est evidemment un continu et on a $A \times T = a$. Donc a est accessible dans R_n c. q. f. d.

6. Un problème ce pose: le théorème reste-t il vrai si on supprime la condition que A est fermé? Le résultat suivant me parait probable: Si l'ensemble $A \subset R_n$ est homéomorphe d'un sous-ensemble vrai du R_{n-1}, alors $R_n - A$ est un semi-continu.

Varsovie, 4/X 1928.

On Continuous Curves which are Homogeneous except for a Finite Number of Points.

By

T. C. Benton (Philadelphia).

1. **Introduction.** It is the author's purpose to classify all the plane continuous curves which are homogeneous except for a finite number of points. This has only been completed in the case where the non-homogeneous points may be made to correspond to each other. If the number of non-homogeneous points is greater than 3 it is found that a bounded continuous curve is the sum of a number of curves which are homogeneous except for two points. It is shown that a continuous curve, homogeneous except for two points is a finite number of arcs (not two) joining the two points. Then the general case is solved by replacing each one of the two point curves, whose sum is the given curve, by an arc joining the two non-homogeneous points. The resulting curve can then be either a simple closed curve or in special cases a more complicated curve which is in one-to-one correspondence with the projection of the edges of one of the regular or semi-regular solids upon one of its faces in such a manner that none of the projections of edges have any intersections except the projections of the vertices of the solid.

The paper is an extension of the results of S. Mazurkiewicz[1]) who proved that a bounded continuous curve which is homogeneous at every point is a simple closed curve.

The author is deeply indebted to Dr. J. R. Kline who suggested the problem and gave invaluable assistance in the working

[1]) S. Mazurkiewicz: *Sur les continus homogènes.* Fund. Math. vol. 5 (1922) p. 131. This article will be quoted by paragraph thus: ⌗16 M.

out of it, and in preparing the paper. He wishes to express his thanks for all that Dr. Kline has done to help him.

2. **Notations.** One-to-one, continuous correspondences having continuous inverses and having the additional property that they cause a set X to correspond to itself[1]) will be denoted by Greek lower case letters with an X subscript, $\pi_{X'}, \varphi_{X'}, \ldots$ Also the notation $\pi_M[x] = y$ means that if x and y are points of M, then x corresponds to y under the correspondence π_M which makes M correspond to itself.

The notation $M \subset N$ [or $p \subset N$] means that M [or p] is a subset [or point] of N.

The notation $\overline{M}$ means M plus all its limit points.

$H[M, N] = M.\overline{N} + \overline{M}.N$, so that $H[M, N] = 0$ means that M and N are mutually separated.

Definitions. (*i*) If for a_1, a_2, *any* pair of points of a continuous curve M, there exists a π_M such that $\pi_M[a_1] = a_2$, M is said to be a homogeneous continuous curve.

(*ii*) If a continuous curve M contains a set of points $c_1, c_2, \ldots, c_n$, such that for each c_i there is some x_i for which no π_M exists giving $\pi_M[x_i] = c_i$; and if for every pair of points a_1, a_2 neither of which belongs to $c_1, c_2, \ldots, c_n$, there does exist a φ_M such that $\varphi_M[a_1] = a_2$; then M is a continuous curve, homogeneous except for the points $c_1, c_2, \ldots, c_n$.

3. **Theorem 1.** *A bounded continuous curve M which is homogeneous except for one point c consists of a finite $[\geqslant 2]$ or countably infinite number of simple closed curves having only c in common and such that only a finite number of these curves are of diameter greater than any assigned positive number.*

3.1 *No homogeneous point of M is a cut point of M.*

Proof. Let x be a homogeneous cut point of M. Let y be another homogeneous point of M. By definition of M there exists a π_M such that $\pi_M(x) = y$. Since x cuts M, $M - x = M_1 + M_2$ where $H(M_1, M_2) = 0$. Then $\pi_M(M - x) = M - y = \pi_M(M_1) + \pi_M(M_2)$. And since π_M is continuous $H\{\pi_M(M_1) . \pi_M(M_2)\} = 0$. Therefore y is a cut point of M. Hence every homogeneous point of M is a cut point of M and c is the only non-cut point. This contradicts the theorem that every bounded continuous curve has at least two non-cut points[1]).

3.2. $\pi_M(c) = c$. This follows as a special case of the more general theorem that in any continuous curve homogeneous except for n points the correspondent of any c-point must be a c-point. To prove this let $\pi_M(c) = x$ where x is some homogeneous point of the curve M. Then if π_M^{-1} is the inverse of π_M, $\pi_M^{-1}(x) = c$. From the definition of homogeneity, if y is any other homogeneous point of M there exists a ψ_M such that $\psi_M(y) = x$. Then $\pi_M^{-1}\{\psi_M(y)\} = c$ or if $\pi_M^{-1}\psi_M = \varphi_M$ we have $\varphi_M(y) = c$. Hence c is a homogeneous point but this contradicts the assumption so the theorem is proved.

3.3*i*. *The point c is a cut point of M.*

Proof. Suppose c not a cut point of M. Let S be the whole plane. Then $S - M = \sum_k \beta_k$ where each β_k is a complementary domain of M. Since every point of M is non-cut if c is (3.1), it follows that the boundary $F(\beta_k)$ of β_k is a simple closed curve by # 13 M.

We shall first show that under the assumption that c is a non-cut point $F(\beta_i) . F(\beta_k)$ must be totally disconnected. Let us suppose that this set contains a connected subset other than a single point. Then it follows that one, and hence all, of the homogeneous points must be ordinary points[2]) and hence must be the ends of at most two arcs which lie in M. As M is bounded one of the two domains β_i and β_k is bounded, let β_i be the bounded one. If β_k is not the whole exterior of β_i, then there is a point and hence at least one domain β_j $(j \neq k)$ outside the simple closed curve $F(\beta_i)$. If β_k should be unbounded then we may invert about a point of β_j keeping $F(\beta_j)$ invariant and get β_k bounded, so we shall assume that β_k is bounded at the outset. Now not all the points of $F(\beta_k)$ are also points of $F(\beta_i)$. Let h be an interior point of the arc Q common to $F(\beta_i)$

[1]) Especial attention is called to the fact that the correspondences are assumed to be on the curve alone and *that they are not correspondences of the whole plane into itself*. For examples of correspondences of a curve into itself which cannot be extended to the whole plane and a discussion of this whole problem see, H. M. Gehman, *On extending a continuous* $(1-1)$ *correspondence of two plane continuous curves to a correspondence of their planes*, Trans. Amer. Math. Soc. Vol. XXVIII, pp. 252–265.

[1]) S. Mazurkiewicz: „*Un théorème sur les lignes de Jordan*“, Fund. Math. vol. 2 (1921) pp. 119–130.

[2]) If p is an interior point of an arc of M and is not a limit point of any points of M except sets lying on that arc, p is said to be an ordinary point of M. This definition is equivalent to that due to Janiszewski; Thesis p. 64.

and $F(\beta_k)$. while k is a point of $F(\beta_i)$. Then on each of the arcs hy_ik there is a point p_i $(i=1, 2)$ such that there goes from this point an arc of $F(\beta_k)$ which has no point of $F(\beta_i)$ on it except p_i. These points must be distinct and hence cannot both be c. Hence there is at least one homogeneous point which is not ordinary. Thus it follows that $F(\beta_i).F(\beta_k)$ is a totally disconnected set if c is not a cut point.

Now by a very simple modification of $\# 16M$ it can be shown that $M \neq \sum_i F(\beta_i)+c$. Then by applying the argument of $\# 17$—$21M$, it may be shown that the removal of $F(\beta_i)$ does not disconnect M while the removal of a simple closed curve which contains a point of $M-(\sum_i F(\beta_i)+c)$ does disconnect M.

3.3 *ii.* Let a point x be taken which is homogeneous and belongs to some $F(\beta_i)$ and also a point y homogeneous but not on the boundary of any domain. Take π_M so that $\pi_M(x)=y$. The curve $F(\beta_i)$ does not cut M, but $\pi_M\{F(\beta_i)\}$, which contains y, must cut M. Hence the correspondence is not continuous and we are led to a contradiction if we assume that c does not cut M.

3.4. Let $M-c=\sum_{j=1}^{n} M_j$, where each M_j is a maximal connected subset of $M-c$. Each of the sets $N_j=M_j+c$ is a continuous curve [1]). Since c cuts M, n is greater than 1.

3.5. *N is a simple closed curve.*

Proof. Take the point c as a center of inversion [2]).

By $\# 23M$, it follows that N_j+c is inverted into an unbounded set which is homogeneous at every point, and hence is an open curve. But by inverting again we see that the original set must be a simple closed curve.

Since $M=c+\sum_i M_i$, there can be only a finite number of the sets N_i of a diameter greater than any assigned positive number since M was a continuous curve. Hence the theorem is proved.

4. **Theorem 2.** *A bounded continuous curve M which is homogeneous except for two points c_1, c_2, consists of a finite number of arcs $[\neq 2]$ joining c_1 to c_2 and having no other points in common.*

[1]) H. M. Gehman: Annals of Math. 1927 p. 111. Theorem 8.

[2]) C. Kuratowski: *„Sur la méthode d'inversion"*. Fund. Math. Vol. 4. pp. 151—163.

Assume that M is more than a single arc from c_1 to c_2. Such a set will of course be a solution to the problem but it will be convenient to rule it out at the start and investigate only the more complicated cases.

4.1. *No homogeneous point of M is a cut point of M.*

Proof. If one homogeneous point of M is a cut point every one is. Then if c_1 and c_2 or c_1 or c_2 are cut points M has no or one non-cut points, which is impossible [1]). If both c_1 and c_2 are non-cut points and every homogeneous point of M cuts M, then M is an arc [2]), which was just ruled out. Hence 4.1. is established.

4.2. *No point of M is a cut point of M.*

Proof. Suppose c_1 is a cut point. Then $M-c_1=\sum_{i=1}^{n} M_i$, where n is greater than 1, and $H[M_i, M_j]=0$, and each M_i is a maximal set. Let M_1 be the set which contains c_2. Now M_1+c_1 is a continuous curve [3]). The set M_1+c_1 contains c_2 so there exists an arc c_1xc_2 in M_1 except for c_1. Let $x_2 \subset M_2$, and take π_M so that $\pi_M[x]=x_2$. Then $\pi_M[c_1]=c_1$ or c_2. *Case i.* $\pi_M[c_1]=c_2$. Then $\pi_M[c_1x]=c_2x_2$. Now the arc c_2x_2 does not contain c_1 for π_M^{-1} would then cause $c_1x \supset c_2$ which is not true for the original choice of x between c_1 and c_2 on the arc. But then x_2 is joined to c_2 by an arc of M_1 not containing c_1. Hence M_1 is not a maximal set contrary to hypothesis.

Case ii. $\pi_M[c_1]=c_1$. Then $\pi_M[c_2]=c_2$, so $\pi_M[xc_2]=x_2c_2$. As before x_2c_2 does not contain c_1 so M_1 is not maximal.

Hence neither case is possible so c_1 is not a cut point. The same proof holds for c_2. Therefore the theorem 4.2 is established.

4.3. *M contains the interior of no simple closed curve.*

Proof. If M has an interior point x, let C be the outer boundary of the domain which contains x. Let y be a point of the boundary C. Then no π_M can make $\pi_M[x]=y$, since a boundary point is a limit point of the exterior of C but an interior point is not. Now C has an infinite number of points so M has more than two non-homogeneous points contrary to hypothesis.

4.4. If S is the whole plane, $S-M=\sum \beta_i$. As no point of M is a cut point of M, $F[\beta_i]$ is a simple closed curve. If $F[\beta_k].F[\beta_i]$

[1]) see ref. under 3.1.

[2]) R. L. Moore: *„Report on continuous curves"*. Bull. Amer. Math. Soc. 1923, p. 334, Definition 1.

contains an arc, every point of $M - c_1 - c_2$ is an ordinary point. Further, the arc common to $F[\beta_k]$ and $F[\beta_i]$ must be an arc from c_1 to c_2 because the end points of this arc are meeting places of at least three arcs and are therefore not ordinary points. (Proofs of these theorems are in # 11—14 *M*).

If $F[\beta_k] . F[\beta_i]$ contains an arc, M is a finite number of arcs joining c_1 to c_2.

Proof. $F[\beta_k]$ and $F[\beta_i]$ both contain c_1 and c_2 since the arc which they have in common does so. Let $x \subset M$ and x is contained in neither $F[\beta_k]$ nor $F[\beta_i]$. Then x can be joined to c_1 by an arc J_1 not containing c_2, because c_2 is not a cut point. Then $J_1 . \{F[\beta_k] + F[\beta_i]\}$ consists of only c_1, since any point of this product cannot be an ordinary point and J_1 does not contain c_2. Also x can be joined to c_2 by an arc J_2 and J_2 does not contain c_1. Then $J_1 . J_2 = x$, since any other point of $J_1 . J_2$ would not be ordinary, but $J_1 . J_2$ does not contain c_1 or c_2 and hence every point of $J_1 . J_2$ must be ordinary. Then x lies on an arc from c_1 to c_2. Since x was any point of M the set M must be composed of arcs from c_1 to c_2 and there are at most a finite number of these arcs because $M - c_1 - c_2$ cannot have more than a finite number of maximal connected subsets which are of diameter greater than any assigned positive constant.

4.5. If no two domains have an arc common to their boundaries, then $F[\beta_k] . F[\beta_i]$ will be totally disconnected. Call this set F_{ki}. By # 16*M*, $M \neq \Sigma F[\beta_i]$. We shall now show that $M \neq c_1 + c_2 + \sum_i F[\beta_i]$. Let us take x in β_1 and y in β_2. There exists a chain C[1]) every link of which is a circle of diameter less than e and no link of which contains either c_1 or c_2. It is clear that no link is disconnected by a totally disconnected closed set lying entirely within it or on its boundary. Arrange the countable sets F_{ki} in some definite order and label them $F_1, F_2, F_3, \dots$ Then using precisely

[1]) If A and B are distinct points, then a *simple chain* from A to B is defined by R. L. Moore as a finite set of regions $R_1, R_2, \dots, R_n$ such that [1] R_i contains A if and only if $n = 1$, [2] R_i contains B if and only if $i = n$, [3] if $1 \leqslant i \leqslant n$ and $1 \leqslant j \leqslant n$ $[i < j]$, then R_i and R_j have a point in common if and only if

$$j = i + 1.$$

The region R_k is said to be the k th region of the chain. See R. L. Moore: *On the foundations of plane Analysis Situs*; Trans. Amer. Math. Soc. vol. 17 [1916] p. 134.

the methods used by Moore[1]) it is possible to obtain a sequence of chains $C_0, C_1, C_2, \dots$ such that [1] the chain $C_0 = C$, [2] each link of the chain C_{n+1} lies, together with its limit points, wholly in some single link of C_n; [3] if a link x of C_{n+1} lies in a link y of C_n; then every link that follows x in C_{n+1} lies in y or in some link that follows y in C_n; [4] every link of C_n is a circle of diameter less than $e/2^n$; [5] no point of the set F_n lies within C_n. The common part of these chains is an arc C^* from x to y which has no points in common with $c_1 + c_2 + F_{ki}$. Now $S = c_1 + c_2 + \Sigma\{\beta_i + F[\beta_i]\}$ so $C^* . S = C^* . [c_1 + c_2] + \sum_i C . \{\beta_i + F[\beta_i]\}$ and $C^* . [c_1 + c_2] = 0$. If $C^* \{\beta_k + F[\beta_k]\}$ be denoted by A_k. then; $A_k . A_i = C[\beta_k . \beta_i] + C . \{F[\beta_k] . \beta_i\} + C \{F[\beta_i] . \beta_k\} + C . \{F[\beta_k] . F[\beta_i]\} = 0$, since the first three terms are always 0, and C was constructed so that the last term was 0. But then $C = \sum_i A_i$ which contradicts Sierpiński's theorem[2]) so $M - \sum_k F[\beta_k]$ must contain more than $c_1 + c_2$ and hence must contain at least one homogeneous point.

4.6. We shall now prove that, under the assumption that no two complementary domains have an arc in common to their boundaries, we arrive at a contradiction by showing that M must be equal to $\sum_i F[\beta_i]$. Let z be a point of $M - \sum_i F[\beta_i]$, different from c_1 and c_2 and let e be less than the smaller of the following numbers, $1/2\, d[z, c_1]$ and $1/2\, d[z, c_2]$, where $d[x, y]$ is the distance between x and y.

By applying an argument similar in all respects to that used by Mazurkiewicz, we may show that z taken with any other point of M, will not cut M; then by using the homogeneity of the point z, we can show that no pair of points of M can cut M unless it be the pair composed of c_1 and c_2. Thus it will follow that of the points common to the boundaries of two complementary domains at most one can be a homogeneous point[3]).

As z is not on the boundary of any complementary domain, there is a simple closed curve Q of M, which is of a diameter less

[1]) Cf. R. L. Moore: *Foundations of Analysis Situs*, loc. cit. Theorem 15, p. 137.

[2]) see 3.2. *iii*.

[3]) The argument referred to here is found in the article of Mazurkiewicz loc. cit. pages 142—143.

than e and has z in its interior [1]). The non-homogeneous points are in the exterior of Q. As M contains the interior of no simple closed curve, there must be a point not in M within Q. Hence there is a complementary domain β_p within Q, and this complementary domain has the property that all its boundary points belong to the homogeneous points of M. Then by applying the argument of Mazurkiewicz in # 20 [2]) we may show that $F[\beta_p]$ does not cut M, for any domain whose boundary has a point in common with $F[\beta_p]$ must have homogeneous points in common and thus can have but a single point in common, the essential fact in the argument referred to.

The point z must be on a simple closed curve of M [3]), since z is not a cut point or an end point. z is not a cut point (4.1). z is not an end point because if it were it would have to have a cut point in its neighborhood [4]) and M has no cut points.

By # 21 M, K, any simple closed curve containing z, must cut M. Take y contained in $F[\beta_p]$ and π_M so that $\pi_M[y]=z$; then $\pi_M\{F[\beta_p]\}$ is a simple closed curve through z and hence must cut M. But β_p does not cut and hence $\pi_M[\beta_p]$ cannot cut. So there is a contradiction and no point of $M-\{c_1+c_2+\Sigma F(B_i)\}$ can exist.

4.7. Now there is a real contradiction between the assumption in the first sentence in 4.5 and the result of 4.6, since the one required the set M to be more than the sum of the boundaries of its complementary domains while the other shows that M is just that sum. Hence there must be some arc common to two boundaries of complementary domains and the result of 4.4 is the only possible set of the required type.

II.

5. Let M be a bounded continuous curve having $c_1, c_2, \ldots c_n$ $(n>2)$, as non-homogeneous points and all other points homogeneous. By 3.2 *ii* it follows that $\pi_M[c_i]=c_j$.

5.1 *M contains at least three non-cut points.*

[1]) See G. T. Whyburn: *Continua in the plane*, Trans. Amer. Math. Soc. vol. 29 [1927] Theorem 7.

[2]) Loc. cit. page 144.

[3]) Loc. cit. Theorem 22.

[4]) H. M. Gehman: *Concerning endpoints of Continuous Curves*. Trans. Amer. Math. Soc. vol. 30 [1928] p. 182, corollary 24 a.

Proof. Every continuous curve has at least two non-cut points and if only two it is an arc joining the two non-cut points. But an arc has only two non-homogeneous points and M is supposed to have n, so the theorem is proved.

5.2. *If every homogeneous point cuts M, M contains no simple closed curve.*

Proof. This follows from the results of Mazurkiewicz [1]).

5.3. *If every homogeneous point cuts M, one and only one of the points $c_1, c_2, \ldots, c_n$, may cut M and M is a set of arcs which have only that point in common.*

Proof. It will first be shown that one point c_i must cut M. Suppose no c_i cuts M. By 5.1 there are three non-cut points in M and by hypothesis they cannot be homogeneous; so let c_1, c_2, c_3, be these points. By 5.2, M is acyclic so the non-cut points of M are end points. Then, since every c_i was supposed non-cut, there must be n endpoints in M. There must be at least one branch point as there are at least three endpoints. But a finite number of end points implies a finite number of branch points [2]). Hence only a finite number of the homogeneous points are branch points and the rest must be ordinary cut points, which is a contradiction since a branch point cannot correspond to a cut point which is ordinary in a continuous correspondence. Hence there must be a cut point among the c-points.

It will now be shown that not more than one of the c-points cuts M. Suppose c_i and c_j are cut points of M, while c_1 is a non-cut point and therefore an end point of M. Join c_1 to c_i by an arc L_1 in M. Since c_i cuts M it is not an end point of M and therefore there is another arc L_2 distinct from L_1 which has c_i for an end point. There can be no point of the c-points on the arc from c_1 to c_i. For if c_g be the first c-point distinct from c_1, and c_k be the last c-point distinct from c_i on the arc $c_1 c_i$, take h a homogeneous point on $c_1 c_g$ and h' a homogeneous point on $c_k c_i$. Let π_M be the correspondence which gives $\pi_M[h]=h'$. Then $\pi_M[c_1]=c_k$ or c_i. But this is impossible since neither c_k or c_i are endpoints of M. Therefore no c_g can exist. Now c_1 and

[1]) S. Mazurkiewicz: „*Un théorème sur les lignes de Jordan*". Fund. Math. vol. 2, p. 119—130.

[2]) K. Menger: „*Über Reguläre Baumkurven*". Math. Annalen. 96 [1927] pp. 572—582. See also Ważewski: Annales Soc. Pol. Math. 2 [1923] p. 49—170.

c_j can be joined by an arc of M, but c_i was the first c-point on the arc $c_1 c_i$ and c_1 was an end point of M, hence c_i must be on $c_1 c_j$ or c_1 would not be an end point. Also c_j is not an end point since it is a cut point. Take the c-point c_n nearest c_j in the order $c_1 c_i c_n c_j$. Then if $h \subset$ arc $c_1 c_i$ and $h' \subset$ arc $c_n c_j$ and π_M is chosen so that $\pi_M[h] = h'$, it follows that $\pi_M[c_1] = c_n$ or c_j, neither of which are end points and hence π_M is not continuous. Hence c_j does not exist.

5.4. *If the homogeneous points of M are non-cut points of M, then M has at most one cut point.*

Proof. Let c_1 cut M so that $M - c_1 = \sum\limits_i M_{1i}$, where each M_{1i} is a maximal connected subset of $M - c_1$ and $i \geqslant 2$. Let us suppose that c_2 is in M_{11} and c_2 cuts M.

Now every one of the sets M_{1i} must contain a c-point which cuts M. For suppose that M_{1i} contains no such c-point. There is an arc lying, except for c_1, entirely within M_{11} and joining c_1 and c_2. Now let $\bar{c}_2$ be the first cutting c-point after c_1 on the arc $c_1 x c_2$. If there are any non-cutting c-points on this arc between c_1 and $\bar{c}_2$, let them be denoted by $c_2', c_3', \ldots, c_k'$, and let them occur in the order $c_1 c_2' c_3' \ldots c_k' \bar{c}_2$. Let x be a homogeneous point between c_1 and c_2' while y is a homogeneous point of M_{1j}; let π_M be a $[1-1]$ continuous correspondence, which has $\pi[x] = y$. Now the connected set $c_1 x c_2' c_3' \ldots c_k' \bar{c}_2 - \bar{c}_2 - c_1$, as it contains no point whose correspondent under π_M can be c_1, must go entirely into the set M_{1j}. But $\pi_M[c_1]$ and $\pi_M[\bar{c}_2]$ must be cut points and must also be contained in $\overline{M_{1j}}$, which we know is $M_{1j} + c_1$. It follows that $\pi_M[c_1]$ and $\pi_M[\bar{c}_2]$ must both be c_1 and the correspondence cannot be $[1-1]$. Thus we have reached a contradiction if each of the sets M_{1i} does not contain a c-point which cuts M.

Let $M_{11} - c_2 = \sum\limits_k M_{11k}$, where some of the sets M_{11k} cannot have c_1 as a limit point, otherwise the set $M - c_2$ would be connected. Let those sets of M_{11k} which have c_1 as a limit point be $\sum\limits_k M_{11k}^*$ while those that fail to have c_1 as a limit point are $\sum\limits_j M_{11j}'$. Then $c_1 + \sum\limits_k M_{11k}^* + \sum\limits_i M_{1i}$ $(i \neq 1)$ is a maximal connected subset of $M - c_2$. As before each of the sets of $\sum\limits_j M_{11j}'$ must contain a c-point which cuts M. Let that c-point which is in M_{111}' and cuts M be c_{11}. By considering $M_{111}' - c_{11}$ and proceeding as before we get c_1 and c_2 lying in the same maximal connected subset of $M - c_{11}$ and also other maximal connected subset of $M - c_{11}$ each of which must contain at least one cutting c-point, and each distinct from the set which contains c_1 and c_2. Continue this process. As the number of c-points is finite, there will be a stage where we reach a c-point, c^*, which when it is removed from M leaves maximal connected sets which are such that not all of them contain cutting c-points. Thus we have reached a contradiction if we assume that more than one c-point can cut M.

5.5. *If $M - c_1 = M_{11} + M_{12} + \ldots + M_{1n}$, each M_{1i} is a homogeneous continuous curve except for the c-points which it contains, and no point of this curve can cut it.*

Proof. $M_{1k} + c_1$ is a continuous curve.

If $[M_{1k}]$ denotes all homogeneous points of M in M_{1k}, $M_{1k} + c_1 = [M_{1k}] + \sum\limits_{k=1}^{n_1} c_k'$ where $\sum\limits_{k=1}^{n_1} c_k'$ gives all the c-points in $M_{1k} + c_1$. The set $M_{1k} + c_1$ is a closed set, so if h and h' are points in $[M_{1k}]$, and π_M is selected so that $\pi_M[h] = h'$, then $\pi_M[M_{1k} + c_1] = [M_{1k}] + \sum\limits_{k=1}^{n_1} \pi_M[c_k']$. Also $\sum\limits_{k=1}^{n_1} \pi_M[c_k'] = \sum\limits_{k=1}^{n_1} c_k'$. For, suppose for some k in M_{1k}, $\pi_M[k] = k'$, where k' is in M_{1j}. $[j \neq k]$. There is in M_{1k} an arc hxk. Evidently $\pi_M[c_1] = c_1$, since a cut point must correspond to a cut point and c_1 is the only cut point in M But any arc from a point of M_{1k} to a point of M_{1j} must contain c_1. Hence $\pi_M[hxk]$ contains c_1. So there is on hxk a point z such that $\pi_M[z] = c$ and then π_M is not (1—1). This is contrary to assumption.

Suppose that c_n cuts $M_{1k} + c_1$. Let $M_{1k} + c_1 = c_n + \sum\limits_j M_{1kj}$ and let $M_{1k1} \supset c_1$. Now;

$$M - c_n = [\sum_{j \neq 1} M_{1kj}] + [M_{11} + \ldots + M_{1,j-1} + M_{1j1} + M_{1,j+1} + \ldots]$$

and $M - c_n$ is disconnected which contradicts 5.4. Hence c_n cannot cut $M_{1k} + c_1$.

5.6. *If M has c_1 a cut point and $M - c_1 = \sum\limits_i M_{1i}$ then each M_{1i} contains just one c-point which will be called c_{i+1}.*

Proof. Suppose $M_{11} \supset c_2$ and c_3. By 5.5 there is an arc from c_2 to c_3 not containing c_1 since c_1 is not a cut point of $M_{11} + c_1$. Also there is an arc from c_1 to c_2. Now c_2 and c_3 are not cut points of M by 5.4. Therefore if $h \subset c_2 c_3$ and $h' \subset c_1 c_2$ and π_M is chosen so that $\pi_M[h] = h'$ there is a contradiction because either $\pi_M[c_2]$ or $\pi_M[c_3]$ must be c_1 which is impossible. Hence only one of the points c_2 and c_3 can exist.

5.7. *If M has a cut point c_1, it is a set of $n-1$ continuous curves $\sum_{i=1}^{n-1} N_i$ homogeneous except for two points c_1 and c_{i+1}, where each N_i contains the same number of arcs joining c_1 to c_{i+1}.*

Proof. This follows easily from 5.5 and 5.6.

5.8 The case where no point of M is a cut point is now to be considered. # 13M shows that the boundary of any complementary domain is a simple closed curve. By a simple modification of 4.5 and 4.6 [use Σc_i instead of $c_1 + c_2$.] it results that two complementary domains must exist which have an arc common to their boundaries and from this result together with # 14M all homogeneous points are ordinary points of M.

5 8.1. *If $M - \Sigma c_j = \Sigma M_i$, then $\overline{M_i}$ contains two and only two c-points. Each of the M_i is a maximal connected subset of $M - \Sigma c_j$.*

Proof. Let h be a homogeneous [ordinary] point of M_i. Then h is on an arc of M_i which precedes to a new c-point, one at either end. Let these be c_2 and c_3. Now the arc $c_2 c_3$ has no other c-points on it. Therfore h cannot be joined to any point of M_i not on $c_2 c_3$ by an arc not containing at least one of the two points c_2 or c_3. Let h' be a point of M_i not on this arc, then h' cannot be joined to h by an arc of M_i hence M_i is disconnected, which is false. Therefore each M_i is an arc joining two c-points.

5 9. *If M contains no cut point then either (i) every c_i can be made to correspond to c_1 by some π_M, or (ii) the c-points fall into two sets such that those of one set correspond but a point of one set cannot be made to correspond to one of the other set.*

Proof. Let $M - \sum_i c_i = \sum_j M_j$. Let π_M be some correspondence which gives $\pi_M[c_1] = c_j$ where $j \neq 1$. Now $\pi_M[c_1] \neq c_1$ for every possible π_M for, if it were, take $h \subset M_j$ and $h' \subset M_j$ and π_M so that $\pi_M[h] = h'$. Then $\pi_M[\overline{M_i}] = \overline{M_j}$ since these are arcs joining two c-points by 5.8. Let c_{i1} and c_{i2} be the c-points of $\overline{M_i}$. Then let $\pi_M[c_{i1}] = c_{j1}$ and $\pi_M[c_{i2}] = c_{j2}$. Suppose $\overline{M_i}$ was chosen so that c_1 was one of the c-points. Then let c_{i1} be c_1. There must be an $\overline{M_j}$ not containing c_1 for if not, c_1 would be a cut point of M. There are then two cases, *i)* every c-point can be made to correspond to c_1,

or *ii)* some point exists which does not correspond to c_1. Suppose c_q is such a point. Then let $\overline{M_q}$ contain c_q. Let c_q' be the other c-point which is a limit point of M_q Let $\overline{M_1}$ contain c_1 and c_1'. There exists a ψ_M which, if $h \subset M_1$ and $h' \subset M_q$ gives $\psi_M[h] = h'$. From this it follows that $\psi_M[c_1] = c_q'$ since it cannot be c_q, and $\psi_M[c_1'] = c_q$. Now let $\overline{M_k}$ contain c_k and c_k' and suppose that both c_k and c_k' can be made to correspond to c_1. Then take $h \subset M_q$ and $h' \subset M_k$ and φ_M so that $\varphi_M[h] = h'$. Then $\varphi_M[c_q]$ has to be either c_k or c_k' but then c_q can be made to correspond to c_1 which is a contradiction. It follows that in this case one c-point of a given M corresponds to c_1 and the other cannot.

III.

Every c-point can be made to correspond to c_1.

6.1. Suppose all the arcs of M which join c_i to c_j, which arcs are M_i of 5.8.2, together with the two points c_i and c_j be called M_{ij} Then if π_M be taken to make two homogeneous points of M_{ij} correspond, it follows easily that $\pi_M[c_i] = c_i$ and $\pi_M[c_j] = c_j$ or $\pi_M[c_i] = c_j$ and $\pi_M[c_j] = c_i$. Hence $\pi_M[M_{ij}] = M_{ij}$ and this set is a continuous curve except for the two c points or else the set is vacuous. Also any M_{ij} has exactly the same number of arcs from c_i to c_j as any other M_{uv} has from c_u to c_v. This results from the fact that M_{ij} can be made to correspond to M_{uv} by some π_M, which is easily shown.

6.2. **Definition.** The set obtained by replacing each set M_{ij} by a single arc joining c_i to c_j will be called the ***skeleton set*** of M and will be denoted by m.

Suppose one of the c-points becomes a homogeneous point in m then, since every c-point can be made to correspond to any other [since both can be made to correspond to c]. every c-point must be homogeneous. Then m is a simple closed curve. The set M must therefore be a number of sets [equal to the number of c-points] homogeneous except for two points and these sets are joined end to end so as to form a ring.

If the c-points do not become homogeneous, the set m will be a set which is homogeneous except for the same n c-points which were non-homogeneous in M.

The set m consists of arcs joining the n c-points, hence m determines a finite number of complementary domains and each homogeneous point is on the boundary of just two of these.

6.3. *The boundaries of two complementary domains of m have at most one arc in common unless their common part is totally disconnected.*

Proof. Let $F[\mathfrak{G}_1].F[\mathfrak{G}_2]$ contain the arc c_1xc_2 where c_1xc_2 is not a proper subset of any other arc from c_1 to c_2 in $F[\mathfrak{G}_1].F[\mathfrak{G}_2]$. Case I). If $S-m=\mathfrak{G}_1+\mathfrak{G}_2$ then as m will have no cut points and is therefore a simple closed curve, M must be a set of the type described in 6.2.

Case II). If $S-m\neq\mathfrak{G}_1+\mathfrak{G}_2$ then $\mathfrak{G}_1$ and $\mathfrak{G}_2$ may both be regarded as bounded domains for if one is unbounded the set may be inverted about a point of $\mathfrak{G}_3$, a domain different from $\mathfrak{G}_1$ or $\mathfrak{G}_2$, with $F[\mathfrak{G}_3]$ for the invariant curve so that $\mathfrak{G}_3$ becomes the new unbounded domain and $\mathfrak{G}_1$ and $\mathfrak{G}_2$ become bounded domains.

Suppose $F[\mathfrak{G}_1]\ F[\mathfrak{G}_2]-c_1xc_2\neq 0$. This set is either totally disconnected or contains a maximal arc whose end points are c points. So in every case this set contains a c-point c_3.

Now let us suppose that $\mathfrak{G}_1$ and $\mathfrak{G}_2$ were chosen so that inside the outer boundary of $\mathfrak{G}_1+\mathfrak{G}_2$ there is no other pair of domains which have more than an arc in common to their boundaries. This choice is possible since there will have to be a new c-point inside every boundary of two domains of the required type and since the total number of c-points is finite there must be a last pair of this type.

The set $m-[x+c_3]$ is not connected. For there exist arcs xy_1c_3, xy_2c_3 lying except for their end points entirely in $\mathfrak{G}_1$ and $\mathfrak{G}_2$ respectively. Now $m-[x+c_3]$ is partly within and partly without the simple closed curve $xy_1c_3y_2x$. Hence this set is disconnected.

Take $h\subset c_1xc_2$ and $h'\subset c_2zc_3$ where z is inside the outer boundary of $\mathfrak{G}_1+\mathfrak{G}_2$. Let π_M be a transformation which gives $\pi_M[h]=h'$. Now $\pi_M[c_1xc_2]$ must be an arc $c_1'x'c_2'$ which is contained in c_2zc_3. Also $c_1'\neq c_3'$, $c_2'\neq c_3'$ since c_3 is not on c_1xc_2 Then $c_1'x'c_2'$ is the common arc of $F[\mathfrak{G}_1]$ and some other boundary of a complementary domain. $F[\mathfrak{G}_3]$ say. The set $m-c_3'$ is connected. There are now two possibilities.

Case I). $F[\mathfrak{G}_3]$ does not contain c_3'. Then $F[\mathfrak{G}_3]-h'$ is a connected set, since the removal of a single point does not disconnect a simple closed curve. Also $m-c_3'-h'$ is connected; for $m-c_3'$ is connected and if $m-c_3'-h'$ were disconnected between the points q and r, consider an arc from q to r which contains h' and to it add the simple closed curve $F[\mathfrak{G}_3]$; then since $m-c_3'\supset F[\mathfrak{G}_3]$ it follows that q and r can still be joined since $F[\mathfrak{G}_3]-h'$ is not disconnected. But then $m-c_3-h$ is disconnected so π_M is not a continuous correspondence, which is a contradiction.

Case II). $F[\mathfrak{G}_3]\supset c_3'$. Then there is a c-point $[c_1'$ or $c_2']$ between c_3' and h' on either arc $c_3'uh'$ or $c_3'vh'$ of $F[\mathfrak{G}_3]$ (since c_1 or c_2 is between c_3 and h going either way around $F[\mathfrak{G}_1]$). Since $F[\mathfrak{G}_1]$ is a simple closed curve, c_1' can be joined to c_2' by an arc G of $F[\mathfrak{G}_1]$ not containing c_3' or h', for c_3' is not on $c_1'x'c_2'$ and from the choice of $\mathfrak{G}_1$ and $\mathfrak{G}_2$, $F[\mathfrak{G}_1].F[\mathfrak{G}_3]$ consists of $c_1'x'c_2'$ only. Then if p and q are points of $m-c_3'-h'$ which cannot be joined by an arc of this set, take the arc L which joins p to q in $m-c_3'$ and add to it the arc G of $F[\mathfrak{G}_1]$ which joins c_1' to c_2' and does not contain h'. Now $L\supset h'$ and therefore $L\supset c_1'h'c_2'$. Hence $L+G$ is connected but $L+G$ is not disconnected by the removal of h' so p and q can be joined in $m-c_3'-h'$ which was supposed false. Hence $m-c_3'-h'$ is connected. Then the same contradiction follows as in case I.

6.4. *If the set of points common to the boundaries of two complementary domains is totally disconnected in m, it consists of only one point.*

Proof. Since there are only a finite number of domains in $S-m$, it is possible to take $\mathfrak{G}_1$ and $\mathfrak{G}_2$ such that no pair of domains other than $\mathfrak{G}_1$ and $\mathfrak{G}_2$ inside the outer boundary of $\mathfrak{G}_1+\mathfrak{G}_2$ will have more than one point or an arc in the common part of their boundaries. If this were not true it would be possible to find an infinite number of c-points in the set. Let $\mathfrak{G}_1$ and $\mathfrak{G}_2$ have c_1 and c_2 as the two c-points which are common to their boundaries such that on the arc of $F[\mathfrak{G}_1]$ which is within the outer boundary of $\mathfrak{G}_1+\mathfrak{G}_2$ there is no common c-point between c_1 and c_2 Let x be a point on $F[\mathfrak{G}_1]$ such that x is not on the outer boundary of $\mathfrak{G}_1+\mathfrak{G}_2$, and let y be a point on $F[\mathfrak{G}_2]$ with the same property. There are now three cases to be discussed: I) c_1xc_2 and c_1yc_2 have no c-point other than c_1 and c_2 on them, II) one of these arcs has a c-point but the other has not, III) both have c points [one or more].

Case I). Take $\mathfrak{Q}_3$, $[\mathfrak{Q}_3 \neq \mathfrak{Q}_1]$ which has $c_1 x c_2$ as part of its boundary. $F[\mathfrak{Q}_3] \supset c_3$, some c-point since $F[\mathfrak{Q}_3]$ is in m and all domains of m have at least three c-points on their boundaries. Take $\mathfrak{Q}_4 [\mathfrak{Q}_4 \neq \mathfrak{Q}_2]$ which has $c_1 y c_2$ for part of its boundary. Now $c_1 c_3 c_2$ on $F[\mathfrak{Q}_3]$ is not part of $F[\mathfrak{Q}_4]$ because c_3 would be homogeneous if it were. If $c_3 \subset F[\mathfrak{Q}_4]$, $F[\mathfrak{Q}_4].F[\mathfrak{Q}_3]$ cannot be an arc because $F[\mathfrak{Q}_4].F[\mathfrak{Q}_3] \supset c_1$ and c_2 and the arc would have to be from c_1 to c_2, which was just shown to be impossible. Then $F[\mathfrak{Q}_3]$ and $F[\mathfrak{Q}_4]$ repeat the situation with which $\mathfrak{Q}_1$ and $\mathfrak{Q}_2$ occured, two domains with at least two points common to their boundaries but no arc common. But this contradicts the way in which $\mathfrak{Q}_1$ and $\mathfrak{Q}_2$ were chosen. So this case is impossible.

Case II). Let $c_1 x c_2$ have a c-point c_3. Suppose the order $c_1 x c_3 c_2$, while $c_1 y c_2$ has no c-point. Then $\mathfrak{Q}_4$ exists with $c_1 y c_2$ as part of its boundary and there must be another arc from c_1 to c_2 on $F[\mathfrak{Q}_4]$. Let this be $c_1 p c_2$. The arc $c_1 p c_2$ must have a c-point c_4 on it since $F[\mathfrak{Q}_4]$ has at least three c-points This case also leads to a contradiction since $\mathfrak{Q}_1$ and $\mathfrak{Q}_4$ exist with c_1 and c_2 their only common points.

Case III). $c_1 x c_2 \supset c_3$; $c_1 y c_2 \supset c_4$. Let the simple closed curve $c_1 c_3 c_2 c_4$ be called 0.

Suppose there is no c-point within 0. Let c_3 denote the first c-point on the arc $c_1 c_2$, common to $F[\mathfrak{Q}_1]$ and 0. Now there must be a complementary domain $\mathfrak{Q}_3$, a proper subset of the interior 0, whose boundary $F[\mathfrak{Q}_3]$ is a simple closed curve containing the subarc $c_1 x c_3$, common to 0 and $F[\mathfrak{Q}]$. Let $c_1 y c_3$ be that arc of $F[\mathfrak{Q}_3]$ which does not contain x. Now there is no connected subset of $\widehat{c_3 y c_1}$, which has c_3 as a limit point and is common to $\widehat{c_3 y c_1}$ of $F[\mathfrak{Q}_3]$ and the arc $c_3 c_2$ of $F[\mathfrak{Q}_1]$ which does not contain c_1; if this were not the case it would be impossible for another arc of M to leave c_3, contrary to the fact that every c-point must have at least four distinct arcs leaving it. Therefore $c_3 y c_1$ of $F[\mathfrak{Q}_3]$ can have no other point other than c_3 in common with the arc $c_3 c_2$ which is common to $F[\mathfrak{Q}_1]$ and 0, otherwise we would contradict 6.3. Now there must be on $\widehat{c_3 y c_1}$ another c-point, c_4, because every domain must have at least three c-points on its boundary. As c_4 is not within 0 it must therefore be on the arc $c_1 k c_2$, common to 0 and $F[\mathfrak{Q}_2]$. Now all points of the arc of $F[\mathfrak{Q}_2]$ from c_1 to c_4 which

does not contain c_2 must belong to $F[\mathfrak{Q}_3]$ otherwise $\mathfrak{Q}_3$ and $\mathfrak{Q}_2$ would have a disconnected set common to their boundaries, contrary to the choice of $\mathfrak{Q}_1$ and $\mathfrak{Q}_2$. Within the interior of $c_3 c_2 c_4 c_3$ there must be another arc leaving c_3 for so far we have accounted for only three arcs leaving c_3. Hence there is a domain $\mathfrak{Q}_4$, a proper subset of the interior of $c_3 c_2 c_4 c_3$, which has as its boundary a simple closed curve J_4 having the arc of $F[\mathfrak{Q}_3]$ from c_3 to c_4, which does not contain c_1, as part of its boundary. Let $c_3 \bar{z} c_4$ denote that arc from c_3 to c_4 of J_4 which is common to $\mathfrak{Q}_3$ and $\mathfrak{Q}_4$ while $c_3 w c_4$ is the other arc of $F[\mathfrak{Q}_4]$ from c_3 to c_4. Now as above there is on $\widehat{c_3 w c_4}$ no connected subset which has c_3 as a limit point and is common to $\widehat{c_3 w c_4}$ and the arc $c_3 h c_2$ of $F[\mathfrak{Q}_1]$ from c_3 to c_2 which does not contain c_1. It follows that $c_3 w c_4$ can have no point in common with $c_3 h c_2$ otherwise we would contradict our choice of $\mathfrak{Q}_1$ and $\mathfrak{Q}_2$. Now there must be another c-point, c_5, on $\widehat{c_3 w c_4}$ for the boundary of any complementary domain has at least three c-points and as no c-point is interior to 0. c_5 must be on the arc $c_2 v c_4$ of $F[\mathfrak{Q}_2]$ which does not contain c_1 and must be different from c_2. Now the subarc $c_5 c_4$ of $c_3 w c_4$ of $F[\mathfrak{Q}_4]$ must coincide with the arc $c_4 c_5$ of $c_2 v c_4$, otherwise $F[\mathfrak{Q}_2]$ and $F[\mathfrak{Q}_4]$ contradict our choice of $\mathfrak{Q}_1$ and $\mathfrak{Q}_2$. But now it becomes impossible for a fourth arc to leave c_4 and hence we are led to a contradiction if no point of the non-homogeneous set is within 0.

If q' is the c-point inside 0 there exists a π_M which gives $\pi_M[c_1] = c_1'$. In this case $\pi_M[c_2]$ must be inside or on 0. For if not $m - c_1'$ is connected and $m - c_1' - c_2'$ has to be connected because any arc from c_1' to c_2' has to have a point of 0 on it and points on one of these arcs can be joined to points on any other by passing around 0. But $m - c_1 - c_2$ is disconnected since $\mathfrak{Q}_1$ and $\mathfrak{Q}_2$ enclose 0 and the exterior of 0 is connected to the interior of 0 only by points c_1 and c_2. Hence $\pi_M[m - c_1 - c_2] = m - c_1' - c_2'$ must be connected. But it was shown that this was false, therefore c_2' is inside or on 0. [It may happen that c_2' is c_1 or c_2].

Now c_1 and c_2 cannot have an arc joining them in m which contains no c-points except c_1 and c_2. For if there were such an arc let h be a homogeneous point of it and let h' be a homogeneous point of the part of $c_1 c_3$ from c_1 to the first c-point c_p on this arc [c_p is different from c_2 since c_3 is already on this arc]. Then if

π_M is so chosen that $\pi_M[h] = h'$, $\pi_M[c_1]$ is either c_1 or c_p and $\pi_M[c_2]$ is the other one of these two. Now $m - [c_1 + c_2]$ falls into at least three separated sets, but $m - [c_1 + c_p]$ can fall into at most two. For suppose $m - [c_1 + c_p] = M_1 + M_2 + M_3 + \dots$ where M_1 contains the arc $c_1 c_p$ and M_2 contains $F[\mathfrak{Q}_1] - c_1 c_p$. It follows that the arc $\widehat{c_2 c_4 c_1}$ of $F[\mathfrak{Q}_2]\cdot 0$ is also in M_2. Then $M_3 + M_4 + \dots$ are inside 0, for, as c_1 does not disconnect, all points outside 0 can be joined to $F[\mathfrak{Q}_1]$ and are therefore in M_2. Now M_3 is in 0 and does not have any points in common with 0 except c_1 and c_p. The arc $c_1 h c_p$ of $F[\mathfrak{Q}_1]$ must be on the boundary of some complementary domain, $\mathfrak{Q}_3$, which is a proper subset of the interior of 0. Let $c_p z c_1$ be the arc of a simple closed curve forming $F[\mathfrak{Q}_3]$ which does not contain $c_1 h c_p$. There can be on $\widehat{c_p z c_1}$ no connected set having c_p as a limit point which is common to $\widehat{c_p z c_1}$ and the arc $c_p \bar{v} c_2$ of $F[\mathfrak{Q}_1]$ from c_p to c_2 which does not contain c_1; otherwise it would be impossible for four arcs to leave c_p. Then $\widehat{c_p z c_1}$ can have no point in common with the arc of $F[\mathfrak{Q}_1]$ from c_p to c_2 which does not contain c_1, otherwise we would contradict our choice of $\mathfrak{Q}_1$ and $\mathfrak{Q}_2$. Suppose $c_p z c_1$ as a point other than c_1 in common with the arc $c_1 c_4 c_2$ of $F[\mathfrak{Q}_2]\,.\,0$; this point will evidently be a non-homogenous point and will be denoted by c_5. Then as $F[\mathfrak{Q}_3]$ and $F[\mathfrak{Q}_2]$ will have c_5 and c_1 in common, from our choice of $\mathfrak{Q}_1$ and $\mathfrak{Q}_2$ it will follow that the arc $c_p z c_1$ must follow $c_2 c_4 c_1$ of $F[\mathfrak{Q}_2]$ from c_5 to c_1. But then it follows that all points of $\widehat{c_p z c_1}$ belong to M_2 and the set M_3 must lie wholly in the interior of $c_p c_5 c_2 c_p$. But then c_1 is not a limit point of M_3 which is contrary to the fact that c_1 and c_p together cut M while neither alone cuts. Hence $\widehat{c_p z c_1}$ must not meet $c_1 c_4 c_2$ of $F[\mathfrak{Q}_2]$. Then there must be a point c_6 on $c_p z c_1$. It may easily be proved that $\widehat{c_1 c_6 c_p}$ is not in either M_1 or M_2. Let us suppose that it is in M_3. Now consider the simple closed curve $c_p c_6 c_1 c_4 c_2 c_p$. The arc $c_p c_6 c_1$ less c_p and c_1 is in M_3 while $\widehat{c_p c_2 c_4 c_1}$ is in M_2. Hence as c_1 and c_p cut M it follows easily that there is a subdomain of the interior of $c_p c_6 c_1 c_4 c_2 c_p$ having c_1 and c_p as boundary points[1]). Then this domain and $\mathfrak{Q}_1$ contradict

[1]) A proof of this fact may be obtained very easily with the use of an unpublished paper of G. T. Whyburn. For statement of Whyburn's theorem see Bull. Amer. Math. Soc. vol. 33 [1927] p. 388.

the choice of $\mathfrak{Q}_1$ and $\mathfrak{Q}_2$. Hence the supposition of a set M_3 leads to a contradiction. Then $M - c_1 - c_p = M_1 + M_2$ and π_M cannot be continuous.

It follows that c_1' and c_2' cannot lie on an arc which has no c-points between c_1' and c_2'. But inside 0 all domains have either one arc or one point which is common to their boundaries. Let c_1' be on the boundary of $\mathfrak{Q}_i$. Then those domains which have an arc or point in common with $F[\mathfrak{Q}_i]$, will have for their outer boundary a simple closed curve J_1. Now if c_2' is on $F[\mathfrak{Q}_i]$ there is a c-point on any arc from c_1' to c_2' so that every point of such an arc can be joined to J_1, hence $m - c_1' - c_2'$ is connected. If c_2' is not on $F[\mathfrak{Q}_i]$ then every point of $m - c_1' - c_2'$ can still be joined to J_1 for $m - c_1'$ does not disconnect $F[\mathfrak{Q}_i]$ and if c_2' is on an arc from $F[\mathfrak{Q}_i]$ to J_1 one part of this arc goes to $F[\mathfrak{Q}_i]$ and the other goes to J_1 since c_1' and c_2' are not extremities of an arc without c-points between them. If c_2' is on J_1 then $J_1 - c_2'$ is still connected and the other arcs can be joined to J_1 by going around the $F[\mathfrak{Q}_i]$ of which they are parts. If c_2' is outside J_1 then every arc from c_1' to c_2' passes through J_1 and hence m is still connected.

Now $m - [c_1 + c_2]$ is disconnected but $m - [c_1' + c_2']$ is not, therefore π_M is not a continuous correspondence which is contrary to hypothesis. From this contradiction the theorem follows.

6.5. *If k be the number of arcs of m which meet at a c-point, k is not greater than 5.*

Proof. Suppose first that each domain of m has only 3 c-points on its boundary. Start with one c-point from which go k arcs. These arcs reach k distinct c-points because if two arcs end in the same c-point there would be either a domain with only two c-points on its boundary, which was ruled out in the definition of m, or a simple closed curve with only two c-points on it but other c-points inside it, which was shown impossible in 6.4. Now each adjacent pair of the k arcs must lie on a complementary domain and the other arcs of these domains form a simple closed curve c_1. Now c_1 has exactly k c-points since each domain has 3 c points on its boundary. From each of these k c-points go k-3 new arcs lying outside c_1. Let c_i and c_j be adjacent c-points on c_1. The first arc from c_i on the side of c_j and the first arc from c_j on the side of c_i must unite to form a new domain or else there would be a domain with more than 3 c-points on its boundary. The c-point where these arcs meet will be spoken of as a double c-point. For each c-point on c_1 there can be only one double c-point for let c_i and c_j give c_{k1} as their first double c-point and let two other arcs, one from c_i and one from c_j give a second double point c_{k2}. Then there can be no c points on $c_i c_{k1}$, $c_i c_{k2}$, $c_j c_{k1}$ or $c_j c_{k2}$: but from c_{k1} go $k-2$ arcs which cannot go into the domain bounded by $c_i c_j c_{k1}$ and therefore must go to c_{k2} but if k is greater than 4 there will result domains with only two c-points on their boundaries, which is impossible in m. Suppose $k>4$ so that there cannot be more than one double c-point coming from a c-point on c_1. Then from each c point on c_1 come $k-5$ new arcs which end in single c-points. Now form the outer boundary of all the domains formed with these arcs as parts of their boundaries; which boundary will be a simple closed curve c_2, having k double c-points and $k[k-5]$ single c-points. By continuing this process there will arise a curve c_i. Let n_i be the number of double c-points on c_i and m_i the number of single c-points on c_i. The total number of arcs leaving c_i is $n_i\,[k-4]+m_i\,[k-3]$. Let n_{i+1} be the number of double c-points on c_{i+1} and m_{i+1} the number of single c-points. Then $n_{i+1}=m_i+n_i$. Hence on c_{i+1} are $n_i[k-4]+m_i[k-3]-[m_i+n_i]$ or $n_i[k-5]+m_i[k-4]$ c-points; or c_{i+1} contains $n_i[k-6]+m_i[k-5]$ *more* c-points than c_i. Hence if $k>5$ there will be an increasing number of c-points and m will have more than n c-points, contrary to hypothesis.

Now suppose any domain has more than three c-points. Suppose this adds a c-point to c_i [if this happens first in c_1, start there]. From this c-point go $k-2$ arcs which must end in c-points. Now only two of these c-points can be points of the previously constructed set, if $k>4$, for only the two outer arcs can unite with arcs from other c-points on c_i. Hence there must be at least $k-4$ new c-points on c_{i+1} due to the existence of this c-point. Hence if the number of c-points on the boundary of some of the domains is greater than 3 there will be more c-points on each of the curves c_i, c_{i+1}, etc. and therefore if $k>5$ the number of c-points increases more rapidly than before. So k cannot be greater than 5.

6.6 $m - F[\mathfrak{Q}_i]$ *is connected for any* i.

Proof. In the interior of $\mathfrak{Q}_i + F[\mathfrak{Q}_i]$ surround this set by a simple closed curve Z such that every point of Z is at a distance less than e from some point of $F[\mathfrak{Q}_i]$ [1]). Take e so small that no c-point in the exterior of $F[\mathfrak{Q}_i]$ is on Z. Let $\mathfrak{Q}_{e1}$, $\mathfrak{Q}_{e2}, \ldots$ be the set of complementary domains of m which have arcs of their boundaries common to $F[\mathfrak{Q}]$; and let $\mathfrak{Q}_{m1}$, $\mathfrak{Q}_{m2} \ldots$ be the set which have points of their boundaries common to $F[\mathfrak{Q}_i]$ but no arcs. These domains are bounded by simple closed curves which lie partly inside and partly outside Z on account of the choice of e. Let $\mathfrak{Q}_{ei1}$, $\mathfrak{Q}_{mi1}$, $\mathfrak{Q}_{mi2}, \ldots$ $\mathfrak{Q}_{min}$, $\mathfrak{Q}_{ei2}$ be the domains which have c_i on their boundaries where c_i is a c-point of $F[\mathfrak{Q}]$. The arc common to $F[\mathfrak{Q}_{ei1}]$ and $F[\mathfrak{Q}_{mi1}]$ will be called $c_{i1}\,c_i$. etc That common to $F[\mathfrak{Q}_{mi1}]$ and $F[\mathfrak{Q}_{mi2}]$ will be called $c_{i2}\,c_i$, etc. Then Z cuts all these arcs between c_{in} and c_i. So any two points of these domains outside of Z may be joined by an arc lying entirely outside Z By allowing e to approach 0, Z will become $F[\mathfrak{Q}_i]$ and therefore $m - F[\mathfrak{Q}_i]$ is a connected set.

[1]) **Zoretti**: *Sur les fonctions analytiques uniformes*. Journ. des Math. [1905] pp. 9—11.

6.7. *If* $F[\mathfrak{Q}_1]$ *is the boundary of* $\mathfrak{Q}_1$, *a complementary domain of* M; *then* $\pi_m\{F[\mathfrak{Q}_1]\}$ *must be a complementary domain boundary.*

Proof. If $\pi_m\{F[\mathfrak{Q}_1]\} = C$, not the boundary of a complementary domain, we have $m - C$ is disconnected, but $m - F[\mathfrak{Q}_1]$ is connected so π_m fails to be continuous, which is false.

6.8. *Either every complementary domain boundary can be made to correspond to any one boundary by some* π_m; *or the boundaries fall into two classes such that the members of a class may correspond to each other by some* π_m *but not to members of the other class, and the domains which have an arc of their boundaries in common with one domain belong to the class of which that domain is not a member.*

Proof. If every $F[\mathfrak{Q}_i]$ cannot be made to correspond to $F[\mathfrak{Q}_1]$, let $F[\mathfrak{Q}_x]$ be one which does not. Then take h' on $F[\mathfrak{Q}_x]$ and h on $F[\mathfrak{Q}_1]$. Now we get π_m so that $\pi_m[h] = h'$. Now the arc $\supset h'$ is on only two boundaries of complementary domains; $F[\mathfrak{Q}_x]$ is one and let the other be called $F[\mathfrak{Q}_y]$. Since $F[\mathfrak{Q}_1]$ does not correspond to $F[\mathfrak{Q}_x]$ we must have $\pi_m\{F[\mathfrak{Q}_1] = F[\mathfrak{Q}_y]\}$. Then take h'' on the next arc of $F[\mathfrak{Q}_x]$ and that must cause $F[\mathfrak{Q}_{y'}]$ [the other boundary $\supset h''$] to correspond to $F[\mathfrak{Q}_1]$. Hence all the bounding domains which have an arc in common with $F[\mathfrak{Q}_x]$ must correspond to $F[\mathfrak{Q}_1]$. Also the domains which bound $F[\mathfrak{Q}_y]$ say, must all correspond to $F[\mathfrak{Q}_x]$ and by this process we can exhaust the whole set with only the two sets since if k is any point of m we can find $\pi_m[h] = k$ and one domain which has k on its boundary must correspond to $F[\mathfrak{Q}_i]$ and the other to the other domain which belongs to the second set.

7. The forms of the set m will now be investigated in the case where every domain can be made to correspond.

Let p be the number of arcs from each c-point. $p \leqslant 5$ by 6.5. Let k be the number of c-points on each domain.

Choose a particular $F[\mathfrak{Q}_i]$ which is divided into k arcs by the k c points on it. Each of these arcs is part of the boundary of some other complementary domain. Such domains will be said to be of type one. Then there are boundaries which have only one c-point in common with $F[\mathfrak{Q}_i]$. These will be said to be of type two. By 6.3 and 6.4 there are no domains other than these two types.

The outer boundary of all of these domains is a simple closed curve C_1 since $m - F[\mathfrak{Q}_i]$ is connected. On C_1 each domain of type one has $k-2$ c-points and each of type two has $k-1$ but on account of overlapping at the ends count each domain of type one as contributing $k-3$ and each of type two as contributing $k-2$. If p arcs leave each point there must be $p-3$ domains of type two at each c-point on C_1, hence the total number of c-points on C_1 must be $k[k-3]+k[p-3][k-2]$. Now consider the domains which have an arc or point in common with C_1 and lie outside C_1. Their outer boundary will be a simple closed curve C_2. Continue this until a simple closed curve C_i is reached.

If $I[C_i]$ denote the interior of C_i, some c-points of C_i will have 3 arcs of $[C_i+I[C_i]].m$ leaving them and others will have only two.

Let

$m_i =$ the number of points with 3 arcs of $\{C_i+I[C_i]\}.m$

$n_i =$ " " " " " 2 " " "

$t_i = m_i + n_i$ the total number of c-points on C_i.

To calculate t_{i+1} notice that each of the m_i+n_i points gives one domain of type one, each of the m_i gives $p-4$ of type two and each of the n_i gives $p-3$ of type two, hence:

$$t_{i+1} = [m_i+n_i][k-3]+[m_i[p-4]+n_i[p-3]].[k-2].$$

also

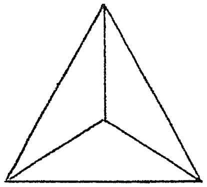
$$m_{i+1} = m_i[p-3]+n_i[p-2];$$

since each of the arcs leaving C_i ends in a c-point of the required kind on C_{i+1}.

Let

$$d_i = t_{i+1}-t_i,$$
$$= [m_i+n_i][k-4]+[m_i[p-4]+n_i[p-3]].[k-2]$$
$$= [m_i+n_i]\{[k-4]+[p-4][k-2]\}+n_i[k-2].$$

Now $n_{i+1} = t_{i+1}-m_{i+1}$ so $n_{i+1}-m_{i+1} = t_{i+1}-2m_{i+1}$.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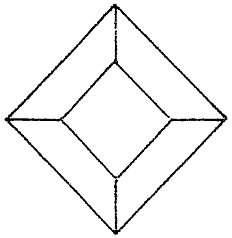
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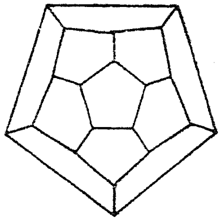
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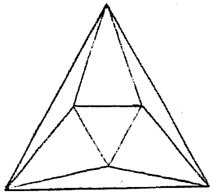
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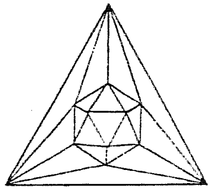
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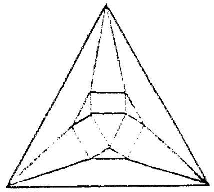
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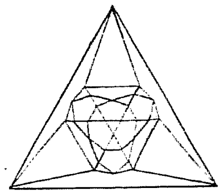
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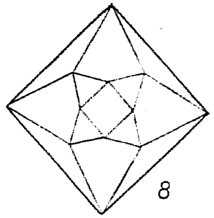
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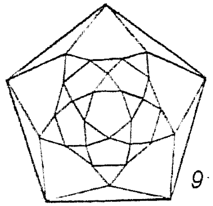
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Therefore

$$n_{i+1}-m_{i+1} = [m_i+n_i][k-3]+m_i\{[p-4][k-2]-2[p-3]\}+ n_i\{[p-3][k-2]-2[p-2]\}.$$

Case I. $p=3$. Then

$$d_i = n_i[k-4]-2m_i;$$
$$n_{i+1}-m_{i+1} = m_i[2k-5]+n_i[k-5].$$

Therefore if k is greater than 5, n_{i+1} is greater than m_{i+1} and d_i is always positive so the number of c-points must increase without limit, which is contrary to the original assumption that the set had only n points of n non-homogeneity. The only possible sets are those for which k is 3, 4, or 5. These sets are shown in figures 1, 2 and 3.

It should be pointed out that $k=5$ is not sufficient to make t_i increase because if $k=5$, $n_{i+1}-m_{i+1}$ may vanish due to $n_i=0$, and this indicates a stopping of the set which actually happens.

Case II. $p=4$.

$$d_i=2n_i[k-3]+m_i[k-4].$$

If $k>4$, d_i is positive. If $k=4$, $n_{i+1}=[m_i+n_i][1]+2n_i$, and there are no minus signs so n_{i+1} will not be 0 and therefore d_i is positive. Hence the only possible case is $k=3$. This set is shown in fig. 4.

Case III. $p=5$.

$$d_i=2[m_i+n_i][k-3]+n_i[k-2].$$

If k is greater than 3, t_i must increase but if $k=3$ and n_i vanished this case could occur and it actually does exist as shown in fig. 5.

7 2. Consider the next case where m contains two classes of complementary domains.

Let

$p=$ the number of arcs from each c-point.
$k=$ „ „ „ c-points on each domain of class I.
$e=$ „ „ „ „ „ „ „ „ „ II.

Then $k\neq e$, and $p\leqslant 5$, $k\geqslant 3$, $e\geqslant 3$.

Build up a simple closed curve C_{i+1} where C_{i+1} is the outer boundary of those domains of class I which have either a point or an arc of their boundary common to the boundaries of domains of class II, which in turn have either a point or an arc in common with C_i. Take a domain of class I as a start and this definition serves as an induction definition to build up the curves. It has previously been shown that domains which have an arc of their boundary common to a domain of class I must be of class II and vice versa [6.8] Hence there must be an even number of domains and hence of arcs meeting at each c-point. It follows that $p=4$.

Let

$m_i=$ no. of c-points on C_i which are of order 4 in $m(C_i+I[C_i])$
$n_i=$ „ „ „ „ „ „ „ „ „ 2 „ „

$D_i=$ the outer boundary of the domains of class II which border on C_i.
$p_i=$ no. of c-points on D_i which are of order 4 in $m(D_i+I[D_i])$.
$q_i=$ „ „ „ „ „ „ „ „ „ 2 „ „
$u_i=$ total number of c-points on D_i.
$t_i=$ „ „ „ „ „ C_i.

Now each point of order 2 in $m(C_i+I[C_i])$ on C_i will give rise to one point of order 4 in $m(D_i+I[D_i])$ on D_i. Therefore $q_i=n_i$.

Then

$$u_i=[n_i-m_i][e-1]+m_i[e-2]=n_i[e-1]-m_i.$$

$$p_i=u_i-q_i=u_i-n_i=n_i[e-2]-m_i.$$

So

$$t_{i+1}=[p_i-q_i][k-1]-q_i[k-2]=p_i[k-1]-q_i.$$
$$=(n_i[e-2]-m_i)[k-1]-n_i=n_i([e-2][k-1]-1)-m_i[k-1].$$
$$m_{i+1}=p_i=n_i(e-2)-m_i.$$
$$n_{i+1}=t_{i+1}-m_{i+1}=n_i([e-2][k-2]-1)-m_i[k-2].$$

Now

$$n_{i+1}-m_{i+1}=n_i([e-2][k-2]-1-[e-2])-m_i[k-1],$$

which can be reduced to

$$=[n_i-m_i][k-1]+n_i(k[e-3]-3[e-2]).$$

Now for $i=1$, $n_i-m_i=k$, hence this formula shows by induction that $n_{i+1}>m_{i+1}$ if $k>3$, $e>3$ because the $(k[e-3]-3[e-2])$ is $+$. Therefore t_{i+1} is always positive and the set cannot have a finite number of c-points.

Consider the cases where k and e are not both greater than 3.

a. $k=3$, $e=4$.

$n_1=3$ $n_2=3$
$m_1=0$ $m_2=6$ $m_3=0$.

Hence this set stops. It is shown in fig. 6.

b. $k=3$, $e=5$.

$n_1=3$ $n_2=6$ $n_3=3$
$m_1=0$ $m_2=9$ $m_3=9$ $m_4=0$

Hence this set stops. It is shown in fig. 7.

c. $k=4$, $e=3$.

$$n_1=4 \qquad n_2=4$$
$$m_1=0 \qquad m_2=4 \qquad m_3=0$$

This set is shown in fig. 8.

d. $k=5$, $e=3$.

$$n_1=5 \qquad n_2=10 \qquad n_3=5$$
$$m_1=0 \qquad m_2=5 \qquad m_3=5 \qquad m_4=0$$

This set is shown in fig. 9.

e. $k=6$, $e=3$.

$$t_{i+1}=n_i[k-2]-m_i[k-1]=[n_i-m_i][k-2]-m_i.$$

Also if $i-1$ is inserted in the formula in place of i, m_i and n_i can be found in terms of m_{i-1} and n_{i-1}. From the old formula for $n_{i+1}-m_{i+1}$ comes the following: $n_i-m_i=m_{i-1}.[2k-5]+ n_{i-1}.[k-5]$, also $m_i=n_{i-1}-m_{i-1}$.

Therefore $t_{i+1}=m_{i-1}[2k^2-9k+11]+n_{i-1}[k^2-7k+9]$,

If $k\geqslant 6$, $2k^2-9k+11\geqslant 19$, $k^2-7k+9\geqslant 3$. Hence t_{i+1} is always positive and the set cannot exist.

f. $k=3$, $e\geqslant 6$.

$$n_{i+1}-n_i=n_i[e-4]-m_i.$$

$$n_{i+2}-n_{i+1}=n_{i+1}[e-4]-m_{i+1}=n_i[e-3][e-4]-m_i[e-4]-n_i[e-2]+m$$

which reduces to

$$=[e-5][n_{i+1}-n_i]+n_i[e-6].$$

From the values of e it follows that the n_i's increase as an arithmetic progression or faster. Hence there are an infinite number of c-points in the set unless the value of m_i can become negative. By the same method a formula can be obtained for the $n_i s$ which shows that they also increase in an arithmetic progression or faster so there is no possibility of a negative value for an m_i. Hence there are no sets of this type.

8. It follows that M has either a simple closed curve or one of these possible m sets for its skeleton set, hence all the sets which are homogeneous except for n c-points and which have the additional property that every c-point can be made to correspond to one c-point have been classified.

9. By the process of inversion it is possible to extend the results of this paper to unbounded continuous curves. By inverting a two point, non-homogeneous, continuous curve in such a way that one of the non-homogeneous points is the center of inversion it can easily be shown that an unbounded continuous curve, homogeneous except for one point is a finite number $[\neq 2]$ of rays with only that point in common.

By inverting any of the sets found previously, other than the two-point set, it can be shown without difficulty that none of the new sets are homogeneous but if an unbounded set with more than one non-homogeneous point existed it would have to invert into a bounded curve which was homogeneous except for one more point than was non-homogeneous in the unbounded curve, and by inverting it again it would have to be possible to recover the original curve which contradicts the first statement. Therefore no other unbounded continuous curve homogeneous except for a finite number of points all of whose non-homogeneous points can be made to correspond, can exist.