Images of arcs — a nonseparable version of the Hahn–Mazurkiewicz theorem

by

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Abstract. Some conditions equivalent to the following one: "X is a Hausdorff space which is a continuous image of some arc", are given. The most important of them are: "X is a locally connected continuum which is a continuous image of some compact linearly ordered topological space" and "X is a continuum which can be approximated by finite dendrons".

1. Introduction. The classical Hahn–Mazurkiewicz theorem ([5], [11]) states that a Hausdorff space X is a continuous image of a metrizable arc (i.e., a space homeomorphic to $[0, 1]$) if and only if X is a locally connected metrizable continuum. The purpose of this paper is to characterize spaces which are continuous images of arcs.

There are two fine survey articles, [9] and [23], dealing with continuous images of arcs and compact linearly ordered spaces. Therefore in this paper we will not try to give any survey of the known results. Note only that there are some new papers, [15], [16], [21], [22] and [30], related to the topics which will be discussed below (however, there is a serious mistake in [15] — the set X constructed there on page 339 is not compact; see also Theorem 4.5 below).

First, we recall some basic definitions and facts and introduce some notation.

A *continuum* is a compact connected Hausdorff space. An *arc* is a continuum with exactly two non-cut points. Arcs are precisely compact connected linearly ordered topological spaces. Each separable arc is homeomorphic to the closed interval $[0, 1]$ of real numbers.

A continuum X is said to be a *dendron* if for any two distinct points p and q of X there is a point $r \in X$ so that p and q lie in distinct components of $X - \{r\}$. Metric dendrons are precisely dendrites. A point x of a dendron X is said to be an *end-point* of X provided x is a non-cut point of X. A dendron is said to be *finite* if it has only finitely many end-points. Dendrons are often called "trees", however, we reserve the word "tree" for quite different mathematical objects (see Chapter 3). An interesting survey of results on dendrons can be found in [12].

We assume that the reader is familiar with the theory of cyclic elements in metrizable locally connected continua (see for example [28]). The theory can be extended to the general case (see [29], [2] and [20]) and we will use immediate generalizations of some of its definitions and theorems without comment.

We often use nets in general topological spaces. All needed definitions and facts concerning nets can be found in [4].

If X is a topological space, $x, y \in X$, and there is exactly one arc in X with end-points x, y, then this arc will be denoted by $[x, y]$ (or $[x, y]_X$ if a confusion is possible). As usual, we write $]x, y] = [x, y]-\{x\}$ and $]x, y[= [x, y]-\{x, y\}$.

Let X be a continuum and J a family of finite dendrons contained in X. We say that J *approximates* X provided: (1) J is directed by inclusion, (2) $\bigcup J$ is dense in X, and (3) if U is an open covering of X, then there exists $T_U \in J$ such that if $T \in J$ and T' is a component of $T-T_U$, then T' is contained in some member of U (this notion was introduced by L. E. Ward in [26]). If, in addition, (2′) $\bigcup J = X$; then we say that J *strongly approximates* X.

Let X be a locally connected continuum and A a subset of X. We say that A is a *T-set* in X if A is closed and each component of $X-A$ has a two-point boundary.

Now we can state the main result of the paper:

1.1. THEOREM. *If X is a continuum, then the following conditions are equivalent:*

(i) *X can be strongly approximated by finite dendrons;*

(ii) *X can be approximated by finite dendrons;*

(iii) *X is a continuous image of some arc;*

(iv) *X is locally connected and is a continuous image of some compact linearly ordered topological space;*

(v) *X is locally connected and for each nondegenerate cyclic element Y of X the following conditions hold:*

(a) *if p, q and r are any points of Y, then there is a separable T-set E in Y such that $p, q, r \in E$,*

(b) *if $E \subset E' \subset Y$ and E' is separable, then E is also separable, and*

(c) *if E' is a continuous and monotone image of Y and E is a separable continuum in E' then E is metrizable;*

(vi) *X is locally connected and if Y is a nondegenerate cyclic element of X and $p, q, r \in Y$, then there is a metrizable T-set A in Y such that $p, q, r \in A$;*

(vii) *X is locally connected and if Y is a nondegenerate cyclic element of X, then there is a collection $\{A_1, A_2, \ldots\}$ of T-sets in Y such that for $n = 1, 2, \ldots$ the following conditions hold:*

(A) *$A_n \subset A_{n+1}$,*

(B) *if Z is a component of $Y-A_n$, then the set of all cut points of $\overline{Z}$ is contained in A_{n+1},*

(C) *if Z is a component of $Y-A_n$ and C is a nondegenerate cyclic element of $\overline{Z}$, then the set $C \cap A_{n+1}$ is metrizable and contains at least three points, and*

(D) *A_1 is metrizable.*

Note that Theorem 1.1 solves some old problems. Namely, the implication (iv) → (iii) answers affirmatively a question of S. Mardešić and P. Papić ([10]; see also [23], Problem 1, p. 97), and the implication (iii) → (ii) shows that a conjecture of L. E. Ward is true ([26], p. 371; see also [23], Problem 2, p. 100).

When the paper was ready, the author got to know that the implication (iv) → (iii) was somewhat earlier shown by L. B. Treybig ([22]; Treybig's proof differs from the one given here).

In the forthcoming papers we will use condition (vii) to obtain further properties of spaces which are continuous images of compact linearly ordered topological spaces.

The proof of Theorem 1.1 will be given in Chapter 6; it is preceded by various auxiliary results. In Chapter 2 we gather some facts on T-sets; most of them have recently been proved by L. B. Treybig in [21]. Chapter 3 contains some simple results on T-sets and open coverings of continua. Chapter 4 deals with special kinds of T-sets and Theorem 4.9 is a key to the proof of Theorem 1.1. The proof of the implication (vii) → (i) of Theorem 1.1 is prepared in Chapter 5.

2. *T*-sets.

2.1. LEMMA ([21], Theorem 6). *Let X be a locally connected continuum and A a T-set in X. There exists an upper semi-continuous decomposition G_A of X into closed sets so that if X_A denotes the quotient space and f_A: $X \to X_A$ is the quotient map, then:*

(i) *X_A is a locally connected continuum;*

(ii) *f_A restricted to A is a homeomorphism from A onto $f_A(A)$;*

(iii) *if Z is a component of $X_A-f_A(A)$, then Z, $\overline{Z}$ are homeomorphic to $]0, 1[$, $[0, 1]$, respectively, and so $f_A(A)$ is a T-set in X_A;*

(iv) *for each component Z of $X_A-f_A(A)$ there is a unique component Y_Z of $X-A$ such that $f_A(Y_Z) \subset \overline{Z}$; this gives a one-to-one and onto correspondence between the family of all components of $X_A-f_A(A)$ and the family of all components of $X-A$.*

The notation of Lemma 2.1 will be used in the sequel without comment.

2.2. LEMMA ([21], Theorem 8). *Let X be a locally connected continuum without cut points and let $\{A_s\colon s \in S\}$ be a family of T-subsets of X, indexed by a well-ordered set S which has no last element, such that $A_s \subset A_t$ if $s < t$. Put $A = \overline{\bigcup_{s\in S} A_s}$, and for each $s \in S$ let Y_s be a component of $X-A_s$ such that $Y_t \subset Y_s$ if $s, t \in S$ and $s < t$. Then the sets $\mathrm{bd}(Y_s)$, $s \in S$, can be labelled $\{a_s, b_s\}$, where there exist points $a, b \in A$ such that the nets $\{a_s\colon s \in S\}$, $\{b_s\colon s \in S\}$ converge to a, b, respectively. Moreover, either $\bigcap_{s\in S} \overline{Y}_s = \{a\} = \{b\}$ or $\bigcap_{s\in S} \overline{Y}_s$ is a nondegenerate continuum which is the disjoint union of $\{a, b\}$, $Z_1, \ldots, Z_n$, where each Z_k is a component of $X-A$ and $\mathrm{bd}(Z_k) = \{a, b\}$, $k = 1, \ldots, n$. Therefore A is a T-set in X.*

2.3. LEMMA ([21], Theorem 7). *Let X be a locally connected continuum, A a T-set in X and a, b two distinct points of A. Let I be an arc in X_A with end-points $f_A(a)$,*

$f_A(b)$, *and put*

$$L = (A \cap f_A^{-1}(I)) \cup \bigcup \{Y\colon Y \text{ is a component of } X-A \text{ such that } f_A(Y) \subset I\}.$$

Then L is a subcontinuum of X such that if $M = \{x\}$ for some $x \in L \cap A - \{a, b\}$, or M is a component of $X-A$ so that $M \subset L$, then $L-M$ is a union of two mutually separated connected sets P Q such that $a \in P$ and $b \in Q$.

2.4. Lemma. *Let X be a locally connected continuum without cut points and let E be any subset of X. If A is a T-set in X such that $E \subset A$, then there exists a minimal (in the sense of inclusion) T-set B in X such that $E \subset B \subset A$.*

Proof. If $|E| < 3$ then let B be any subset of A such that B consists of exactly two points and $E \subset B$. Suppose that E contains three distinct points p, q and r. By the Kuratowski–Zorn Lemma, to prove the existence of B it suffices to show that if $(S, <)$ is a well-ordered set of indices without last element and A_s, $s \in S$, are T-sets in X such that $E \subset A_s$ and $A_t \subset A_s$ for all $s, t \in S$, $s < t$, then $A' = \bigcap_{s \in S} A_s$ is a T-set in X.

Let x be a point of $X - A'$. Hence there is an $s_0 \in S$ so that $x \notin A_s$ for each $s \in S$, $s > s_0$. Let Y_s denote the component of $X - A_s$, $s > s_0$, such that $x \in Y_s$ and let $\mathrm{bd}(Y_s) = \{a_s, b_s\}$ and $B_s = \overline{Y}_s$. Note that each component of $X - B_s$ has a two-point boundary. Moreover, B_s is closed and $B_S \subset B_t$ provided $s_0 < s < t$. By Lemma 2.2, we may assume that the nets $\{a_s\colon s > s_0\}$ and $\{b_s\colon s > s_0\}$ converge to some points a and b, respectively. Observe that $Y^x = \bigcup_{s > s_0} Y_s$ is a component of $X - A'$ so that $x \in Y^x$ — indeed, the connected sets $Y^{x'}$, $x' \in X - A'$, constitute an open covering of $X - A'$ and if $Y^{x'} \cap Y^{x''} \neq \emptyset$, then $Y_s^{x'} \cap Y_s^{x''} \neq \emptyset$ for some $s \in S$ and therefore $Y_s^{x'} = Y_s^{x''}$; so $Y^{x'} = Y^{x''}$. Moreover, $a, b \in A'$ and $\{a, b\} = \mathrm{bd}(Y^x)$.

Now, it suffices to show that $a \neq b$. Suppose that $a = b$. Since X has no cut points, it follows that $Y^x = X - \{a\}$. Moreover, $A' \cap Y^x = \emptyset$, and so $A' \subset \{a\}$, a contradiction because $E \subset A'$.

3. Remarks on open coverings and T-sets.

3.1. Lemma. *Let X be a locally connected continuum, A a T-set in X, S a directed set, and $\{Y_s\colon s \in S\}$ a family of components of $X - A$. If the net $\overline{Y}_s$, $s \in S$, is not eventually constant and converges, then $\lim_{s \in S} \overline{Y}_s$ is a single point which belongs to A.*

Proof. Obviously, $Y = \lim_{s \in S} \overline{Y}_s$ is a subcontinuum of X. Suppose that Y is nondegenerate. Let $\overline{Y}_t$, $t \in T$ be a net finer than $\overline{Y}_s$, $s \in S$, such that $\lim_{t \in T} a_t = a$ and $\lim_{t \in T} b_t = b$ exist, where $\mathrm{bd}(Y_t) = \{a_t, b_t\}$. Note that $Y = \lim_{t \in T} \overline{Y}_t$. Let c be any point of $Y - \{a, b\}$ and let U be a connected neighbourhood of c in X such that $a, b \notin \overline{U}$. There is an element $t_0 \in T$ such that $\overline{Y}_t \cap U \neq \emptyset$ and $a_t, b_t \notin U$ for $t > t_0$. Hence $\emptyset \neq \overline{Y}_t \cap U \subset Y_t$ and therefore U is not connected, a contradiction.

3.2. Lemma. *Let X be a locally connected continuum, U an open covering of X and A a T-set in X. Then the family*

$$\{Y\colon Y \text{ is a component of } X-A \text{ and } \overline{Y} \text{ is not contained in any member of } U\}$$

is finite.

Proof. Let $U = \{U_1, \ldots, U_n\}$. Suppose that there are infinitely many distinct components $Y_1, Y_2, \ldots$ of $X - A$ so that $\overline{Y}_k$ is not contained in any U_l, for $k = 1, 2, \ldots$ and $l = 1, \ldots, n$. Since the hyperspace $C(X)$ of all subcontinua of X is compact, there is a convergent net $\overline{Y}_s$, $s \in S$, which is finer than the net $\overline{Y}_k$, $k \in \{1, 2, \ldots\}$. By Lemma 3.1, $\lim_{s \in S} \overline{Y}_s$ consists of a single point a. Let $U_m \in U$ be such that $a \in U_m$. Since $\lim_{s \in S} \overline{Y}_s = \{a\}$, there exists an $s_0 \in S$ such that $\overline{Y}_s \subset U_m$ for $s > s_0$. Hence there is a positive integer k such that $\overline{Y}_k \subset U_m$, a contradiction.

Recall that a *tree* is a partially ordered set T such that, for each $t \in T$, the set $\{s \in T\colon s \leqslant t\}$ is well-ordered. Recall also the following simple fact:

3.3. Lemma. (König's Lemma, see for example [8]). *If T is an infinite tree with finitely many minimal elements and each element of T has finitely many immediate successors, then T contains an infinite linearly ordered subset.*

3.4. Lemma. *Let X be a locally connected continuum without cut points and let $A_1, A_2, \ldots$ be T-sets in X such that $A_1 \subset A_2 \subset \ldots$ and if Y is a component of $X - A_n$ for some $n \in \{1, 2, \ldots\}$, and Z is a nondegenerate cyclic element of $\overline{Y}$ then $\{y \in Y\colon y \text{ cuts } \overline{Y}\} \subset A_{n+1}$ and $|Z \cap A_{n+1}| > 2$.*

Then $\bigcup_{n \geqslant 1} A_n$ is dense in X and for each open covering U of X there is a positive integer n_U so that the closure of each component Y of $X - A_{n_U}$ is contained in some member of U.

Proof. Put $A = \overline{\bigcup_{n \geqslant 1} A_n}$ and suppose that $x \in X - A$. For each n let Y_n be the component of $X - A_n$ so that $x \in Y_n$. Put $H = \bigcap_{n \geqslant 1} \overline{Y}_n$. By Lemma 2.2 A is a T-set in X, $\mathrm{bd}(Y_n) = \{a_n, b_n\}$, the limits $a = \lim a_n$ and $b = \lim b_n$ exist, $a \neq b$, and $H = \{a, b\} \cup H_1 \cup \ldots \cup H_k$, where $H_1, \ldots, H_k$ are components of $X - A$ such that $\mathrm{bd}(H_i) = \{a, b\}$, for $i = 1, \ldots, k$. For each positive integer n, let Z_n denote the cyclic element of $\overline{Y}_n$ such that $x \in Z_n$. Since X has no cut points, $\overline{Y}_n$ is a cyclic chain from a_n to b_n ([28], p. 71), and so $\mathrm{bd}(Z_n) = \{z_n, z_n'\}$ for some $z_n, z_n' \in \overline{Y}_n$, $z_n \neq z_n'$. Since $\{y \in Y_n\colon y \text{ cuts } \overline{Y}_n\} \cup \{a_n, b_n\} \subset A_{n+1}$, it follows that $\mathrm{bd}(Z_n) \subset A_{n+1}$ and therefore $a_{n+1}, b_{n+1} \in Z_n$. Let F_n denote the component of $Z_n - \{a_{n+1}, b_{n+1}\}$ such that $x \in F_n$. Observe that F_n is also a component of $X - A_{n+1}$; so $F_n = Y_{n+1}$. Since $|Z_n \cap A_{n+1}| > 2$ and $F_n \cap A_{n+1} = \emptyset$, there is a component G_n of $Z_n - \{a_{n+1}, b_{n+1}\}$ such that $x \notin G_n$. Note that $G = \mathrm{Ls}\, G_n$ is a subcontinuum of X and $a, b \in G \subset \bigcap_{n \geqslant 1} Z_n \subset \bigcap_{n \geqslant 1} \overline{Y}_n = H$. Hence there exists a point $y \in G \cap (H - \{a, b\})$. Then $V = H - \{a, b\}$

is a neighbourhood of y such that $V \cap G_n = \emptyset$ for each n. Therefore $y \notin G$, a contradiction.

Let U be an open covering of X. For each n put

$$S_n = \{Y\colon\ Y \text{ is a component of } X - A_n \text{ so that } \overline{Y} \text{ is not contained in any member of } U\}.$$

Put $S = \bigcup_{n \geqslant 1} S_n$ and observe that S ordered by reverse inclusion is a tree. The sets S_n are levels of $(S, \supset)$. By Lemma 3.2, $(S, \supset)$ has only finitely many minimal elements and each member of S has only finitely many immediate successors.

Suppose that S is infinite. By Lemma 3.3, there is $S' \subset S$ such that S' is infinite and $(S', \supset)$ is linearly ordered. We may assume that S' is maximal. Therefore $S' \cap S_n = \{V_n\}$ for each n. Since $A = X$, the set $\bigcap_{n \geqslant 1} \overline{V}_n$ consists of a single point v. Let $U_l \in U$ be such that $v \in U_l$. There is an integer m such that $\overline{V}_m \subset U_l$. Hence $V_m \notin S$, a contradiction.

We have thus shown that S is finite. Since no component of $X - A_n$ is a component of $X - A_k$ when $n < k$, the families S_n and S_k are disjoint when $n \neq k$. Therefore there is a positive integer n_U such that $S_{n_U} = \emptyset$.

3.5. Lemma. *Let X and Y be locally connected continua and let A, B be T-sets in X, Y, respectively, such that all components of $X - A$ and all components of $Y - B$ are homeomorphic to $]0, 1[$. Suppose that there is a homeomorphism $g\colon A \to B$ such that there is a bijection*

$$f\colon \{U\colon\ U \text{ is a component of } X - A\} \to \{V\colon\ V \text{ is a component of } Y - B\}$$

such that $g(\mathrm{bd}(U)) = \mathrm{bd}(f(U))$ for each component U of $X - A$. Then there is a homeomorphism $G\colon X \to Y$ such that $G|_A = g$ and $G(U) = f(U)$ for each component U of $X - A$.

Proof. For each component U of $X - A$, write $\mathrm{bd}(U) = \{a_U, a'_U\}$ and let $g_U\colon \overline{U} \to \overline{f(U)}$ be any homeomorphism such that $g_U(a_U) = g(a_U)$. If $x \in X$, then put $G(x) = g(x)$ provided $x \in A$ and $G(x) = g_U(x)$ if $x \in U$ for a component U of $X - A$. Note that G is one-to-one and onto, and use Lemma 3.1 to show that G is continuous.

3.6. Lemma. *Let X be a locally connected continuum, A a T-set in X, and H any family of components of $X - A$ so that $\overline{V} \cap \overline{W} = \emptyset$ if $V, W \in H$, $V \neq W$. Let G be the decomposition of X into points and the sets $\overline{W}$ for $W \in H$. Then G is upper semi-continuous.*

Proof. This follows from Lemma 3.1.

3.7. Lemma. *Let X be a locally connected continuum, A a T-set in X and $U = \{U_1, \ldots, U_n\}$ an open covering of X. For every component Z of $X_A - f_A(A)$* let Y_Z *be the unique component of $X - A$ so that $f_A(Y_Z) \subset \overline{Z}$. Write $\mathrm{bd}(Y_Z) = \{a_Z, b_Z\}$ and let*

p_Z, q_Z, r_Z be points of Z such that $f_A(a_Z) < p_Z < q_Z < r_Z < f_A(b_Z)$ in a natural ordering of $\overline{Z}$. Let

$$H = \{Z\colon\ Z \text{ is a component of } X_A - f_A(A)\},$$

$$U_k^A = f_A(A \cap U_k) \cup \bigcup \{Z \in H\colon\ \overline{Y}_Z \subset U_k\} \cup$$
$$\cup \bigcup \{]f_A(a_Z), q_Z[_{\overline{Z}}\colon\ Z \in H \text{ and } a_Z \in U_k\} \cup$$
$$\cup \bigcup \{]q_Z, f_A(b_Z)[_{\overline{Z}}\colon\ Z \in H \text{ and } b_Z \in U_k\}$$

for $k = 1, \ldots, n$, and

$$U_0^A = \bigcup \{]p_Z, r_Z[_{\overline{Z}}\colon\ Z \in H \text{ and } \overline{Y}_Z \text{ is not contained in any member of } U\}.$$

Then $U^A = \{U_0^A, \ldots, U_n^A\}$ is an open covering of X_A.

Proof. It is obvious that U_0^A is open. We show that $X_A - U_k^A$ is closed for $k = 1, \ldots, n$. Suppose that

$$z \in \overline{X_A - U_k^A} - (X_A - U_k^A) = \overline{X_A - U_k^A} \cap U_k^A.$$

Note that $z \in f_A(A)$ — indeed, each point $z' \in Z$ of some component Z of $X_A - f_A(A)$ such that $z' \in U_k^A$ has an open neighbourhood V so that $V \subset Z \cap U_k^A$. There is a net z_s, $s \in S$, of points of $X_A - U_k^A$ which converges to z. Observe that $(X_A - U_k^A) \cap f_A(A)$ is closed, and so we may assume that each z_s belongs to some component Z_s of $X_A - f_A(A)$. Using nets finer than $\overline{Z}_s$, $s \in S$, one can easily show that (by Lemma 3.1) $\lim_{s \in S} \overline{Z}_s$ exists and is equal to $\{z\}$. Let x be the unique point of $A \cap f_{A|}^{-1}(z)$; so $x \in U_k$. It follows that $\lim_{s \in S} \overline{Y}_{Z_s}$ exists and is equal to $\{x\}$. Observe that, for each $s \in S$. there exists a point $x_s \in \overline{Y}_{Z_s}$ so that $x_s \notin U_k$. Note that $\lim_{s \in S} x_s = x \in U_k$. Thus U_k is not open, a contradiction.

We show that U^A covers X_A. Observe that

$$f_A(A) = f_A\Big(\bigcup_{k=1}^{n} U_k \cap A\Big) \subset \bigcup_{k=1}^{n} U_k^A.$$

Suppose that $z \in Z$ for some component Z of $X_A - f_A(A)$. If $\overline{Y}_Z \subset U_k$ for some k then $z \in U_k^A$, and if $\overline{Y}_Z$ is not contained in any member of U and $z \neq q_Z$, then $z \in U_k^A$ for each k such that $a_Z \in U_k$ (resp. $b_Z \in U_k$) provided $z \in]f_A(a_Z), q_Z[_{\overline{Z}}$ (resp. $z \in]q_Z, f_A(b_Z)[_{\overline{Z}}$). If $z = q_Z$ and $\overline{Y}_Z$ is not contained in any member of U, then $z \in U_0^A$.

4. Metrizability of special T-sets.

4.1. Lemma. *If X is a locally connected continuum and E is a metrizable T-subset of X, then $X - E$ has only countably many components.*

Proof. Let ϱ be a metric on E. Suppose that $X - E$ has uncountably many components. For some positive integer m, we can find a sequence $W_1, W_2, \ldots$ of distinct components of $X - E$ such that $\mathrm{bd}(W_n) = \{a_n, b_n\}$, $\varrho(a_n, b_n) \geqslant 1/m$, $\lim a_n = a$ and $\lim b_n = b$. Hence $\varrho(a, b) \geqslant 1/m$; so $a \neq b$. Let $\overline{W}_s$, $s \in S$, be a net finer than

$\overline{W}_n$, $n \in \{1, 2, \ldots\}$, such that $W = \lim_{s \in S} \overline{W}_s$ exists. Note that $a, b \in W$. This contradicts Lemma 3.1.

4.2. LEMMA. *If X is a locally connected continuum and E is a metrizable T-set in X such that each component of $X-E$ is homeomorphic to* $]0, 1[$, *then X is metrizable.*

Proof. By Lemma 4.1, the set $X-E$ has countably many components. One can find a countable basis for a Hausdorff topology on X weaker than the original topology of X (for points of E, define basic neighbourhoods similarly to the definition of the sets U_k^A in Lemma 3.7). Since X is compact, the new topology of X coincides with the original one.

Recall that a continuum X is said to be *hereditarily locally connected* provided each subcontinuum of X is locally connected. It is not difficult to show that each hereditarily locally connected continuum is arcwise connected ([24], Corollary 4, p. 125).

If X is a continuum such that, for each open covering U of X, every family H of pairwise disjoint subcontinua of X, none of which is contained in a member of U, is finite, then we say that X is a *finitely Suslinian continuum*. A continuum X is said to be *rim-finite* if each point of X has arbitrarily small open neighbourhoods with finite boundary. Recall that every rim-finite continuum is finitely Suslinian and every finitely Suslinian continuum is hereditarily locally connected ([24]). Note that a continuum X is finitely Suslinian if and only if, for any two disjoint closed sets $F, G \subset X$, each family H of pairwise disjoint subcontinua of X with the property that $Y \cap F \neq \emptyset \neq Y \cap G$ for $Y \in H$ is finite. Observe also that if a Hausdorff space Z is the image of a finitely Suslinian continuum X under a continuous and monotone mapping, then Z is also a finitely Suslinian continuum.

4.3. LEMMA. *If X is a locally connected continuum and A is a zero-dimensional T-set in X such that each component of $X-A$ is homeomorphic to* $]0, 1[$, *then X is rim-finite, and so finitely Suslinian.*

Proof. Let $x \in X$. If $x \in X-A$ then x has arbitrarily small neighbourhoods with two-point boundary. Suppose that $x \in A$ and let U be any open neighbourhood of x. Since A is zero-dimensional, there is an open set V such that $x \in V \subset U$ and $\mathrm{bd}(V) \cap A = \emptyset$. By Lemma 3.1, $\mathrm{bd}(V)$ intersects only finitely many components of $X-A$. Let W be the component of V so that $x \in W$. Since X is locally connected, W is open. Moreover, $\mathrm{bd}(W) \subset \mathrm{bd}(V)$. Thus $\mathrm{bd}(W) \cap A = \emptyset$ and $\mathrm{bd}(W)$ intersects only finitely many components of $X-A$. Since W is connected and the components of $X-A$ are homeomorphic to $]0, 1[$, $\mathrm{bd}(W)$ is finite.

Remark. The result of Lemma 4.3 was used in the proof of Lemma 2 in [20], p. 85, without being proved there. See also [15], p. 340, for a weaker result.

4.4. LEMMA. *Let X be a hereditarily locally connected continuum, A a closed subset of X, W a component of $X-A$, x a point of W, y a point of $\mathrm{bd}(W)$, and U an open set such that $y \in U$. Then there exist a point $z \in A \cap U$ and an arc $I \subset X$ with end-points x, z such that $I \cap A = \{z\}$.*

Proof. We may assume that $X = A \cup W$ and $A = \mathrm{bd}(W)$. Note that for any two points $p, q \in W$ there is a subcontinuum Y of X such that $p, q \in Y \subset W$ (see for example [6], Theorem 3.7). Thus for each $p \in W$ there is an arc $I_p \subset W$ with end-points x, p. Let V be a connected open set such that $y \in V \subset \overline{V} \subset U$. Since $y \in \mathrm{bd}(W)$, there is a point q such that $q \in V \cap W$. Let J be any arc in V with end-points q, y. Observe that there is an arc $I' \subset I_q \cup J$ with end-points x, y. Moreover, $I' \cap A \subset J \cap A \subset \overline{V} \cap A \subset U \cap A$. Finally, there is an arc $I \subset I'$ with end-points x, z such that $I \cap A = \{z\}$.

4.5. THEOREM. *Let X be a hereditarily locally connected continuum. If E is a metrizable closed subset of X, then the family*

$$H = \{W\colon\ W \text{ is a component of } X-E \text{ and } |\mathrm{bd}(W)| \geqslant 2\}$$

is countable.

Proof. Suppose that H is uncountable. By Lemma 4.4, for each component W of $X-E$ such that $|\mathrm{bd}(W)| > 1$, there is an arc $J_W \subset \overline{W}$ with end-points a_W, b_W such that $J_W \cap E = \{a_W, b_W\}$. Let ϱ be a metric on E. There are distinct components $W_1, W_2, \ldots$ of $X-E$ such that $|\mathrm{bd}(W_n)| > 1$, and for some positive integer m, $\varrho(a_{W_n}, b_{W_n}) \geqslant 1/m$, and the limits $\lim a_{W_n} = a$, $\lim b_{W_n} = b$ exist; so $\varrho(a, b) \geqslant 1/m$. Let U, V be open neighbourhoods of a, b, respectively, such that $\overline{U} \cap \overline{V} = \emptyset$. We may assume that $J_{W_n} \cap U \neq \emptyset \neq J_{W_n} \cap V$, for $n = 1, 2, \ldots$ Hence, for each n, there is a subarc I_n of J_{W_n} such that $I_n \subset W_n$ and $I_n \cap U \neq \emptyset \neq I_n \cap V$. Since $W_1, W_2, \ldots$ are pairwise disjoint open subsets of X, it follows that $I_k \cap \overline{\bigcup_{n \neq k} I_n} = \emptyset$ for $k = 1, 2, \ldots$ By [17], Theorem 4, p. 246, X is not hereditarily locally connected, a contradiction.

Remark. If X is a dendron which contains exactly one point x so that $X-\{x\}$ has more than two components (i.e., X is a "fan") and, moreover, $X-\{x\}$ has uncountably many components, then $E = \{x\} \times [0, 1]$ is a closed metrizable subset of a locally connected continuum $Y = X \times [0, 1]$ such that $Y-E$ has uncountably many components. However, if we assume in addition that E is zero-dimensional, then Theorem 4.5 remains true for locally connected continua (it suffices to modify the proof of Lemma 4.1).

4.6 LEMMA. *Let Z be a finitely Suslinian continuum, J a separable arc in Z with end-points j_0, j_1, $<$ the natural ordering of J from j_0 to j_1, and U a countable subset of J. Put*

$$T = \{t \in J\colon \text{ there are pairwise disjoint arcs } J_n \text{ in } Z,\ n = 1, 2, \ldots, \text{ with end-points } a_n, b_n \text{ such that } J \cap J_n = \{a_n, b_n\} \text{ and } a_n < t < b_n\}.$$

Then:

(a) *if $Z-J$ has exactly one component W_0 and $\mathrm{bd}(W_0) = J$, then $J-T$ is countable;*

(b) *if $Z-J$ has exactly one component W_0, then $\mathrm{bd}(W_0)-T$ is countable;*

(c) *T is always a* $G_{\delta\sigma}$*-set in* J;

(d) *if* $J-T$ *is uncountable, then there are two distinct points* $c, d \in J-\{j_0, j_1\}$ *such that either* $\mathrm{bd}(V) \subset [c, d]_J$ *or* $\mathrm{bd}(V) \cap]c, d[_J = \emptyset$ *for each component* V *of* $Z-J$;

(e) *if* $T = \emptyset$, *then there are two distinct points* $c, d \in J-(U \cup \{j_0, j_1\})$ *such that either* $\mathrm{bd}(V) \subset [c, d]_J$ *or* $\mathrm{bd}(V) \cap]c, d[_J = \emptyset$ *for each component* V *of* $Z-J$.

Proof. We may assume that, for each component V of $Z-J$, the set $\mathrm{bd}(V)$ contains at least two points. By Theorem 4.5, $Z-J$ has countably many components denoted by $W_0, W_1, \ldots$

(a). Let $p \in W_0$. Let A_0 be a family of arcs in Z which is maximal (in the sense of inclusion) with respect to the property: if $K \in A_0$ then the end-points of K are $p_K = p$ and q_K with $J \cap K = \{q_K\}$, and if $K, L \in A_0$, $K \neq L$, then $K \cap L = \{p\}$. Since Z is finitely Suslinian, A_0 is finite. Suppose that, for some n, countable families $A_0, \ldots, A_n$ of arcs in Z are constructed such that $\bigcup (A_0 \cup \ldots \cup A_n)$ is a closed subset of Z. Let A_{n+1} be a family of arcs in Z which is maximal with respect to the property: if $K \in A_{n+1}$ then the end-points of K are p_K and q_K with $\bigcup (A_0 \cup \ldots \cup A_n) \cap K = \{p_K\}$ and $J \cap K = \{q_K\}$, and if $K, L \in A_{n+1}$, $K \neq L$, then $K \cap L \subset \{p_K, q_K\} \cap \{p_L, q_L\}$. The assumption that Z is finitely Suslinian shows that A_{n+1} is countable and $\bigcup (A_0 \cup \ldots \cup A_n \cup A_{n+1})$ is closed in Z.

Put $A = \bigcup_{n \geq 0} A_n$, $W = \bigcup A - \{q_K\colon K \in A\}$ and $Y = W \cup J$. Observe that $W \cap J = \emptyset$, W is connected and $\mathrm{bd}(W) \subset J$; so Y is a continuum and W is the unique component of $Y-J$. It is not difficult to show that $\mathrm{bd}(W) = J$ (use Lemma 4.4). Note that, for any two distinct points $x, y \in W$, there is exactly one arc $[x, y]_W$ in W with end-points x, y (observe that W is even a rim-finite dendritic space; see [13], [14] and [25] for some properties of such spaces which can be useful below). We use the last fact to introduce a partial ordering $\prec$ on W: if $x, y \in W$ then let $x \prec y$ provided either $x = y$ or $x \in [p, y]_W$. For each $x \in W$ write $W_x = \{y \in W\colon x \prec y\}$; note that W_x is closed in W.

Put $S = \{p_K\colon K \in A\}$ (so S contains all "ramification" points of W). For each $n \in \{0, 1, \ldots\}$, write

$$S_n = \{x \in S\colon x \text{ has exactly } n \text{ predecessors in } (S, \prec)\}.$$

Since Z is finitely Suslinian, it follows that $S = \bigcup_{n \geq 0} S_n$. Moreover, it is easy to give an inductive proof that every set S_n is finite.

Take any points $x_0, x_1, \ldots \in S$ such that $x_n \in S_n$ and $x_n \prec x_{n+1}$ for $n = 0, 1, \ldots$, and put $N = \bigcup_{n \geq 0} [x_n, x_{n+1}]_W$. Since Z is finitely Suslinian, it follows that $\overline{N} - N$ consists of exactly one point x_∞; so $x_\infty = \lim x_n$, $x_\infty \in J$ and $\overline{N}$ is an arc from x_0 to x_∞. We show that, moreover,

$$\{x_\infty\} = \bigcap_{n \geq 0} \overline{W}_{x_n} = \bigcap_{n \geq 0} \mathrm{bd}(W_{x_n}).$$

Suppose that there is a point $x \neq x_\infty$, $x \in \bigcap_{n \geq 0} \mathrm{bd}(W_{x_n})$. Let F, G be open sets such that $\overline{F} \cap \overline{G} = \emptyset$, $\overline{N} \subset F$ and $x \in G$. By Lemma 4.4, there is an arc $M_n \subset \overline{W}_{x_n}$ with end-points x_n, y_n, satisfying $M_n \cap J = \{y_n\}$ and $y_n \in G$ for $n = 0, 1, \ldots$ Since $x \neq x_\infty$ and $\lim x_n = x_\infty$, it follows that, for each n, the set $M_n \cap \{x_n, x_{n+1}, \ldots\}$ is finite, and so $M_n \cap W_{x_m} = \emptyset$ for some integer m. Now, it is easy to find pairwise disjoint arcs $L_{n_1}, L_{n_2}, \ldots$, $n_1 < n_2 < \ldots$, such that $L_{n_k} \subset M_{n_k}$ and $\overline{F} \cap L_{n_k} \neq \emptyset \neq G \cap L_{n_k}$ for $k = 1, 2, \ldots$ Thus Z is not finitely Suslinian, a contradiction.

For each $x \in S$, put

$$R_x = \{t \in J\colon t \text{ is an end-point of some component of } J - \mathrm{bd}(W_x)\};$$

so R_x is countable. Put $R = \{j_0, j_1\} \cup \bigcup_{x \in S} R_x$; hence R is also countable. We now show that $J - T \subset R$, which will finish the proof of (a).

Let $y \in J - R$. Since S_n are finite and $\mathrm{bd}(W) = J$, it follows that $J = \bigcup_{x \in S_n} \mathrm{bd}(W_x)$ for every n. Thus $S' = \{x \in S\colon y \in \mathrm{bd}(W_x)\}$ is infinite. By König's Lemma (see Lemma 3.3 above), there are $x_0, x_1, \ldots \in S'$ such that $x_n \in S_n$ and $x_n \prec x_{n+1}$ for $n = 0, 1, \ldots$ Since $y \notin R$, there are $u_0, v_0 \in \mathrm{bd}(W_{x_0})$ such that $u_0 < y < v_0$. Arguing as in the above proof that $\overline{N}$ is an arc, it is easy to find arcs I_0, K_0 in $W_{x_0} \cup \{u_0, v_0\}$ with end-points u_0, v_0 and x_0, v_0, respectively. There is an arc $J_0 \subset I_0 \cup K_0$ with end-points x_0, u_0. There is an integer n_1 such that $x_{n_1} \notin J_0$ and $u_0, v_0 \notin \mathrm{bd}(W_{x_{n_1}})$; so $J_0 \cap \overline{W}_{x_{n_1}} = \emptyset$. Apply the above argument to find an arc J_1 in $W_{x_{n_1}} \cup \{u_1, v_1\}$ with end-points $u_1, v_1 \in J$ such that $u_1 < y < v_1$. Note that $J_0 \cap J_1 = \emptyset$. Proceeding by induction we find a sequence $J_0, J_1, \ldots$ of arcs which is required to show that $y \in T$.

(b). We may assume that $\mathrm{bd}(W_0)$ is uncountable. Let A denote the set of all condensation points of $\mathrm{bd}(W_0)$ (see [7]); so $A \subset \mathrm{bd}(W_0) \subset J$ is a closed set without isolated points and $\mathrm{bd}(W_0) - A$ is countable. Put $B = \{\overline{F}\colon F \text{ is a component of } J - A\}$ and let G be the decomposition of Z into points and sets which belong to B. Note that G is upper semi-continuous and monotone. Let $g\colon Z \to Z/G = Y$ be the quotient map and $I = g(J)$; so I is homeomorphic to $[0, 1]$, Y is a finitely Suslinian continuum, and $g(W_0)$ is the unique component of $Y - I$. Moreover, $\mathrm{bd}(g(W_0)) = I$. Put $S = \{t \in I\colon g^{-1}(t) \text{ is nondegenerate}\}$; note that $g^{-1}(S) \cap \mathrm{bd}(W_0) = (\bigcup B) \cap \mathrm{bd}(W_0)$ is countable. Put

$$T' = \{t \in I\colon \text{there are pairwise disjoint arcs } I_n \text{ in } Y,\ n = 1, 2, \ldots, \text{ with end-points } a_n,\ b_n \text{ such that } I \cap I_n = \{a_n, b_n\} \text{ and } a_n < t < b_n\},$$

where $<$ denotes also a natural ordering of I. By (a), $S' = I - T'$ is countable (one can also prove that $S \subset S'$). Observe that $\mathrm{bd}(W_0) - T \subset \mathrm{bd}(W_0) \cap g^{-1}(S \cup S')$. Finally, $\mathrm{bd}(W_0) \cap g^{-1}(S \cup S')$ is countable.

(c). For each $k \in \{0, 1, \ldots\}$, let $[c_k, d_k]_J$ denote the smallest subarc of J such that $\operatorname{bd}(W_k) \subset [c_k, d_k]_J$, $c_k < d_k$, and put

$T_k = \{t \in J$: there are pairwise disjoint arcs J_n in $J \cup W_k$, $n = 1, 2, \ldots$, with end-points a_n, b_n such that $J \cap J_n = \{a_n, b_n\}$ and $a_n < t < b_n\}$

$T' = \{t \in J$: there are pairwise disjoint arcs J_n in Z, $n = 1, 2, \ldots$, with end-points a_n, b_n such that $J \cap J_n = \{a_n, b_n\}$ and $a_n < t < b_n$, and if $n \neq m$ then J_n and J_m do not intersect the same component of $Z - J\}$.

Note that $T = T' \cup \bigcup_{k \geqslant 0} T_k$. By (b), each T_k is a G_δ-set. Moreover,

$$T' = \bigcap_{m \geqslant 0} \bigcup_{k \geqslant m}]c_k, d_k[_J ,$$

and so T' is a G_δ-set. Thus T is a $G_{\delta\sigma}$-set.

(d). Let T', T_k and $[c_k, d_k]_J$ for $k = 0, 1, \ldots$ be as in the proof of (c). Suppose that for any $c, d \in J - \{j_0, j_1\}$, $c \neq d$, there is a component W_k of $Z - J$ such that

$$\operatorname{bd}(W_k) \cap]c, d[_J \neq \emptyset \neq \operatorname{bd}(W_k) - [c, d]_J ;$$

so

$$[c_k, d_k]_J \cap]c, d[_J \neq \emptyset \neq [c_k, d_k]_J - [c, d]_J .$$

First, we show that

(*) if $p_n, q_n \in [0, 1]$, $p_n < q_n$, $n = 0, 1, \ldots$, and for all $c, d \in]0, 1[$, $c < d$, there is a nonnegative integer n such that $[p_n, q_n] \cap]c, d[\neq \emptyset \neq [p_n, q_n] - [c, d]$, then $A =]0, 1[- \bigcup_{n \geqslant 0}]p_n, q_n[$ contains at most one point.

Suppose that $c, d \in A$, $c < d$. If, for some n, $[p_n, q_n] \cap]c, d[\neq \emptyset$, then $[p_n, q_n] \subset [c, d]$ (because $c, d \notin]p_n, q_n[$), and so $[p_n, q_n] - [c, d] = \emptyset$, a contradiction. If $[p_n, q_n] - [c, d] \neq \emptyset$, then $[p_n, q_n] \cap]c, d[= \emptyset$, again a contradiction.

Put $A_0 =]j_0, j_1[_J - \bigcup_{k \geqslant 0}]c_k, d_k[_J$. By (*), $(A_0) \leqslant 1$. Suppose that for some integer $l \geqslant 0$ we have already constructed countable sets $A_0, \ldots, A_l$ such that

$$B_l = \operatorname{bd}(W_0) \cup \ldots \cup \operatorname{bd}(W_{l-1}) \cup A_0 \cup \ldots \cup A_l$$

is a closed set (we put $W_{-1} = \emptyset$). Let C_l denote the family of all components of $]j_0, j_1[_J - B_l$. If $D =]p, q[_J \in C_l$ then, by (*), the set $A_D = D - \bigcup_{k \geqslant l}]c_k, d_k[_J$ contains at most one point. Put $A_{l+1} = \bigcup_{D \in C_l} A_D$, $B_{l+1} = \operatorname{bd}(W_0) \cup \ldots \cup \operatorname{bd}(W_l) \cup A_0 \cup \ldots \cup A_{l+1}$; so A_{l+1} is countable and B_{l+1} is closed (because $\bar{A}_{l+1} \subset B_l \cup A_{l+1}$).

Put $B = \bigcup_{l \geqslant 0} B_l$ and note that $J - B \subset T'$. By (b), the sets $E_k = \operatorname{bd}(W_k) - T_k$, $k = 0, 1, \ldots$, are countable. Since

$$B - \bigcup_{k \geqslant 0} (E_k \cup A_k) \subset \bigcup_{k \geqslant 0} T_k \quad \text{and} \quad T = T' \cup \bigcup_{k \geqslant 0} T_k ,$$

it follows that $J - T$ is also countable.

(e). Assume that $T = \emptyset$. Let $f\colon J \to [0, 1] \times \{0\}$ be any homeomorphism. Let R denote (for a moment) the set of real numbers. First, we construct a subcontinuum Y of the plane R^2.

Let $v_1, v_2, \ldots$ be a sequence of points of $f(U) - \{(0, 0), (1, 0)\}$ such that each point appears in the sequence infinitely many times and $0 \leqslant v_k - 2^{-(k+2)} < v_k + 2^{-(k+2)} \leqslant 1$. Put

$$Y = [0, 1] \times \{0\} \cup \bigcup_{k \geqslant 1} \{(x, y) \in R^2\colon (x - v_k)^2 + y^2 = 2^{-2(k+2)} \text{ and } y \geqslant 0\} ,$$

$S = \{(t, 0) \in [0, 1] \times \{0\}$: there are pairwise disjoint arcs J_n in Y, $n = 1, 2, \ldots$, with end-points $(a_n, 0)$, $(b_n, 0)$, where $[0, 1] \times \{0\} \cap J_n = \{(a_n, 0), (b_n, 0)\}$ and $a_n < t < b_n\}$,

$$S' = \Big(\bigcup_{k \geqslant 1} [v_k - 2^{-(k+2)}, v_k + 2^{-(k+2)}]\Big) \times \{0\} .$$

Note that $f(U) - \{(0, 0), (1, 0)\} \subset S \subset S'$ and $[0, 1] \times \{0\} - S'$ is uncountable (because $\sum_{k \geqslant 1} 2 \cdot 2^{-(k+2)} = \frac{1}{2} < 1$).

We may assume that $Z \cap Y = \emptyset$. Let G be the decomposition of $Z \cup Y$ into points and the sets $\{t, f(t)\}$, $t \in J$; so G is upper semi-continuous. Put $X = (Z \cup Y)/G$ and let $g\colon Z \cup Y \to X$ denote the quotient map. Observe that X is a finitely Suslinian continuum and $g(J) = g([0, 1] \times \{0\})$ is an arc in X. Put

$T' = \{t \in g(J)$: there are pairwise disjoint arcs J_n in X, $n = 1, 2, \ldots$, with end-points a_n, b_n such that $g(J) \cap J_n = \{a_n, b_n\}$ and $a_n < t < b_n\}$.

Note that $T' = g(T) \cup g(S) = g(S)$; so $g(J) - T'$ is uncountable. Use (d) to find $c', d' \in g(J) - g(\{j_0, j_1\})$ such that either $\operatorname{bd}(V) \subset [c', d']_{g(J)}$ or

$$\operatorname{bd}(V) \cap]c', d'[_{g(J)} = \emptyset$$

for each component V of $X - g(J)$. Observe that $c' \notin g(U)$ and $d' \notin g(U)$. Finally, let $c, d \in J$ be the unique points such that $g(c) = c'$ and $g(d) = d'$.

4.7. LEMMA ([7], § 24, Section VII, Theorem 3, p. 265). *Let X be a separable metric space and $\{A_t\colon t \in [0, 1]\}$ a family of closed subsets of X such that $A_t \subset A_s$ if $t < s$. Then the equality $A_t = \overline{\bigcup_{s<t} A_s}$ holds for every $t \in [0, 1]$ except for a countable set of indices.*

4.8. LEMMA. *Let I be an arc and C a subset of I such that the end-points of I belong to C, C is compact, separable and zero-dimensional, and $I - C$ has uncountably many components. Let R_C be the relation in I defined as follows: $x R_C y$ provided $x = y$ or $[x, y] - (C \cap [x, y])$ has countably many components. Then R_C is an equivalence relation whose classes are subarcs of I (possibly degenerate) with end-points in C. Thus the decomposition G^C of I defined by R_C is upper semi-continuous and the quotient space $J = I/G^C$ is homeomorphic to $[0, 1]$. Let $g_C\colon I \to J$ be the quotient map and put*

$P = \{x \in J\colon |g_C^{-1}(x) \cap C| > 2\}$, $Q = \{x \in J\colon |g_C^{-1}(x) \cap C| = 2\}$. *Then P is countable and $[x, y] \cap Q$ is uncountable for any $x, y \in J$, $x \neq y$.*

Proof. Since all decreasing and all increasing sequences of points of C (in a natural ordering of I) are countable, it follows that R_C is indeed an equivalence relation whose classes are arcs with end-points in C. Note that $J = g_C(C)$ is separable and nondegenerate; so J is homeomorphic to $[0, 1]$. For each $x \in P$, the set $(g_C^{-1}(x) - \mathrm{bd}(g_C^{-1}(x))) \cap C$ is open in C and nonempty. Since these sets are pairwise disjoint and C is separable, it follows that P is countable.

Let $<$ denote natural orderings on I and J such that g_C is an increasing map. Suppose that $x, y \in J$, $x < y$, and $[x, y] \cap Q$ is countable. Let u (resp. v) be the first (resp. last) point of I such that $g_C(u) = x$ (resp. $g_C(v) = y$). Since $[x, y] \cap Q$ is countable and P is countable, it follows that $[u, v] - (C \cap [u, v])$ has countably many components. Thus $u R_C v$, and so $g_C(u) = g_C(v)$, a contradiction.

4.9. Theorem. *Let X be a locally connected continuum without cut points such that:*

(a) *if $E \subset E' \subset X$ and E' is separable, then E is also separable, and*

(b) *if E' is a continuous and monotone image of X and E is a separable continuum in E', then E is metrizable.*

Suppose that A is a closed metrizable subset of X, and A' is a separable T-set in X such that $A \subset A'$. Then there exists a metrizable T-set B in X such that $A \subset B \subset A'$. More precisely, each separable T-subset of X which is minimal with respect to the property "contains A" is metrizable.

Proof. Let B be any separable and minimal T-subset of X so that $A \subset B \subset A'$ (by Lemma 2.4 and assumption (a), B does exist). Because of Lemma 2.1, we may assume that each component of $X - B$ is homeomorphic to $]0, 1[$.

Let G be the decomposition of X into the components of B and points. Note that G is upper semi-continuous and monotone (see [20], Lemma 2, p. 85). Let Y denote the quotient space $Y = X/G$ and $f\colon X \to Y$ the quotient map. The set $f(B)$ is zero-dimensional and each component of $Y - f(B)$ is homeomorphic to $]0, 1[$. By Lemma 4.3, Y is finitely Suslinian. Hence Y is arcwise connected.

Let Y_1 be a nondegenerate cyclic element of Y and let $r\colon Y \to Y_1$ denote the unique monotone retraction from Y onto Y_1. Since X has no cut points, it follows that $rf(B) = f(B) \cap Y_1$; so if $rf(B)$ is nondegenerate, then $rf(B)$ is a separable zero-dimensional T-subset of Y_1 such that each component of $Y_1 - rf(B)$ is homeomorphic to $]0, 1[$. Let $\{y_0, y_1, \ldots\}$ be a countable dense subset of $rf(B)$. For each positive integer n choose an arc $I_n \subset Y_1$ from y_0 to y_n. Put $Y_1' = \overline{\bigcup_{n \geq 1} I_n} \subset Y_1$; so Y_1' is a subcontinuum of Y_1 such that $rf(B) \subset Y_1'$. By (b), Y_1' is metrizable if and only if for each positive integer n the set $I_n - rf(B)$ has countably many components.

Suppose that Y_1' is not metrizable. Hence there is an n_0 such that $I_{n_0} - rf(B)$ has uncountably many components. Since $rf(A)$ is metrizable, $I_{n_0} - rf(A)$ has only countably many components. Therefore there is an arc I contained in I_{n_0} with end-points in $rf(B)$ such that $I \cap rf(A) = \emptyset$ and $I - rf(B)$ still has uncountably many components. Put $C = rf(B) \cap I$; so C is separable, compact and zero-dimensional. Let G_1 be the decomposition of Y_1 into points and the classes of the decomposition G^C of the arc I (see Lemma 4.8). Note that G_1 is upper semi-continuous and monotone. Let Z denote the quotient space $Z = Y_1/G_1$ and $g\colon Y_1 \to Z$ the quotient map. Note that g is monotone and $g|_I = g_C$ (Lemma 4.8). Moreover, Z is finitely Suslinian. Put $J = g(I)$; so J is homeomorphic to $[0, 1]$. Let j_0, j_1 denote the end-points of J, put $J^0 = J - \{j_0, j_1\}$, and let $<$ be the natural ordering of J from j_0 to j_1. Let also $<$ denote the natural ordering of I such that $g|_I$ is an increasing map. Put

$$P = \{t \in J\colon |g^{-1}(t) \cap C| > 2\},$$
$$Q = \{t \in J^0\colon |g^{-1}(t) \cap C| = 2\}.$$

Note that for each $q \in Q$ there is a unique component W_q of $X - B$ so that $f(W_q)$ is a component of $Y_1 - f(B)$ and $gf(W_q) = \{q\}$. Let H be the decomposition of X into points and the sets $\overline{W}_q$, $q \in Q$. By Lemma 3.6, H is upper semi-continuous. Put $X_0 = X/H$ and let $h\colon X \to X_0$ be the quotient map; so h is monotone. Let $k\colon X_0 \to Z$ be the mapping defined by the formula: $k(x) = z$ provided $grf(h^{-1}(x)) = \{z\}$. It is easy to see that k is well-defined, continuous (see [3], Theorem 6.3.2, p. 123) and monotone. Put $X' = f^{-1}(I)$ and $X_0' = h(X')$. We show that X_0' is metrizable.

For each $p \in P$ put $X_p = f^{-1}(g^{-1}(p))$; so X_p is a subcontinuum of X'. Note that each X_p is a union of some subset of B and of countably many components of $X - B$. By (a), X_p is separable for each $p \in P$. Observe that

$$X_0' = h\Big(\bigcup_{p \in P} X_p\Big) \cup h(B \cap X').$$

Since P is countable (Lemma 4, 8) it follows that X_0' is separable and so metrizable (by (b)). Let ϱ denote a metric on X_0'.

Observe that $k(X_0') = J$ and define $k'\colon X_0' \to J$ as $k' = k|_{X_0'}$; so k' is continuous. Note that k' is also monotone and write $M_t = (k')^{-1}(t)$ for each $t \in J$. For each positive integer n put $R_n = \{t \in J\colon \mathrm{diam} M_t \geq 1/n\}$, $R = \bigcup_{n \geq 1} R_n$. Note that R_n are closed. Put

$$T = \{t \in J\colon \text{there are pairwise disjoint arcs } J_n \text{ in } Z,\ n = 1, 2, \ldots, \text{ with end-points } a_n, b_n \text{ such that } J \cap J_n = \{a_n, b_n\} \text{ and } a_n < t < b_n\}.$$

Suppose that $t \in T \cap Q$. Let J_n, $n = 1, 2, \ldots$, be pairwise disjoint arcs in Z with end-points a_n, b_n such that $J \cap J_n = \{a_n, b_n\}$ and $a_n < t < b_n$. Let a (resp. b) denote the first (resp. last) point of I which is mapped by g onto t (so $a \neq b$), and put $I_a = \{y \in I\colon y \leq a\}$, $I_b = \{y \in I\colon b \leq y\}$. Note that $g^{-1}(J_n)$, $n = 1, 2, \ldots$, are pairwise disjoint continua in Y such that $g^{-1}(J_n) \cap I_a \neq \emptyset \neq g^{-1}(J_n) \cap I_b$. Since I_a, I_b are closed and disjoint, Y is not finitely Suslinian, a contradiction.

This shows that $T\cap Q=\emptyset$. Now, we prove that $R\cap Q$ is countable.

For each $t\in J$ put $N_t^- = (k')^{-1}([j_0, t]_J)$, $N_t^+ = (k')^{-1}([t, j_1]_J)$; so $M_t = N_t^- \cap N_t^+$. Take any point $t\in Q$. Then $h(\overline{W}_t)$ consists of a single point $m_t\in M_t$ such that m_t cuts X_0' between M_{j_0} and M_{j_1}. Write $g^{-1}(t)\cap C = \{a_t, b_t\}$, where $a_t<b_t$, and put $hf^{-1}(a_t) = M_t^-$, $hf^{-1}(b_t) = M_t^+$. Note that $M_t = M_t^-\cup M_t^+$ and $M_t^-\cap M_t^+ = \{m_t\}$. Moreover, $\overline{\bigcup_{s<t} N_t^-}\cap M_t^+ = \{m_t\}$ and $\overline{\bigcup_{t<s} N_s^+}\cap M_t^- = \{m_t\}$. Therefore if $M_t^-\neq\{m_t\}$ then $\overline{\bigcup_{t<s} N_s^+}\neq N_t^+$, and if $M_t^+\neq\{m_t\}$ then $\overline{\bigcup_{s<t} N_s^-}\neq N_t^-$. Thus if $t\in R\cap Q$, then either $\overline{\bigcup_{t<s} N_s^+}\neq N_t^+$ or $\overline{\bigcup_{s<t} N_s^-}\neq N_t^-$. By Lemma 4.7, $R\cap Q$ is countable.

Since $I\cap rf(A)=\emptyset$, it follows that $J\cap grf(A)=\emptyset$. Therefore there are only finitely many components of Z–J which intersect the closed set $grf(A)$. Moreover, the boundary of each such component is nowhere dense in J. Indeed, if W is a component of $Z-J$ so that $\mathrm{bd}(W)$ contains a nondegenerate subarc J' of J, then $J'-T$ is countable (see Lemma 4.6(a)), $T\cap Q=\emptyset$, and $J'\cap Q$ is uncountable (Lemma 4.8), a contradiction.

Thus we can find a nondegenerate subarc K of J with end-points k_0 and k_1, where $j_0<k_0<k_1<j_1$, such that $\mathrm{bd}(W)\cap K=\emptyset$ for each component W of $Z-J$ so that $grf(A)\cap W\neq\emptyset$.

Note that $K-(R\cup T)$ is a Borel set in K, because R is an F_σ-set and T is a $G_{\delta\sigma}$-set (Lemma 4.6(c)). Since $Q\cap R$ is countable, $Q\cap T=\emptyset$, and $K\cap Q$ is uncountable (Lemma 4.8), it follows that the set $K-(R\cup T)$ is uncountable. Therefore there is a subset K' of $K-(R\cup T)$ which is homeomorphic to the Cantor set (see for example [7], § 37, Section I, Theorem 3 (of Alexandrov–Hausdorff), p. 447). We may assume that $k_0, k_1\in K'$.

Let H' be the decomposition of Z into points and the sets $\overline{W}$ for each component W of $K-K'$. Thus H' is upper semi-continuous and monotone. Put $Z_1 = Z/H'$ and let $l\colon Z\to Z_1$ denote the quotient map. Then Z_1 is a finitely Suslinian continuum. Put $L = l(K) = l(K')$; so L is a separable (nondegenerate) arc in Z_1. Let $<$ denote the natural ordering of L such that $l|_K$ is an increasing map and put $l_0 = l(k_0)$, $l_1 = l(k_1)$; so l_0, l_1 are the end-points of L and $l_0<l_1$. Put

$$T' = t\in L\colon \text{there are pairwise disjoint arcs } J_n \text{ in } Z_1,\ n=1,2,\ldots,\ \text{with end-points } a_n,\, b_n \text{ such that } L\cap J_n = \{a_n, b_n\} \text{ and } a_n<t<b_n\}\,.$$

Since $T\cap K'=\emptyset$, it follows that $T'=\emptyset$.

Put $U_1 = \{t\in L\colon l^{-1}(t) \text{ is nondegenerate}\}$; so U_1 is countable. Observe that $P\subset R$. Therefore $l(P\cap K)\subset l(R\cap K)\subset U_1$. Put

$$U_2 = \{t\in L\colon \text{there is a component } W \text{ of } Z_1-L \text{ such that } t\in\mathrm{bd}(W) \text{ and } \mathrm{bd}(W) \text{ is nondegenerate}\}\,.$$

Because of Lemma 4.6(b), the boundary of each component of Z_1-L is countable (since $T'=\emptyset$). By Theorem 4.5, U_2 is countable. Now, put $U = U_1\cup U_2$; so U is countable.

By Lemma 4.6(e), there are $c, d\in L-(U\cup\{l_0, l_1\})$, $c<d$, such that either $\mathrm{bd}(W)\subset[c, d]_L$ or $\mathrm{bd}(W)\cap\,]c, d[_L=\emptyset$ for each component W of Z_1-L. Moreover, if W is a component of Z_1-L such that $\{c, d\}\cap\mathrm{bd}(W)\neq\emptyset$, then either $\mathrm{bd}(W)=\{c\}$ or $\mathrm{bd}(W)=\{d\}$ (because $c, d\notin U_2$). Put

$$V = \,]c, d[_L\cup\bigcup\{W\colon W \text{ is a component of } Z_1-L \text{ such that } \mathrm{bd}(W)\subset\,]c, d[_L\}$$

and note that V is a component of $Z_1-\{c, d\}$.

Since $c, d\notin U_1$, each of the sets $l^{-1}(c)$ and $l^{-1}(d)$ consists of exactly one point which does not belong to P. Therefore each of the sets $g^{-1}l^{-1}(c)\cap C$ and $g^{-1}l^{-1}(d)\cap C$ consists of either one or two points (which obviously belong to I). Put $c_1 = \max(g^{-1}l^{-1}(c)\cap C)$, $d_1 = \min(g^{-1}l^{-1}(d)\cap C)$; so $c_1<d_1$ Recall that $l(R\cap K)\subset U_1$. Hence $c_1, d_1\notin g^{-1}(R)$, and so $f^{-1}(c_1)=\{c_2\}$ and $f^{-1}(d_1)=\{d_2\}$ for some points $c_2, d_2\in B$.

We show that $V_1 = g^{-1}l^{-1}(V)$ is a component of $Y_1-\{c_1, d_1\}$. Since g, l are monotone mappings, it follows that V_1 is a component of $Y_1-g^{-1}l^{-1}(\{c, d\})$. It suffices to prove that $\mathrm{bd}(V_1)=\{c_1, d_1\}$. Observe that

$$V_1 = \,]c_1, d_1[_I\cup\bigcup\{W\colon W \text{ is a component of } Y_1-g^{-1}l^{-1}(L) \text{ such that } \mathrm{bd}(W)\subset\,]c_1, d_1[_I\}$$

(because $]c_1, d_1[_I = g^{-1}l^{-1}(]c, d[_L)$). Since Y_1 is locally connected, it follows that $\overline{V}_1 = V_1\cup\{c_1, d_1\}$, and so $\mathrm{bd}(V_1)=\{c_1, d_1\}$.

Now, we prove that $V_2 = f^{-1}r^{-1}(V_1)$ is a component of $X-\{c_2, d_2\}$. Since r, f are monotone mappings, it follows that V_2 is a component of $X-f^{-1}r^{-1}(\{c_1, d_1\})$. Since $f^{-1}(c_1)=\{c_2\}$ and $f^{-1}(d_1)=\{d_2\}$, it suffices to show that $r^{-1}(c_1)=\{c_1\}$ and $r^{-1}(d_1)=\{d_1\}$. We check that $r^{-1}(c_1)=\{c_1\}$; the proof of the second equality is analogous.

Suppose that $r^{-1}(c_1)$ is nondegenerate. Hence c_1 is a cut point of Y. Let W be any component of $Y-\{c_1\}$. Since f is monotone, $f^{-1}(W)$ is a component of $X-f^{-1}(c_1) = X-\{c_2\}$. Thus X has cut points, a contradiction.

Put $D = B-V_2$. Recall that $\mathrm{bd}(W)\cap K=\emptyset$ for each component W of $Z-J$ such that $grf(A)\cap W\neq\emptyset$ and, moreover, $K\cap grf(A)=\emptyset$. Therefore $lgrf(A)\cap V=\emptyset$, and so $A\subset D$ (because $A\subset B$). Since $lgrf(B)\cap\,]c, d[_L\neq\emptyset$, it follows that $D\neq B$. Moreover, $B-D\subset V_2$ and $c_2, d_2\in D$. Hence if W is a component of $X-D$, $W\neq V_2$, then $\overline{W}\cap(B-D)=\emptyset$, and so W is also a component of $X-B$. Thus D is a T-set in X such that $A\subset D\subset B$ and $D\neq B$. This contradicts the minimality of B.

We have finished the proof that Y_1' is metrizable. By Theorem 4.5, Y_1-Y_1' has countably many components. By (b) (see also Lemma 4.2), Y_1 is metrizable. We have shown that each cyclic element of Y is metrizable.

From now on, the only notations from the above part of the proof which will have the same meaning as previously are: X, A, B, f, Y.

Let $\{y_0, y_1, \ldots\}$ be a countable dense subset of $f(B)$. For each positive integer n, let I_n be an arc in Y with end-points y_0, y_n. Put $Y' = \overline{\bigcup_{n\geq 1} I_n}$ and observe that (by (b)) Y' is metrizable if and only if for each positive integer n the set $I_n - f(B)$ has countably many components.

Suppose that Y' is not metrizable. Hence there is an n_0 such that $I_{n_0} - f(B)$ has uncountably many components. Put $C = f(B) \cap I_{n_0}$; so C is separable, compact and zero-dimensional. Let G^C be the decomposition of I_{n_0} as in Lemma 4.8, write $J = I_{n_0}/G^C$, let $g_C\colon I_{n_0} \to J$ be the quotient map and $Q = \{t \in J\colon |g_C^{-1}(t) \cap C| = 2\}$. Put $X' = f^{-1}(I_{n_0})$; so X' is a subcontinuum of X. Note that for each $q \in Q$ there is a unique component W_q of $X' - B$ so that $g_C f(W_q) = \{q\}$ (W_q is also a component of $X - B$). Let H be the decomposition of X' into points and the sets $\overline{W}_q$, $q \in Q$. By Lemma 3.6, H is upper semi-continuous. Put $X_0' = X'/H$ and let $h\colon X' \to X_0'$ be the quotient map; so h is monotone. As in the first part of the proof, X_0' is separable, and so metrizable. Let $k\colon X_0' \to J$ be the mapping defined by the formula: $k(x) = z$ provided $g_C f(h^{-1}(x)) = \{z\}$. Then k is well-defined, continuous and monotone. Put $M_t = k^{-1}(t)$ for $t \in J$, and $R = \{t \in J\colon M_t$ is nondegenerate$\}$. As previously, $R \cap Q$ is countable (by Lemma 4.7). Since Q is uncountable (Lemma 4.8), there exists a point $a \in Q - R$ which is not an end-point of J. Since M_a is degenerate, it follows that if $b \in g_C^{-1}(a) \cap C$, then $f^{-1}(b)$ is also degenerate, say $f^{-1}(b) = \{c\}$. We show that c is a cut point of X.

Let L denote the cyclic chain from y_0 to y_{n_0} in Y, i.e. $L = D \cup S$, where $D = \{y_0, y_{n_0}\} \cup \{y \in Y\colon y$ separates y_0 and $y_{n_0}\}$ and S is a union of the family T of nondegenerate cyclic elements of Y such that $Z \in T$ if and only if Z contains exactly two points d_Z, d_Z' from the set D. Note that $\{y \in Y\colon \{y\}$ is a cyclic element of $L\} \subset D \subset C \subset I_{n_0} \subset L$ ($D \subset C$ because X has no cut points) and recall that each nondegenerate cyclic element of L is a cyclic element of Y. Since each cyclic element of Y is metrizable, it follows that if Z is a cyclic element of L, then the set $I_{n_0} \cap Z$ is a separable arc with end-points d_Z, d_Z'. Hence the set $g_C(I_{n_0} \cap Z)$ consists of a single point e_Z for each cyclic element Z of L. Define $g\colon L \to J$ by the formula $g(y) = e_Z$ provided Z is a cyclic element of L such that $y \in Z$. Obviously, $g_C^{-1}(a) = g^{-1}(a) \cap I_{n_0}$. Since $a \in Q$ and $D \subset C$, it follows that $Z = g^{-1}(a)$ is a cyclic element of L and $b \in \{d_Z, d_Z'\}$. Since a is not an end-point of J, b is a cut point of L. Therefore b is a cut point of Y. Since f is monotone, it follows that c is a cut point of X, a contradiction.

We have thus proved that Y' is metrizable. Hence $Y' - f(B)$ has countably many components. Put $Z = f^{-1}(Y')$ and note that Z is a subcontinuum of X, $B \subset Z$, and $Z - B$ has countably many components (each homeomorphic to $]0, 1[$). Hence Z is separable and, by (b), metrizable. Therefore B is also metrizable.

4.10. Corollary. *If X is a locally connected continuum without cut points which is a continuous image of some compact linearly ordered topological space, and E is a closed metrizable subset of X, then there is a metrizable T-set E' in X such that $E \subset E'$.*

Proof. See Chapter 6, the proof of the implication (iv) → (v) of Theorem 1.1.

5. Approximation by finite dendrons. Let X be a continuum and J a family of finite dendrons which approximates X. Recall that X is locally connected (this follows from a theorem of L. E. Ward, [26], Theorem 1, p. 370).

If $a, b \in \bigcup J$ then there is a unique arc $[a, b]_J$ contained in X with end-points a, b such that $[a, b]_J \subset T$ for each $T \in J$ with $a, b \in T$. Write

$$J^s(a) = \{\bigcup_{k=1}^{n} [a, b_k]_J\colon b_k \in \bigcup J,\ k = 1, \ldots, n,\ n = 1, 2, \ldots\}$$

for each $a \in \bigcup J$, and

$$J^s = \bigcup \{J^s(a)\colon a \in \bigcup J\} = \{T\colon T \text{ is a finite dendron and } T \subset T' \text{ for some } T' \in J\}.$$

It can easily be shown that $J \subset J^s$, $\bigcup J = \bigcup J^s(a) = \bigcup J^s$, $[x, y]_J = [x, y]_{J^s(a)} = [x, y]_{J^s}$ for all $x, y \in \bigcup J$, and each of the families $J^s(a)$ and J^s consists of finite dendrons and approximates X.

Now, we recall some definitions and facts from [26]. X is still a continuum approximated by a family J of finite dendrons. If T and T' are dendrons and $T \subset T'$ then there is a unique monotone retraction $r\colon T' \to T$. Hence J is an inverse system of dendrons, where the ordering of J is inclusion and bonding maps are monotone retractions. Write $T_J = \operatorname{liminv} J$; so T_J is a dendron. If $(x_T) \in T_J$ then (x_T) is a convergent net in X. Moreover, the function $g_J\colon T_J \to X$ defined by $g_J((x_T)) = \lim_{T \in J} x_T$ is a continuous surjection.

If $T \in J$ then we can treat T as a subdendron of T_J:

$$T = \{(x_S) \in T_J\colon x_S = x_T \text{ if } T \subset S\}.$$

Put $T_\omega = \bigcup J$; so $T_\omega \subset T_J$, and note that $T_J - T_\omega$ is a subset of the set of all end-points of T_J.

5.1. Lemma. *Let X be a continuum which is approximated by a family J of finite dendrons, and let A be a subset of X such that $\bigcup J \subset A$. Then there is a family J' of finite dendrons such that J' approximates X and $\bigcup J' = A$. In particular, X can be strongly approximated by finite dendrons.*

Proof. Let a be a fixed point of $\bigcup J$. For each $x \in A - \bigcup J$ choose $(x_T) \in T_J$ so that $g_J((x_T)) = x$. Note that the image $g_J([a, (x_T)])$ of the subarc $[a, (x_T)]$ of T_J

is an arc in X — indeed, $g_J|_{[a,(x_T)]}$ is one-to-one and continuous. Let

$$J' = \{\bigcup_{k=1}^{n} L_k\colon \text{ either } L_k = [a, a^k]_J \text{ for some } a^k \in \bigcup J \text{ or } L_k = g_J([a, (x_T^k)]) \text{ for some } x^k \in A - \bigcup J;\ k = 1, \ldots, n;\ n = 1, 2, \ldots\}$$

and observe that J' is a family of finite dendrons such that $J^s(a) \subset J'$ and $\bigcup J' = A$; so $\bigcup J'$ is dense in X. Obviously J' is directed by inclusion.

Let $U = \{U_1, \ldots, U_l\}$ be an open covering of X and let $V = \{V_1, \ldots, V_l\}$ be a shrink of U, i.e., V is an open covering of X so that $\overline{V}_i \subset U_i$ for $i = 1, \ldots, l$ (see [4], Theorem 1.5.18, p. 67). Let $T_V \in J^s(a)$ be such that, for each $T \in J^s(a)$, each component of $T - T_V$ is contained in some member of V. We show that if $S \in J'$, then each component S' of $S - T_V$ is contained in $\overline{V}_i \subset U_i$ for some i. This will finish the proof.

Let $s_1, \ldots, s_m$ denote all end-points of S. There is a directed set B and dendrons $S_b \in J^s(a)$, $b \in B$, such that $S - \{s_1, \ldots, s_m\} \subset \bigcup_{b \in B} S_b \subset S$ and $S_b \subset S_c$ if $b < c$. For each $b \in B$ let S'_b denote the component of $S_b - T_V$ so that $S'_b \subset S'$. Hence $S'_b \subset S'_c$ if $b < c$. Therefore there is an index $i_0 \in \{1, \ldots, l\}$ such that $S'_b \subset V_{i_0}$, for $b \in B$. Since $S' - \{s_1, \ldots, s_m\} \subset \bigcup_{b \in B} S'_b \subset S'$, it follows that $S' \subset \overline{V}_{i_0}$.

5.2. Lemma. *Let X be a locally connected continuum. The following conditions are equivalent*:

(i) *X can be strongly approximated by finite dendrons,*

(ii) *each cyclic element of X can be approximated by finite dendrons.*

Proof. We show that (ii) implies (i). The implication (i) → (ii) will not be used in this paper (it can be derived from Theorem 1.1).

For each cyclic element Y of X let J_Y denote a family of finite dendrons such that J_Y strongly approximates Y (Lemma 5.1) and $J_Y = J_Y^s$. Let a be a fixed point of X. For each point $b \in X$ let $C(a, b)$ be the cyclic chain from a to b, i.e.,

$$C(a, b) = \{a, b\} \cup \{x \in X\colon x \text{ separates } a \text{ and } b\} \cup C,$$

where C is the union of the family H of nondegenerate cyclic elements of X defined as follows: $Y \in H$ if and only if Y contains exactly two points x_Y and y_Y from the set $\{a, b\} \cup \{x\colon x \text{ separates } a \text{ and } b\}$. Put

$$[a, b] = \{a, b\} \cup \{x\colon x \text{ separates } a \text{ and } b\} \cup \bigcup_{Y \in H} [x_Y, y_Y]_{J_Y}$$

and note that $[a, b]$ is an arc (see [2], p. 254). Observe that

$$J = \{\bigcup_{k=1}^{n} [a, b_k]\colon b_k \in X,\ k = 1, \ldots, n,\ n = 1, 2, \ldots\}$$

is a directed family of finite dendrons and $\bigcup J = X$. Now, use [2], Lemma 2, p. 256, to finish the proof that J approximates X.

5.3. Lemma. *Let X be a locally connected continuum and let $a, b \in X$ be such that X is a cyclic chain from a to b. Suppose that*

(i) *each cyclic element of X which does not contain a can be approximated by finite dendrons, and*

(ii) *either $\{a\}$ is a degenerate cyclic element of X or the unique cyclic element Y of X which contains a can be approximated by a family J of finite dendrons such that $\bigcup J = Y - \{a\}$.*

Then X can be approximated by a family K of finite dendrons such that $\bigcup K = X - \{a\}$.

Proof. The family K can be constructed as in the proof of Lemma 5.2.

5.4. Lemma. *Let X be a metrizable locally connected continuum. If a is a non-cut point of X, then there exists a family J of finite dendrons such that J approximates X and $\bigcup J = X - \{a\}$.*

Proof. It can easily be shown that $X - \{a\}$ has property S (observe that if a is a local separating point, then $U - \{a\}$ has only finitely many components for each connected open neighbourhood U of a). By [1], Theorem 1, p. 1103 and Theorem 5, p. 1104, then set $X - \{a\}$ is partitionable (for the definition see [1] or [27]). Now, it is easy to apply the methods of L. E. Ward, [27], pp. 286–287, to find an increasing sequence $D_1, D_2, \ldots$ of finite dendrites such that $\{D_1, D_2, \ldots\}$ approximates X and $a \notin \bigcup_{n \geq 1} D_n$. By Lemma 5.1, there is a family J of finite dendrites such that J approximates X and $\bigcup J = X - \{a\}$.

5.5. Lemma. *Let X be a continuum, J a family of finite dendrons which strongly approximates X, and A a T-set in X such that each component of $X - A$ is homeomorphic to $]0, 1[$. Then there is a family J' of finite dendrons such that*:

(i) *J' strongly approximates X,*

(ii) *for each component Y of $X - A$ there is a point $d_Y \in \mathrm{bd}(Y)$ such that the arc $[d_Y, y]_{J'}$ is contained in $Y \cup \{d_Y\}$ for each $y \in Y$,*

(iii). *if Y is a component of $X - A$ and $d \in \mathrm{bd}(Y)$ is a point such that, for each $y \in Y$, $[d, y]_J \subset Y \cup \{d\}$, then also, for each $y \in Y$, $[d, y]_{J'} \subset Y \cup \{d\}$ (i.e., one can take $d_Y = d$), and*

(iv) *$[a, b]_J = [a, b]_{J'}$ for all $a, b \in A$.*

Proof. Let H be the family of all components Y of $X - A$ such that there are $x, y \in Y$ with the property $[a_Y, x]_J \not\subset Y \cup \{a_Y\}$ and $[b_Y, y]_J \not\subset Y \cup \{b_Y\}$, where $\mathrm{bd}(Y) = \{a_Y, b_Y\}$. Hence there is a unique point $z_Y \in Y$ such that either

(1) $[a_Y, z_Y]_J \subset Y \cup \{a_Y\}$ and $[a_Y, x]_J \not\subset Y \cup \{a_Y\}$ for $x \in Y - [a_Y, z_Y]_J$,

or

(2) $[b_Y, z_Y]_J \subset Y \cup \{b_Y\}$ and $[b_Y, y]_J \not\subset Y \cup \{b_Y\}$ for $y \in Y - [b_Y, z_Y]_J$.

Define d_Y to be b_Y if (1) holds, and a_Y if (2) holds. Let $c \in A$ be fixed and put

$$J' = \{\bigcup_{k=1}^{n} L_k\colon \text{ either } L_k = [c, x_k]_J \text{ for some } x_k \in X - \bigcup H \text{ or } L_k = [c, d_{Y_k}]_J \cup [d_{Y_k}, x_k]_{\overline{Y}_k} \text{ for some } x_k \in Y_k \in H;\ k = 1, \ldots, n;\ n = 1, 2, \ldots\}.$$

It can easily be shown that J' is a family of finite dendrons which approximates X (use Lemma 3.2), $\bigcup J' = X$, and J' fulfils (ii)–(iv).

5.6. LEMMA. *Let X be a locally connected continuum without cut points and let A be a T-set in X. Suppose that*

(i) *X_A can be strongly approximated by a family J of finite dendrons such that for each component Z of $X_A - f_A(A)$ there is a point $d_Z \in \mathrm{bd}(Z)$ so that, for each $z \in Z$, the arc $[d_Z, z]_J$ is contained in $Z \cup \{d_Z\}$, and*

(ii) *if Y is a component of $X - A$, then $\overline{Y}$ can be approximated by a family J'_Z of finite dendrons such that $\bigcup J'_Z = Y - \{b_Z\}$, where Z is the unique component of $X_A - f_A(A)$ such that $f_A(Y) \subset \overline{Z}$, and $b_Z \in \mathrm{bd}(Y)$ is the unique point of $A \cap f_A^{-1}(\mathrm{bd}(Z) - \{d_Z\})$.*

Then X can be strongly approximated by a family K of finite dendrons such that $f_A([p, q]_K) = [f_A(p), f_A(q)]_J$ for all $p, q \in A$.

Proof. For each component Z of $X_A - f_A(A)$ let Y_Z denote the unique component of $X - A$ so that $f_A(Y_Z) \subset \overline{Z}$ and let $\mathrm{bd}(Z) = \{d_Z, e_Z\}$, $\mathrm{bd}(Y_Z) = \{a_Z, b_Z\}$, so $f_A(a_Z) = d_Z$, and $f_A(b_Z) = e_Z$. Let H be the family of all components Z of $X_A - f_A(A)$ such that $[d_Z, e_Z]_J \neq \overline{Z}$. Now, put $J_Z = J'_Z$ if $Z \in H$; for $Z \notin H$ let J_Z be any family of finite dendrons which strongly approximates $\overline{Y}_Z$ (see Lemma 5.1).

Let $a \in A$. Observe that if Z is a component of $X_A - f_A(A)$, then either $e_Z \notin [f_A(a), d_Z]_J$ or $d_Z \notin [f_A(a), e_Z]_J$. For every $x \in X$ put:

(1) $L(x) = (A \cap f_A^{-1}([f_A(a), f_A(x)]_J)) \cup \bigcup \{[a_Z, b_Z]_{J_Z}\colon Z \subset [f_A(a), f_A(x)]_J$ and Z is a component of $X_A - f_A(A)\}$

if $x \in A$,

(2) $L(x) = L(a_Z) \cup [a_Z, x]_{J_Z}$ if $x \in Y_Z$ for a component Z of $X_A - f_A(A)$ so that $e_Z \notin [f_A(a), d_Z]_J$, and

(3) $L(x) = L(b_Z) \cup [b_Z, x]_{J_Z}$ if $x \in Y_Z$ for a component Z of $X_A - f_A(A)$ so that $d_Z \notin [f_A(a), e_Z]_J$ (note that in this case $Z \notin H$).

We show that the sets $L(x)$, $x \in X$, are arcs.

First consider the case $x \in A$. Suppose that there is a point $y \in \overline{L(x)} - L(x)$. If $y \notin A$, then there is a component Y of $X - A$ such that $y \in Y$, $Y = Y_Z$ for some component Z of $X_A - f_A(A)$. Observe that $Z \subset [f_A(a), f_A(x)]_J$ because $x \in A$. Therefore $L(x) \cap \overline{Y}_Z = [a_Z, b_Z]_{J_Z}$ and $Y_Z - [a_Z, b_Z]_{J_Z}$ is a neighbourhood of y disjoint from $L(x)$, a contradiction which shows that $y \in A$. Note that

$$f_A^{-1}([f_A(a), f_A(x)]_J) \cap A = L(x) \cap A$$

is closed and $f_A(y) \in \overline{[f_A(a), f_A(x)]_J} = [f_A(a), f_A(x)]_J$; so $y \in L(x) \cap A$, again a contradiction. Thus $L(x)$ is compact. Put

$$L = (A \cap f_A^{-1}([f_A(a), f_A(x)]_J)) \cup \bigcup \{Y_Z\colon Z \text{ is a component of } X_A - f_A(A) \text{ and } Z \subset [f_A(a), f_A(x)]_J\}.$$

By Lemma 2.3, L is a continuum such that

(*) if either $M = \{y\}$ and $y \in L \cap A - \{a, x\}$, or M is a component of $X - A$ so that $M \subset L$, then $L - M$ is a union of two mutually separated connected sets P, Q such that $a \in P$ and $x \in Q$.

Observe that $L(x) \subset L$ and $L(x) \cap A = L \cap A$. Suppose that $L(x)$ is not connected, i.e. there are two closed disjoint and nonempty sets F, G such that $L(x) = F \cup G$. Put

$$F' = F \cup \bigcup \{Y_Z\colon [a_Z, b_Z]_{J_Z} \subset L(x) \text{ and } [a_Z, b_Z]_{J_Z} \cap F \neq \emptyset\},$$

$$G' = G \cup \bigcup \{Y_Z\colon [a_Z, b_Z]_{J_Z} \subset L(x) \text{ and } [a_Z, b_Z]_{J_Z} \cap G \neq \emptyset\},$$

and note that F', G' are closed, disjoint and $L = F' \cup G'$, a contradiction. By (*), it follows that each point of $L(x) - \{a, x\}$ is a cut point of $L(x)$, and so $L(x)$ is an arc ([6], Theorem 2–25).

If $x \notin A$, then $L(x)$ is a union of two arcs with exactly one common point which is an end-point of each of them. We have thus shown that $L(x)$ is an arc for each $x \in X$.

Let $x, x' \in X$. We show that $L(x) \cap L(x')$ is connected. Note that if suffices to consider the case $x, x' \in A$. In this case, $f_A(L(x)) = [f_A(a), f_A(x)]_J$ and $f_A(L(x')) = [f_A(a), f_A(x')]_J$. There is a $z \in X_A$ such that

$$[f_A(a), f_A(x)]_J \cap [f_A(a), f_A(x')]_J = [f_A(a), z]_J.$$

Obviously, $z \in f_A(A)$. Let x_0 be the unique point of $f_A^{-1}(z) \cap A$; note that $L(x) \cap L(x') = L(x_0)$. Therefore $L(x_1) \cup \ldots \cup L(x_n)$ is a finite dendron for any $x_1, \ldots, x_n \in X$, $n = 1, 2, \ldots$ Put

$$K = \{L(x_1) \cup \ldots \cup L(x_n)\colon x_1, \ldots, x_n \in X,\ n = 1, 2, \ldots\}.$$

Then K is a family of finite dendrons such that $\bigcup K = X$ and K is directed by inclusion. Moreover, the equality $[f_A(a), f_A(x)]_J = f_A(L(x))$ for $x \in A$ implies that $f_A([p, q]_K) = [f_A(p), f_A(q)]_J$ for all $p, q \in A$ (because $L(x) = [a, x]_K$ for $x \in X$).

Let $U = \{U_1, \ldots, U_n\}$ be an open covering of X. For each component Z of $X_A - f_A(A)$ choose $p_Z, q_Z, r_Z \in Z$ such that $d_Z < p_Z < q_Z < r_Z < e_Z$ in a natural ordering of $\overline{Z}$. Let $U^A = \{U_0^A, \ldots, U_n^A\}$ be an open covering of X_A constructed in Lemma 3.7. Put

$$D = \{Z\colon Z \text{ is a component of } X_A - f_A(A) \text{ such that } \overline{Y}_Z \text{ is not contained in any member of } U\};$$

so D is finite (Lemma 3.2). For each $Z \in D$ let $T_Z^U \in J_Z$ be such that, for each $T_Z \in J_Z$ and each component T'_Z of $T_Z - T_Z^U$, T'_Z is contained in some member of the open covering $\{U_1 \cap \overline{Y}_Z, \ldots, U_n \cap \overline{Y}_Z\}$ of $\overline{Y}_Z$.

Let $S_U \in J$ be such that for each $S \in J$ and each component S' of $S - S_U$ there is a k such that $S' \subset U_k^A$. We may assume that $S_U \cap Z \neq \emptyset$ for each $Z \in D$, and that

$$S_U = \bigcup_{i=1}^{l} [f_A(a), f_A(a_i)]_J \cup \bigcup_{i=1}^{m} [f_A(a), q_{Z_i}]_J$$

for some $a_1, \ldots, a_l \in A$ and some components $Z_1, \ldots, Z_m$ of $X_A - f_A(A)$ such that $Z_1, \ldots, Z_m \in H$. Put

$$T_U = \bigcup_{i=1}^{l} L(a_i) \cup \bigcup_{i=1}^{m} L(a_{Z_i}) \cup \bigcup \{L(x)\colon x \text{ is an end-point of } T_Z^U \text{ for some } Z \in D\};$$

so $T_U \in K$. Let T' be a component of $T - T_U$ for some $T \in K$, and let t be the unique point such that T' is a component of $T - \{t\}$; so $t \in T \cap T_U$. We show that T' is contained in some member of U.

First, suppose that $t \notin A$. Hence $t \in Y_Z$ for some component Z of $X_A - f_A(A)$. If $Z \in H$ then $a_Z \in R$ for each $R \in K$ such that $R \cap Y_Z \neq \emptyset$. Moreover, $b_Z \notin T'$, and so $T' \subset Y_Z$. If $Z \notin H$ then $a_Z, b_Z \in T_U$. Thus $T' \subset Y_Z$. If $Z \in H$ then $Z \in D$, so $T_Z^U \subset T_U$. Therefore T' is contained in some component T'' of $(\overline{Y}_Z \cap T) - T_Z^U$ (note that $\overline{Y}_Z \cap T$ is a dendron belonging to J_Z), and there is an $i \in \{1, \ldots, n\}$ such that $T' \subset T'' \subset \overline{Y}_Z \cap U_i \subset U_i$. If $Z \notin D$ then $T' \subset Y_Z \subset \overline{Y}_Z \subset U_i$ for some i.

Now, suppose that $t \in A$. We may assume that $T' \cap Y_Z = \emptyset$ for each component Z of $X_A - f_A(A)$ such that $Z \cap S_U \neq \emptyset$ (because if $T' \cap Y_Z \neq \emptyset$ for some Z which intersects S_U, then the same argument as in the case $t \notin A$ can be applied to show that $T' \subset Y_Z$, and so $T' \subset U_i$ for some i). Hence $T' \cap Y_Z = \emptyset$ for each $Z \in D$. Put

$$S' = \bigcup \{]f_A(t), f_A(c)]_J\colon c \text{ is an end-point of } \overline{T'} \text{ and } c \in A\} \cup \bigcup \{]f_A(t), q_Z]_J\colon \text{some end-point of } \overline{T'} \text{ belongs to } Y_Z\}$$

and note that $S = S' \cup S_U \in J$ and S' is a component of $S - S_U$. Hence there is an $i \in \{0, \ldots, n\}$ such that $S' \subset U_i^A$. Observe that $i \neq 0$. We show that $T' \subset U_i$.

Note that $U_i \cap A = f_A^{-1}(U_i^A) \cap A$ and $T' \cap A = f_A^{-1}(S') \cap A$; so $T' \cap A \subset U_i \cap A \subset U_i$. Take any $x \in T' - A$ and let Z be the component of $X_A - f_A(A)$ such that $x \in Y_Z$. Observe that $Z \cap S' \neq \emptyset$. If $Z \subset S'$ then $Z \subset U_i^A$, and so $x \in Y_Z \subset \overline{Y}_Z \subset U_i$. If $Z \not\subset S'$, then $q_Z \in S'$ (because $Z \cap S' \neq \emptyset$). Thus $q_Z \in U_i^A$ and $i \neq 0$. Therefore $\overline{Y}_Z \subset U_i$.

5.7. Lemma. *Let X be a locally connected continuum without cut points, and let $A_1, A_2, \ldots$ be T-sets in X such that $A_1 \subset A_2 \subset \ldots$ and if Y is a component of $X - A_n$ for some $n \in \{1, 2, \ldots\}$ and Z is a nondegenerate cyclic element of $\overline{Y}$, then $\{y \in Y\colon y \text{ cuts } \overline{Y}\} \subset A_{n+1}$ and $|Z \cap A_{n+1}| > 2$. Suppose that, for each n, there is a family J_n of finite dendrons which strongly approximates X_{A_n} such that for each component Z of $X_{A_n} - f_{A_n}(A_n)$ there is a $d_Z \in \mathrm{bd}(Z)$ so that, for each $z \in Z$, the arc $[d_Z, z]_{J_n}$ is contained in $Z \cup \{d_Z\}$. Put*

$$L_n(p, q) = \big(f_{A_n}^{-1}([f_{A_n}(p), f_{A_n}(q)]_{J_n}) \cap A_n\big) \cup \bigcup \{Y\colon Y \text{ is a component of } X - A_n \text{ such that } f_{A_n}(Y) \subset [f_{A_n}(p), f_{A_n}(q)]_{J_n}\}$$

for each $n \in \{1, 2, \ldots\}$ and all $p, q \in A$.

If there is an $a \in A_1$ such that $L_{n+1}(a, p) \subset L_n(a, p)$ for each $n \in \{1, 2, \ldots\}$ and each $p \in A_n$, then X can be approximated by finite dendrons.

Proof. Put $A = \bigcup_{n \geq 1} A_n$; so A is dense in X (Lemma 3.4). If $p \in A_n$ for some n, then let $L(p) = \bigcap_{k \geq n} L_k(a, p)$. Thus $L(p)$ is defined for each $p \in A$.

Take any $p \in A$ and let n be so that $p \in A_n$. Since $L(p)$ is an intersection of a decreasing sequence of continua, it is a subcontinuum of X. Obviously, $a, p \in L(p)$. If $x \in L(p) \cap A$ and $a \neq x \neq p$, then x is a cut point of $L(p)$ — indeed, x is a cut point of $L_m(a, p)$ provided $x \in A_m$ and $m \geq n$. Suppose that $x \in L(p) - A$. For each k, $k \geq n$, let Y_k be the component of $X - A_k$ so that $x \in Y_k$. Note that $Y_k \subset L_k(a, p)$ for each $k \geq n$. By Lemma 2.2, $\bigcap_{k \geq n} \overline{Y}_k = \{x\}$. Moreover, the sets $\mathrm{bd}(Y_k)$, $k = n, n+1, \ldots$, can be labelled $\{a_k, b_k\}$ in a unique manner such that there are components G_k of $L(p) - \{a_k\}$ and H_k of $L(p) - \{b_k\}$ so that $H_k \cap G_k = \emptyset$, $a \in G_k$, and $p \in H_k$ (see Lemma 2.3). Note that G_k, H_k are open in $L(p)$. Put $G = \bigcup_{k \geq n} G_k$ and $H = \bigcup_{k \geq n} H_k$. Observe that $G \cup H = L(p) - \{x\}$ (because $G_k \cup H_k \cup \{a_k, b_k\} \subset L(p) \subset G_k \cup H_k \cup \{a_k, b_k\} \cup Y_k$ and $G_k \subset G_{k'}$, $H_k \subset H_{k'}$ if $k \leq k'$), $G \cap H = \emptyset$, and G, H are both open in $L(p)$ and connected. We have shown that each point of $L(p) - \{a, p\}$ is a cut point of $L(p)$. Thus $L(p)$ is an arc ([6], Theorem 2–25; see also [21], the proof of Theorem 9).

Take any $p, q \in A$. We show that $L(p) \cap L(q)$ is connected. Let n be such that $p, q \in A_n$. Observe that

$$[f_{A_n}(a), f_{A_n}(p)]_{J_n} \cap [f_{A_n}(a), f_{A_n}(q)]_{J_n} = [f_{A_n}(a), f_{A_n}(r)]_{J_n}$$

for some $r \in A_n$. Thus $L_n(a, p) \cap L_n(a, q) = L_n(a, r)$. Since

$$L(p) \cap A = \bigcup_{k \geq n} (L_k(a, p) \cap A_k),$$

it follows that $r \in L(p) \cap L(q)$. Therefore, for each $k \geq n$, $r \in L_k(a, p) \cap L_k(a, q)$. It follows that $L_k(a, r) = L_k(a, p) \cap L_k(a, q)$ for each $k \geq n$. Thus $L(r) = L(p) \cap L(q)$, and so $L(p) \cap L(q)$ is connected. Put

$$K = \{\bigcup_{i=1}^{m} L(p_i)\colon p_i \in A;\ i = 1, \ldots, m;\ m = 1, 2, \ldots\}.$$

Hence K is a family of finite dendrons which is directed by inclusion. Moreover, $A \subset \bigcup K$, and so $\bigcup K$ is dense in X.

Let $U = \{U_1, \ldots, U_k\}$ be an open covering of X. By Lemma 3.4, there is an integer n so that $\overline{Y}$ is contained in some member of U, for each component Y of $X - A_n$.

Let Z be a component of $X_{A_n} - f_{A_n}(A_n)$. Write $\mathrm{bd}(Z) = \{d_Z, e_Z\}$, where $[d_Z, z]_{J_n} \subset Z \cup \{d_Z\}$ for each $z \in Z$. Let Y_Z be the unique component of $X - A_n$ such that $f_{A_n}(Y_Z) \subset \overline{Z}$, and write $\mathrm{bd}(Y_Z) = \{a_Z, b_Z\}$, where $f_{A_n}(a_Z) = d_Z$ and

$f_{A_n}(b_Z) = e_Z$. Let $p_Z, q_Z, r_Z \in Z$ be such that $d_Z < p_Z < q_Z < r_Z < e_Z$ in a natural ordering of $\bar{Z}$.

Let $U^{A_n} = \{U_0^{A_n}, \ldots, U_k^{A_n}\}$ be an open covering of X_{A_n} constructed in Lemma 3.7. Observe that $U_0^{A_n} = \emptyset$. Let $S_U \in J_n$ be such that for each $S \in J_n$ and each component S' of $S - S_U$ there is an m such that $S' \subset U_m^{A_n}$. We may assume that

$$S_U = \bigcup_{i=1}^{j} [f_{A_n}(a), f_{A_n}(a_i)]_{J_n} \cup \bigcup_{i=1}^{l} [f_{A_n}(a), q_{Z_i}]_{J_n}$$

for some $a_1, \ldots, a_j \in A_n$ and some components $Z_1, \ldots, Z_l$ of $X_{A_n} - f_{A_n}(A_n)$ such that

$$Z_1, \ldots, Z_l \in F = \{Z\colon Z \text{ is a component of } X_{A_n} - f_{A_n}(A_n) \text{ such that } [d_Z, e_Z]_{J_n} \neq \bar{Z}\}$$

Put

$$T_U = \bigcup_{i=1}^{j} L(a_i) \cup \bigcup_{i=1}^{l} L(a_{Z_i})\,;$$

so $T_U \in K$. Let T' be a component of $T - T_U$ for some $T \in K$ and let t be the unique point such that T' is a component of $T - \{t\}$; so $t \in T \cap T_U$. We show that T' is contained is some member of U.

Suppose that $t \notin A_n$. Hence $t \in Y_Z$ for some component Z of $X_{A_n} - f_{A_n}(A_n)$. Observe that $Z \notin F$; so $a_Z, b_Z \in T_U$. Thus $T' \subset Y_Z$ and therefore T' is contained in some $U_i \in U$.

Suppose now that $t \in A_n$. We may assume that $T' \cap Y_Z = \emptyset$ for each component Z of $X_{A_n} - f_{A_n}(A_n)$ such that $Z \cap S_U \neq \emptyset$ (because if not, then $T' \subset Y_Z \subset U_i$ for some i). Put

$$S' = \bigcup \{]f_{A_n}(t), f_{A_n}(c)]_{J_n}\colon c \text{ is an end-point of } \bar{T}' \text{ and } c \in A_n\} \cup \\ \cup \bigcup \{]f_{A_n}(t), q_Z]_{J_n}\colon \text{some end-point of } \bar{T}' \text{ belongs to } Y_Z\}$$

and note that $S = S' \cup S_U \in J_n$ and S' is a component of $S - S_U$. Hence there is an $i \in \{1, \ldots, k\}$ such that $S' \subset U_i^{A_n}$. An argument analogous to that used at the end of the proof of Lemma 5.6 shows that $T' \subset U_i$. We have thus shown that K is a family of finite dendrons which approximates X.

6. Proof of Theorem 1.1. (i) → (ii) is obvious.

(ii) → (iii) was shown by L. E. Ward, [26], Theorem 1, p. 370.

(iii) → (iv). Recall that each arc is a compact linearly ordered topological space. Moreover, each arc is locally connected; so each Hausdorff space which is a continuous image of some arc is locally connected ([6], Theorem 3-22).

(iv) → (v). Let Y be any cyclic element of X; so Y is a locally connected continuum without cut points which is a continuous image of some compact linearly ordered space. L. B. Treybig has shown ([21], Theorem 4) that for each countable subset E' of Y there is a separable T-set E in Y such that $E' \subset E$. Thus we get (a). Condition (b) follows immediately from [18], Lemma 2, p. 867; see [4], Problem (read: Exercise) 3.12.4 (c), p. 281, for more general facts. Condition (c) follows immediately from a result of L. B. Treybig ([19], Theorem 1, p. 417).

(v) → (vi) Follows from Theorem 4.9.

(vi) → (vii). Let Y be any nondegenerate cyclic element of X. Let p, q, r be any distinct points of Y and let A_1 be a metrizable T-set in Y such that $p, q, r \in A_1$; so (D) holds.

Suppose that we have already constructed a T-set A_n in Y for some positive integer n. Put $H = \{Z\colon Z$ is a component of $Y - A_n\}$ and, for each $Z \in H$, put $H_Z = \{C\colon C$ is a nondegenerate cyclic element of $\bar{Z}\}$, $\{a_Z, b_Z\} = \mathrm{bd}(Z)$ (recall that $\bar{Z}$ is a cyclic chain from a_Z to b_Z), and $A'_Z = \{a_Z, b_Z\} \cup \{y \in Z\colon y$ cuts $\bar{Z}\}$ (recall that A'_Z is a closed subset of Y).

Take any $Z \in H$ and $C \in H_Z$. Note that $C \cap A'_Z$ consists of exactly two points p_C and q_C. Let r_C be any point of $C - \{p_C, q_C\}$. Let A'_C be a metrizable T-set in Y such that $p_C, q_C, r_C \in A'_C$, and put $A_C = C \cap A'_C$. Observe that A_C is a metrizable T-set in C such that $p_C, q_C, r_C \in A_C$. Put $A_Z = A'_Z \cup \bigcup \{A_C\colon C \in H_Z\}$. Then A_Z is a T-set in $\bar{Z}$ — indeed, A_Z is closed and each component of $\bar{Z} - A_Z$ is a component of $C - A_C$ for some $C \in H_Z$.

Now, put $A_{n+1} = A_n \cup \bigcup \{A_Z\colon Z \in H\}$. By Lemma 3.1, A_{n+1} is a closed subset of Y. Note that if Z' is a component of $Y - A_{n+1}$, then there is a $Z \in H$ such that Z' is a component of $\bar{Z} - A_Z$ (because $A_n \subset A_{n+1}$). Therefore A_{n+1} is a T-set in Y. It is obvious that conditions (B) and (C) hold.

(vii) → (i). Let Y be a nondegenerate cyclic element of X. By Lemma 5.2, it suffices to prove that Y can be approximated by finite dendrons. By Lemma 5.7, we have to show that, for each positive integer n, there is a family J_n of finite dendrons which strongly approximates Y_{A_n} in such a manner that:

(1) for each component V of $Y_{A_n} - f_{A_n}(A_n)$ there is a $d_V \in \mathrm{bd}(V)$ so that, for each $v \in V$ the arc $[d_V, v]_{J_n}$ is contained in $V \cup \{d_V\}$, and

(2) there is an $a \in A_1$ so that, for each $n \in \{1, 2, \ldots\}$ and $b \in A_n$, the following inclusion holds: $L_{n+1}(b) \subset L_n(b)$, where

$$L_n(b) = (f_{A_n}^{-1}([f_{A_n}(a), f_{A_n}(b)]_{J_n}) \cap A_n) \cup \bigcup \{U\colon U \text{ is a component of } Y - A_n \text{ such that } f_{A_n}(U) \subset [f_{A_n}(a), f_{A_n}(b)]_{J_n}\}.$$

Choose any point $a \in A_1$. Since A_1 is metrizable, Y_{A_1} is a metrizable locally connected continuum (Lemma 4.2). By [26], Theorem 2, p. 373, Y_{A_1} can be approximated by finite dendrons. By Lemma 5.5, there is a family J_1 of finite dendrons which strongly approximates Y_{A_1} in such a manner that (1) holds.

Suppose that for some positive integer n a family J_n of finite dendrons is already constructed such that J_n strongly approximates Y_{A_n} and (1), (2) hold.

Note that $A = f_{A_{n+1}}(A_n)$ is a T-set in $Z = Y_{A_{n+1}}$ and let $g\colon Z \to Z/G_A = Z_A$ be the quotient map (here G_A is as in Lemma 2.1). By Lemma 3.5, there is a homeomorphism $h\colon Y_{A_n} \to Z_A$ such that $hf_{A_n}(b) = gf_{A_{n+1}}(b)$ for each $b \in A_n$, and moreover:

(*) if U is a component of $Y-A_n$, if V is the unique component of $Z-A$ such that $f_{A_{n+1}}(U)\subset\overline{V}$ and if W is the unique component of $Z_A-g(A)$ such that $g(V)\subset\overline{W}$, then W is the unique component of $Z_A-g(A)$ such that both $gf_{A_{n+1}}(U)\subset\overline{W}$ and $hf_{A_n}(U)\subset\overline{W}$.

Put $J=\{h(T)\colon T\in J_n\}$; so J is a family of finite dendrons which strongly approximates Z_A and, by (1), for each component N of $Z_A-g(A)$ there is a $d_N\in\mathrm{bd}(N)$ so that, for each $z\in N$, the arc $[d_N,z]_J$ is contained in $N\cup\{d_N\}$.

Let N be a component of $Z_A-g(A)$. Let M_N be the unique component of $Z-A$ so that $g(M_N)\subset\overline{N}$ and let a_N be the unique point of $A\cap g^{-1}(d_N)$. Hence $a_N\in\mathrm{bd}(M_N)$; let b_N be the second point of $\mathrm{bd}(M_N)$. Note that M_N is a cyclic chain from a_N to b_N. Let C be a nondegenerate cyclic element of $\overline{M}_N$. Put

$$A_C=C\cap f_{A_{n+1}}(A_{n+1}).$$

Note that if U is the unique component of $Y-A_n$ such that $f_{A_{n+1}}(U)\subset\overline{M}_N$, then there is a unique cyclic element C' of $\overline{U}$ such that $f_{A_{n+1}}(C'\cap A_{n+1})=A_C$. By (C), A_C is metrizable. Since A_C is a T-set in C and each component of $C-A_C$ is homeomorphic to $]0,1[$, it follows that C is metrizable (Lemma 4.2). Thus C is a metrizable locally connected continuum. By [26], Theorem 2, p. 373, C can be approximated by finite dendrons. Moreover, if $b_N\in C$, then b_N is a non-cut point of C; so C can be approximated by a family I of finite dendrons such that $\bigcup I=C-\{b_N\}$ (Lemma 5.4). By Lemma 5.3, M_N can be approximated by a family J'_N of finite dendrons such that $\bigcup J'_N=M_N-\{b_N\}$.

By Lemma 5.6, Z can be strongly approximated by a family K of finite dendrons such that $g([p,q]_K)=[g(p),g(q)]_J$ for all $p,q\in A$. By Lemma 5.5, there is a family K' of finite dendrons which strongly approximates Z such that $[p,q]_K=[p,q]_{K'}$ for all $p,q\in A$ and, moreover, for each component V of $Z-f_{A_{n+1}}(A_{n+1})=Y_{A_{n+1}}-f_{A_{n+1}}(A_{n+1})$ there is a $d_V\in\mathrm{bd}(V)$ so that, for each $v\in V$, the arc $[d_V,v]_{K'}$ is contained in $V\cup\{d_V\}$.

Put $J_{n+1}=K'$; so (1) holds. Moreover, $g([p,q]_{J_{n+1}})=[g(p),g(q)]_J$ for all $p,q\in A=f_{A_{n+1}}(A_n)$ (Lemma 5.6).

Let $b\in A_n$ be a fixed point. We show that $L_{n+1}(b)\subset L_n(b)$. Observe that $g([f_{A_{n+1}}(a),f_{A_{n+1}}(b)]_{J_{n+1}})=[gf_{A_{n+1}}(a),gf_{A_{n+1}}(b)]_J$. It follows that

$$(**)\quad h^{-1}g([f_{A_{n+1}}(a),f_{A_{n+1}}(b)]_{J_{n+1}})=h^{-1}([gf_{A_{n+1}}(a),gf_{A_{n+1}}(b)]_J)=$$
$$=[h^{-1}gf_{A_{n+1}}(a),h^{-1}gf_{A_{n+1}}(b)]_{J_n}=[f_{A_n}(a),f_{A_n}(b)]_{J_n}.$$

Therefore $L_{n+1}(b)\cap A_n=L_n(b)\cap A_n$. Take any point $y\in L_{n+1}(b)-A_n$. We show that $y\in L_n(b)$.

Let U be the component of $Y-A_n$ such that $y\in U$, let V be the unique component of $Y_{A_{n+1}}-f_{A_{n+1}}(A_n)=Z-A$ such that $f_{A_{n+1}}(U)\subset\overline{V}$, and let W be the unique component of $Z_A-g(A)$ such that $g(V)\subset\overline{W}$. Thus W is the unique component of $Z_A-g(A)$ such that $gf_{A_{n+1}}(U)\subset\overline{W}$.

Since $L_{n+1}(b)\cap A_n=L_n(b)\cap A_n$, U is a component of $Y-A_n$, and $U\cap L_{n+1}(b)\neq\emptyset$, it follows that $\mathrm{bd}(U)\cap L_{n+1}(b)\neq\emptyset$. Suppose that $\mathrm{bd}(U)=\{c,d\}$ is not contained in $L_{n+1}(b)$. We may assume that $c\in L_{n+1}(b)$ and $d\notin L_{n+1}(b)$. Suppose that $c\notin\{a,b\}$. By Lemma 2.3, $L_{n+1}(b)-\{c\}$ has exactly two components P, Q so that $a\in P$ and $b\in Q$. Note that the nonempty set $U\cap L_{n+1}(b)$ is a union of some family of components of $L_{n+1}(b)-\{c\}$ (since $d\notin L_{n+1}(b)$) but $a,b\notin U$, a contradiction. Suppose that $c\in\{a,b\}$. Hence $L_{n+1}(b)-\{c\}$ is connected; so $L_{n+1}(b)\subset U\cup\{c\}$, again a contradiction (because $L_{n+1}(b)$ is nondegenerate, and so $a\neq b$). We have thus proved that $\mathrm{bd}(U)\subset L_{n+1}(b)$.

Since $L_{n+1}(b)$ is a continuum, $\mathrm{bd}(U)\subset L_{n+1}(b)$, and $y\in U\cap L_{n+1}(b)\neq\emptyset$, it follows that $L_{n+1}(b)\cap\overline{U}$ is a continuum. Since W is homeomorphic to $]0,1[$, $\overline{W}=gf_{A_{n+1}}(\overline{U})$, and $\mathrm{bd}(W)=g(\mathrm{bd}(V))=gf_{A_{n+1}}(\mathrm{bd}(U))$, it follows that $gf_{A_{n+1}}(L_{n+1}(b)\cap\overline{U})=\overline{W}$. Observe that, by (**),

$$\overline{W}\subset gf_{A_{n+1}}(L_{n+1}(b))=g([f_{A_{n+1}}(a),f_{A_{n+1}}(b)]_{J_{n+1}})=h([f_{A_n}(a),f_{A_n}(b)]_{J_n}),$$

and so $h^{-1}(\overline{W})\subset[f_{A_n}(a),f_{A_n}(b)]_{J_n}$. By (*), $h^{-1}(W)$ is the unique component of $Y_{A_n}-f_{A_n}(A_n)$ such that $f_{A_n}(U)\subset\overline{h^{-1}(W)}=h^{-1}(\overline{W})$. Therefore $y\in U\subset L_n(b)$. This proves (2).

Adeed in proof. We have pointed out a mistake in the proof of the main result of [15] (see Introduction, above). Recently we have proved that each hereditarily locally connected continuum is a continuous image of an arc, thus the proof in [15] cannot be repaired.

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On irreducibility and indecomposability of continua

by

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Abstract. Kuratowski (1927) showed that in metric continua their points of indecomposability are always points of irreducibility. The aim of this paper is to exhibit a general form of those Hausdorff continua for which the result of Kuratowski does not hold.

1. Introduction and preliminaries. In this paper X will always be a *Hausdorff continuum*, shortly a $\mathcal{T}_2$-*continuum*, i.e. a connected and compact topological space which satisfies the $\mathcal{T}_2$-axiom of separability. A point a of X is said to be a *point of indecomposability* of X if there is no decomposition of X into two proper subcontinua which both contain a, i.e. for every two subcontinua K_1 and K_2 of X

(1.1) $a \in K_1 \cap K_2$ and $K_1 \cup K_2 = X$ imply $K_1 = X$ or $K_2 = X$.

A point a of X is said to be a *point of irreducibility* of X if there is $b \in X$ such that no proper subcontinuum of X contains both a and b, i.e for every subcontinuum K of X

(1.2) $a \in K$ and $b \in K$ imply $K = X$;

X is then said to be *irreducible between a and b.*

Directly by the above two definitions, every point of irreducibility is a point of indecomposability, and the converse assertion:

(1.3) Every point of indecomposability is a point of irreducibility

has been proved for metric continua in [10] (Théorème XIX, p. 270). In connection with some fixed point theorems [4], [12], [13] and [15], the assertion (1.3) has been proved for hereditarily decomposable $\mathcal{T}_2$-continua in [14] (Theorem 1, p. 52, where in fact no axiom of separability is used). In [2], a $\mathcal{T}_2$-continuum has been constructed which is indecomposable but not irreducible, i.e. its every point is a point of indecomposability but no point is a point of irreducibility, so that the above assertion (1.3) is not true for an arbitrary $\mathcal{T}_2$-continuum X.

In the present paper, we shall characterize those $\mathcal{T}_2$-continua X which have this singularity, i.e. such X that

(1.4) There exists a point of indecomposability of X which is not a point of irreducibility of X;