f(\mathcal{P})$ at the point of $\mathcal{E}$ of position $P$ and sense $\Theta$.*

Given a function of position $F(P)$ whose limit in each open quadrant in a set $E$ is finite and unique, this limit constitutes a *function of each strictly monotone subsequence of $E$*, depending solely on the limit and sense, say $P$ and $-\Theta$, of this subsequence. If $P$ denote the directed point of position $P$ and sense $\Theta$, the limit in question constitutes precisely a function of $\mathcal{P}$, say $f(\mathcal{P})$, in the set $\mathcal{E}'$ of points each having the position $P$ of a limiting real point of $E$, — one which does not correspond merely to *not strictly* monotone subsequences of $E$ — and the sense of a strictly monotone subsequence of $E$, of limit $P$; or, in the language adopted for the second mode of representation of directed points, in the set $\mathcal{E}'$ of directed points „represented“ by subsequences of $E$. Using also the criterion (p. 250) which serves to interpret in this language the notion of limit, we easily prove that:

*This function $f(\mathcal{P})$ is continuous in $\mathcal{E}'$.*

For, if $\mathcal{P}_i$ describes any monotone sequence of points of $\mathcal{E}$ tending to a point $\mathcal{P}_0$ of $\mathcal{E}'$, of position $P_0$, the criterion says that, if we choose any real point $P_i'$ sufficiently near to $P_0$ in the strictly

[1]) In the theory of functions of position, the analogue of Theorem III, from which the uniform continuity theorem is deduced, may conversely be inferred from the latter, because the oscillation of a continuous function of position in the sum of two sets between which the distance is 0, is always at most equal to the sum of its oscillations in the two sets. In our case, Theorem III is more precise than IV, and cannot be deduced from it.

monotone subsequence of the points of $E$ representing $P_i$, every monotone subsequence of the points $P_i'$ represents $\mathscr{P}_0$.

By the definition of $f(\mathscr{P}_i)$, we may choose $P_i'$ so near $P_0$ that

$$|F(P_i') - f(\mathscr{P}_i)| < \frac{1}{i};$$

and, by the definition of $f(\mathscr{P}_0)$,

$$\lim_{i\to\infty} F(P_i') = f(\mathscr{P}_0),$$

(because this is true for every monotone subsequence of the points $P_i'$).

Hence also

$$\lim_{i\to\infty} f(\mathscr{P}_i) = f(\mathscr{P}_0).$$

Q. E. D.

Thus there is a complete correspondence between the continuous functions of a directed point and the functions of position with unique limits in every open quadrant at each point.

*Each continuous function of a directed point represents the limit, in the corresponding open quadrant, of a function of position with unique limits in each open quadrant at every point. And each function of position of this type defines uniquely a continuous function of a directed point representing its limits in the open quadrants.*

# Concerning triodic continua in the plane.

By

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In a recent paper [1]) I defined the term *triod* and showed that there does not exist, in the plane, an uncountable set of mutually exclusive triods. In the present paper I will generalize this notion and establish a correspondingly more general theorem.

Lemma. *If the metric space $S$ contains a countable collection of compact point sets $S_1, S_2, S_3, \ldots$ such that every compact subset of $S$ is contained in the sum of a finite number of point sets of this collection, then every uncountable collection of closed and compact subsets of $S$ contains an uncountable subcollection $G$ such that if $e$ is any positive number and $g_0$ is any point set of the collection $G$ then there exist uncountably many point sets $g$ of $G$ such that every point of $g$ is at a distance less than $e$ from some point of $g_0$ and every point of $g_0$ is at a distance less than $e$ from some point of $g$.*

Proof.[2]). Let $E_n$ denote the compact point set $S_1 + S_2 + \ldots + S_n$. For each pair of natural numbers $m$ and $n$, $E_n$ contains a finite point set $S_{mn}$ such that every point of $E_n$ is at a distance less than $1/m$ from some point of $S_{mn}$. Let $T$ denote the collection of all point sets $X$ such that, for some $m$ and $n$, $X$ is a subset of $S_{mn}$. Each

[1]) *Concerning triods in the plane and the junction points of plane continua*, Proceedings of the National Academy of Sciences, vol. 14 (1928), pp. 85—88.

[2]) For the case where the space $S$ is Euclidean space of a finite number of dimensions this lemma may be proved (even though the requirement that the point sets of the collection $G$ be compact is removed) by a modification of an argument given by Zarankiewicz to prove a related theorem. Cf. Casimir Zarankiewicz, *Sur les points de division dans les ensembles connexes*, Fundamenta Mathematicae, vol. IX (1927), Theorem 2, page 6.