Images of arcs — a nonseparable version of the Hahn–Mazurkiewicz theorem

by

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Abstract. Some conditions equivalent to the following one: "X is a Hausdorff space which is a continuous image of some arc", are given. The most important of them are: "X is a locally connected continuum which is a continuous image of some compact linearly ordered topological space" and "X is a continuum which can be approximated by finite dendrons".

1. Introduction. The classical Hahn–Mazurkiewicz theorem ([5], [11]) states that a Hausdorff space X is a continuous image of a metrizable arc (i.e., a space homeomorphic to [0, 1]) if and only if X is a locally connected metrizable continuum. The purpose of this paper is to characterize spaces which are continuous images of arcs.

There are two fine survey articles, [9] and [23], dealing with continuous images of arcs and compact linearly ordered spaces. Therefore in this paper we will not try to give any survey of the known results. Note only that there are some new papers, [15], [16], [21], [22] and [30], related to the topics which will be discussed below (however, there is a serious mistake in [15] — the set X constructed there on page 339 is not compact; see also Theorem 4.5 below).

First, we recall some basic definitions and facts and introduce some notation.

A continuum is a compact connected Hausdorff space. An arc is a continuum with exactly two non-cut points. Arcs are precisely compact connected linearly ordered topological spaces. Each separable arc is homeomorphic to the closed interval [0, 1] of real numbers.

A continuum X is said to be a dendron if for any two distinct points p and q of X there is a point r \in X so that p and q lie in distinct components of X\setminus \{r\}. Metric dendrons are precisely dendrites. A point x of a dendron X is said to be an end-point of X provided x is a non-cut point of X. A dendron is said to be finite if it has only finitely many end-points. Dendrons are often called "trees", however, we reserve the word "tree" for quite different mathematical objects (see Chapter 3).

An interesting survey of results on dendrons can be found in [12].
We assume that the reader is familiar with the theory of cyclic elements in metrizable locally connected continuum (see for example [28]). The theory can be extended to the general case (see [29], [2] and [20]) and we will use immediate generalizations of some of its definitions and theorems without comment.

We often use nets in general topological spaces. All needed definitions and facts concerning nets can be found in [4].

If \( X \) is a topological space, \( x, y \in X \), and there is exactly one arc in \( X \) with end-points \( x, y \), then this arc will be denoted by \([x, y]\) or \([x, y]\) if a confusion is possible. As usual, we write \([x, y] = [x, y] - [x, y] - [x, y] \). Let \( X \) be a continuum and \( J \) a family of finite dendrons contained in \( X \). We say that \( J \) approximates \( X \) provided: (1) \( J \) is directed by inclusion, (2) \( \bigcup J \) is dense in \( X \), and (3) if \( U \) is an open covering of \( X \), then there exists \( T_j \in J \) such that if \( T \in J \) and \( T \) is a component of \( T - T_j \), then \( T \) is contained in some member of \( U \) (this notion was introduced by L. E. Ward in [26]). If, in addition, (2) \( \bigcup J = X \); then we say that \( J \) strongly approximates \( X \).

Let \( X \) be a locally connected continuum and \( A \) a subset of \( X \). We say that \( A \) is a \( T \)-set in \( X \) if \( A \) is closed and each component of \( X - A \) has a two-point boundary.

Now we can state the main result of the paper:

1.1. THEOREM. If \( X \) is a continuum, then the following conditions are equivalent:

(i) \( X \) can be strongly approximated by finite dendrons;
(ii) \( X \) can be approximated by finite dendrons;
(iii) \( X \) is a continuous image of some arc;
(iv) \( X \) is locally connected and is a continuous image of some compact linearly ordered topological space;
(v) \( X \) is locally connected and for each non-degenerate cyclic element \( Y \) of \( X \) the following conditions hold:

(a) if \( x, y, z \in Y \) are any points of \( Y \), then there is a separable \( T \)-set \( E \) in \( Y \) such that \( p, q, r \in E \),
(b) if \( E, E' \) are separable, then \( E \) is also separable, and
(c) if \( E' \) is a continuous and monotone image of \( E \) and \( E \) is a separable continuum in \( E' \) then \( E \) is metrizable.

(vi) \( X \) is locally connected and if \( Y \) is a non-degenerate cyclic element of \( X \) and \( p, q, r \in Y \), then there is a metrizable \( T \)-set \( A \) in \( Y \) such that \( p, q, r \in A \);
(vii) \( X \) is locally connected and if \( Y \) is a non-degenerate cyclic element of \( X \), then there is a collection \( \{A_1, A_2, \ldots \} \) of \( T \)-sets in \( X \) such that for \( n = 1, 2, \ldots \) the following conditions hold:

(A) \( \bigcup A_n \) is metrizable;
(B) \( \bigcap A_n \) is a component of \( X - A_n \);
(C) \( \bigcap A_n \) is a non-degenerate cyclic element of \( X \), then the set \( C \cap A_n \) is metrizable and contains at least three points, and
(D) \( A_n \) is metrizable.

Note that Theorem 1.1 solves some old problems. Namely, the implication (iv) → (ii) answers affirmatively a question of S. Mardell and F. Papé ([10]; see also [23], Problem 1, p. 97), and the implication (iii) → (ii) shows that a conjecture of L. E. Ward is true ([26], p. 371; see also [23], Problem 2, p. 100).

When the paper was ready, the author got to know that the implication (iv) → (ii) was somewhat earlier shown by L. B. Treybig ([12]; Treybig's proof differs from the one given here).

In the forthcoming papers we will use condition (vii) to obtain further properties of spaces which are continuous images of compact linearly ordered topological spaces.

The proof of Theorem 1.1 will be given in Chapter 6; it is preceded by various auxiliary results. In Chapter 2 we gather some facts on \( T \)-sets; most of them have recently been proved by L. B. Treybig in [21]. Chapter 3 contains some simple results on \( T \)-sets and open coverings of continua. Chapter 4 deals with special kinds of \( T \)-sets and Theorem 4.9 is a key to the proof of Theorem 1.1. The proof of the implication (vii) → (i) of Theorem 1.1 is prepared in Chapter 5.

2. \( T \)-sets.

2.1. LEMMA ([21], Theorem 6). Let \( X \) be a locally connected continuum and \( A \) a \( T \)-set in \( X \). There exists an upper semi-continuous decomposition \( G_X \) of \( X \) into closed sets so that if \( X_A \) denotes the quotient space and \( f_A : X \to X_A \) is the quotient map, then:

(i) \( X_A \) is a locally connected continuum;
(ii) \( f_A \) restricted to \( A \) is a homeomorphism from \( A \) onto \( f_A(A) \);
(iii) if \( Z \) is a component of \( X_A - f_A(A) \), then \( Z, Z \) are connected to \( 0, 1 \), respectively, and so \( f_A(A) \) is a \( T \)-set in \( X_A \);
(iv) for each component \( Z \) of \( X_A - f_A(A) \) there is a unique component \( Y \) of \( X - A \) such that \( f_A(Y) \cap 0 = Z \); this gives a one-to-one correspondence between the family of all components of \( X_A - f_A(A) \) and the family of all components of \( X - A \).

The notation of Lemma 2.1 will be used in the sequel without comment.

2.2. LEMMA ([21], Theorem 8). Let \( X \) be a locally connected continuum without cut points and let \( \{A_s : s \in S \} \) be a family of \( T \)-subsets of \( X \), indexed by a well-ordered set \( S \) which has no last element, such that \( A_s \subset A_t \) if \( s < t \). Put \( A = \bigcup_{s \in S} A_s \), and for each \( s \in S \) let \( Y_s \) be a component of \( X - A_s \), such that \( Y_s \subset Y_s \) if \( s \in S \) and \( s < t \). Then the sets \( \{Y_s : s \in S \} \) can be labelled \( \{a_k, b_k \} \) where there exist points \( a, b \) such that the sets \( \{a_k, s \in S \} \), \( \{b_k : s \in S \} \) converge to \( a, b \), respectively. Moreover, either \( \bigcap_{s \in S} Y_s = \{a \} = \{b \} \) or \( \bigcap_{s \in S} Y_s \) is a non-degenerate continuum which is the disjoint union of \( \{a, b \}, Z_1, \ldots, Z_n \), where each \( Z_k \) is a component of \( X - A \) and \( \bigcap_{s \in S} Y_s = \{a \} = \{b \} \), \( k = 1, \ldots, n \). Therefore, \( A \) is a \( T \)-set in \( X \).

2.3. LEMMA ([21], Theorem 7). Let \( X \) be a locally connected continuum, \( A \) a \( T \)-set in \( X \) and \( a, b \) two distinct points of \( A \). Let \( I \) be an arc in \( X_A \) with end-points \( f_A(a), f_A(b) \),
Let \( Y \) be a component of \( X - A \) such that \( f_0(Y) \subseteq I \).

Then \( L \) is a subcontinuum of \( X \) such that if \( M = \{ x \} \), for some \( x \in L \cap A - \{ a, b \} \), or \( M \) is a component of \( X - A \) so that \( M \subseteq L \), then \( L - M \) is an union of two mutually separated connected sets \( P, Q \) such that \( a \in P \) and \( b \in Q \).

2.4. Lemma. Let \( X \) be a locally connected continuum without cut points and let \( E \) be any subset of \( X \). If \( A \) is a \( T \)-set in \( X \) such that \( E \subseteq A \), then there exists a minimal (in the sense of inclusion) \( T \)-set \( B \) in \( X \) such that \( E \subseteq B \).

Proof. If \( |E| < 3 \) then let \( B \) be any subset of \( A \) such that \( B \) consists of exactly two points and \( E \subseteq B \). Suppose that \( E \) contains three distinct points \( p, q, r \). By the Kuratowski–Zorn Lemma, to prove the existence of \( B \) it suffices to show that if \( (S, <) \) is a well-ordered set of indices without last element and \( A_s, s \in S \), are \( T \)-sets in \( X \) such that \( E \subseteq A_s \) and \( A_s \subseteq A_t \) for all \( s, t \in S \), \( s < t \), then \( A = \bigcap_{s \in S} A_s \) is a \( T \)-set in \( X \).

Let \( x \) be a point of \( X - A \). Hence there is an \( x \in S \) so that \( x \not\in A \) for each \( s \in S \), \( s > x_0 \). Let \( Y_s \) denote the component of \( X - A_{x} \), \( s > x_0 \), such that \( x \in Y_s \) and let \( b(Y_s) = \{ a_0, b_0 \} \) and \( b_s = Y_s \). Note that each component of \( X - B \) has a two-point boundary. Moreover, \( B_s \) is closed and \( b_s \in B_s \) provided \( s < x_0 \). By Lemma 2.2, we may assume that the nets \( \{ a_s : s > x_0 \} \) and \( \{ b_s : s > x_0 \} \) converge to some points \( a \) and \( b \), respectively. Hence, \( Y = \bigcap_{s \in S} Y_s \) is a component of \( X - A \) so that \( x \in Y \) — indeed, the connected sets \( Y_s, x \in X - A \), constitute an open covering of \( X - A \) and if \( X - A \cap X' \not= \emptyset \), then \( Y_{s'} \cap Y_{s} \cap X' \not= \emptyset \) for some \( s, s' \in S \) and therefore \( Y_{s'} = Y_{s'}^{s} \); so \( Y_s = \bigcap_{s \in S} Y_s \).

Now, it suffices to show that \( a \neq b \). Suppose that \( a = b \). Since \( X \) has no cut points, it follows that \( Y_a = X - (a) \). Moreover, \( A' \cap Y_a = \emptyset \), and so \( A' \not\subseteq \{ a \} \), a contradiction because \( E \subseteq A' \).

3. Remarks on open coverings and \( T \)-sets.

3.1. Lemma. Let \( X \) be a locally connected continuum, \( A \) a \( T \)-set in \( X, A \) a directed set, and \( \{ Y_s : s \in S \} \) a family of components of \( X - A \). If the net \( Y_s, s \in S \), is not eventually constant and converges, then \( \lim_{s \rightarrow s} Y_s \) is a single point which belongs to \( A \).

Proof. Obviously, \( Y = \lim_{s \rightarrow s} Y_s \) is a subcontinuum of \( X \). Suppose that \( Y \) is nondegenerate. Let \( Y_s, s \rightarrow s \) be a net finer than \( Y \), \( s \in S \), such that \( \lim a = a \) and \( \lim b = b \) exist, where \( b(Y_s) = \{ a_s, b_s \} \). Note that \( Y = \lim_{s \rightarrow s} Y_s \). Let \( c \) be any point of \( Y - \{ a, b \} \) and let \( U \) be a connected neighbourhood of \( c \) in \( X \) such that \( a, b \notin U \). There is an element \( t \in T \) such that \( Y_s \cap U \not= \emptyset \) and \( a_s, b_s \notin U \) for \( s > t \). Hence \( \emptyset \not\subseteq Y, U \subseteq Y \), and therefore \( U \not\subseteq \emptyset \), a contradiction.

3.2. Lemma. Let \( X \) be a locally connected continuum, \( U \) an open covering of \( X \) and \( A \) a \( T \)-set in \( X \). Then the family

\[ \{ Y : Y \text{ is a component of } X - A \text{ and } Y \text{ is not contained in any member of } U \} \]

is finite.

Proof. Let \( U = \{ U_1, \ldots, U_n \} \). Suppose that there are infinitely many distinct components \( Y_1, Y_2, \ldots \) of \( X - A \) so that \( Y_k \) is not contained in any \( U_i \), for \( k = 1, 2, \ldots \), and \( i = 1, \ldots, n \). Since the hyperspace \( C(X) \) of all subcontinua of \( X \) is compact, there is a convergent net \( \{ \gamma_i, s \in S \} \), which is finer than the net \( \{ \gamma_i, s \in S \} \), for \( k = 1, 2, \ldots \). By Lemma 3.1, \( \lim_{s \rightarrow s} Y_s \) consists of a single point \( a \). Let \( U \in U \) be such that \( a \in U \). Since \( \lim_{s \rightarrow s} Y_s \), there exists an \( x_0 \in S \) such that \( Y_0 \subseteq U \), a contradiction.

Recall that \( a \) is a partially ordered set \( T \) such that, for each \( s \in T \), the set \( \{ s \in T : s \leq t \} \) is well-ordered. Recall also the following simple fact:

3.3. Lemma. (König's Lemma, see for example [8]). If \( T \) is an infinite tree with finitely many minimal elements and each element of \( T \) has finitely many immediate successors, then \( T \) is an infinite linearly ordered set.

3.4. Lemma. Let \( X \) be a locally connected continuum without cut points and let \( A_1, A_2, \ldots \) be \( T \)-sets in \( X \) such that \( A_1 \subseteq A_2 \subseteq \ldots \) and if \( Y \) is a component of \( X - A_1 \) for some \( n \in \{ 1, 2, \ldots \} \), and \( Z \) is a nondegenerate cyclic element of \( Y \) then \( \{ y \in Y : y \text{ cuts } Y \} \subseteq A_{n+1} \), and \( |Z \cap A_{n+1}| > 2 \).

Then \( \bigcup_{s \in S} A_s \) is dense in \( X \) and for each open covering \( U \) of \( X \) there is a positive integer \( n_0 \) so that the closure of each component \( Y \) of \( X - A_{n_0} \) is contained in some member of \( U \).

Proof. Put \( A = \bigcup_{s \rightarrow s} A_s \) and suppose that \( x \in X - A \). For each \( n \) let \( B_n \) be the component of \( X - A_n \) so that \( x \in B_n \). Put \( H = \bigcup_{s \rightarrow s} Y_s \). By Lemmas 2.2 and 2.4, \( A \) is a \( T \)-set in \( X \), \( b(Y_s) = \{ a_s, b_s \} \), the limits \( a = \lim_{s \rightarrow s} a_s \) and \( b = \lim_{s \rightarrow s} b_s \) exist, \( a \neq b \), and \( H = \{ a, b \} \cup H_1 \cup \ldots \cup H_k \), where \( H_1, \ldots, H_k \) are components of \( X - A \) such that \( b(Y_s) = \{ a, b \} \) for \( s \in T \). For each positive integer \( n \), let \( Z_n \) denote the cyclic element of \( Y_n \) such that \( x \in Z_n \). Since \( X \) has no cut points, \( Y_n \) is a cyclic chain from \( a \) to \( b \), (28), p. 71), and so \( b(Y_s) = \{ a, b \} \) for some \( a, b \in Y_n \), \( a \neq b \). Since \( \{ y \in Y : y \text{ cuts } Y \} \subseteq A_{n+1} \), it follows that \( b(Y_s) = \{ a, b \} \) and therefore \( a_{n+1}, b_{n+1} \in Z_n \). Let \( F_n \) denote the component of \( Z_n - \{ a_{n+1}, b_{n+1} \} \) such that \( x \in F_n \). Observe that \( F_n \) is also a component of \( X - A_{n+1} \) and \( F_n = Y_n \). Since \( |Z_n \cap A_{n+1}| > 2 \) and \( F_n \cap A_{n+1} = \emptyset \), there is a component \( G_n \) of \( Z_n - \{ a_{n+1}, b_{n+1} \} \) such that \( x \in G_n \). Note that \( G = L \in Z_n \) is a subcontinuum of \( X \) and \( a, b \in \bigcap_{n \rightarrow n} Z_n \). Hence there exists a point \( y \in G \cap (H - \{ a, b \}) \). Then \( Y = H - \{ a, b \} \).
is a neighbourhood of \( y \) such that \( V \cap G_n = \emptyset \) for each \( n \). Therefore \( y \notin G \), a contradiction.

Let \( U \) be an open covering of \( X \). For each \( n \) put
\[
S_n = \{ Y : Y \text{ is a component of } X - A_n \text{ so that } Y \text{ is not contained in any member of } U \}.
\]

Put \( S = \bigcup_{n \geq 1} S_n \) and observe that \( S \) ordered by reverse inclusion is a tree. The sets \( S_n \) are levels of \( (S, \supseteq) \). By Lemma 3.2, \( (S, \supseteq) \) has only finitely many minimal elements and each member of \( S \) has only finitely many immediate successors.

Suppose that \( S \) is infinite. By Lemma 3.3, there is \( S' \subseteq S \) such that \( S' \) is infinite and \((S', \supseteq)\) is linearly ordered. We may assume that \( S' \) is maximal. Therefore \( S' \cap S_n = \{ Y_n \} \) for each \( n \). Since \( A = X \), the set \( \bigcap_{n \geq 1} U_n \) consists of a single point \( v \).

Let \( U_k \in U \) be such that \( v \in U_k \). There is an integer \( n \) such that \( V_n \subseteq U_k \). Hence \( V_n \subseteq S \), a contradiction.

We have thus shown that \( S \) is finite. Since no component of \( X - A_n \) is a component of \( X - A_k \) when \( n < k \), the families \( S_n \) and \( S_k \) are disjoint when \( n \neq k \). Therefore there is a positive integer \( n_0 \) such that \( S_{n_0} = \emptyset \).

3.5. Lemma. Let \( X \) and \( Y \) be locally connected continua and let \( A, B \) be T-sets in \( X \), \( Y \), respectively, such that all components of \( X - A \) and all components of \( Y - B \) are homeomorphic to \( [0,1] \). Suppose that there is a homeomorphism \( g : A \rightarrow B \) such that there is a bijection:
\[
f : \{ U : U \text{ is a component of } X - A \} \rightarrow \{ V : V \text{ is a component of } Y - B \}
\]

such that \( g(\partial(U)) = bd(f(U)) \) for each component \( U \) of \( X - A \). Then there is a homeomorphism \( G : X \rightarrow Y \) such that \( G|_A = g \) and \( G(U) = f(U) \) for each component \( U \) of \( X - A \).

Proof. For each component \( U \) of \( X - A \), write \( \partial(U) = \{ a_U, a'_U \} \) and let \( g_U : U \rightarrow f(U) \) be any homeomorphism such that \( g_U(a_U) = g(a_U) \) and \( g_U(a'_U) = g(a'_U) \). If \( x \in X \), then define \( G(x) = f(x) \) provided \( x \in A \) and \( G(x) = g(x) \) if \( x \notin U \) for a component \( U \) of \( X - A \).

Since \( G \) is one-to-one and onto, and uses Lemma 3.1 to show that \( G \) is continuous.

3.6. Lemma. Let \( X \) be a locally connected continuum, \( A \) a T-set in \( X \), and \( H \) any family of components of \( X - A \) so that \( V \cap \overline{W} = \emptyset \) if \( V \subseteq H \) and \( W \subseteq H \). Let \( G \) be the decomposition of \( X \) into points and the sets \( \overline{W} \) for \( W \subseteq H \). Then \( G \) is upper semi-continuous.

Proof. This follows from Lemma 3.1.

3.7. Lemma. Let \( X \) be a locally connected continuum, \( A \) a T-set in \( X \) and \( U = \{ U_1, \ldots, U_k \} \) an open covering of \( X \). For every component \( Z \) of \( X - f(A) \) let \( Y_Z \) be the unique component of \( X - A \) so that \( f(A)(Y_Z) = Z \). Write \( bd(Y_Z) = \{ a_Z, b_Z \} \) and let \( p_Z, q_Z, r_Z \) be points of \( Z \) such that \( f(A)(p_Z) < p_Z < q_Z < r_Z < f(A)(b_Z) \) in a natural ordering of \( Z \). Let
\[
H = \{ Z : Z \text{ is a component of } X - f(A) \},
\]

\[
U_A^k = f(A) \cap U_k \cup \{ Z \in H : Y_Z \subseteq U_k \} \cup \bigcup \{ f(A)(a_Z), q_Z] : Z \in H \text{ and } a_Z \subseteq U_k \} \cup \bigcup \{ [r_Z, f(A)(b_Z)) : Z \in H \text{ and } b_Z \subseteq U_k \}
\]

for \( k = 1, \ldots, n \), and
\[
U_A^n = \bigcup \{ [p_Z, q_Z) : Z \in H \text{ and } Y_Z \not\subseteq U_k \}.
\]

Then \( U_A^k \) is an open covering of \( X - f(A) \).

Proof. It is obvious that \( U_A^k \) is open. We show that \( X - U_A^k \) is closed for \( k = 1, \ldots, n \). Suppose that
\[
ze \in X - U_A^k \quad \text{indeed, each point } z \in Z \text{ of some component } Z \text{ of } X - f(A),
\]
such that \( z \in U_A^k \) has an open neighbourhood \( V \) so that \( V \cap Z \neq \emptyset \). There is a net \( z_n \), \( z \in S \), of points of \( X - U_A^k \) which converges to \( z \). Observe that \( (X - U_A^k) \cap f(A) \) is closed, and so we may assume that each \( z_n \) belongs to some component \( Z_n \) of \( X - f(A) \).

Using nets finer than \( z_n \), \( z \in S \), one can easily show that \( \lim Z_n \) exists and is equal to \( \{ z \} \). Let \( x \) be the unique point of \( A \cap Z_n \) (so \( x \in U_k \)).

It follows that \( \lim Y_k \) exists and is equal to \( \{ x \} \). Observe that, for each \( x \in S \), there exists a point \( y_n \in Y_k \) so that \( x_n \not\in U_k \). Note that \( \lim(x_n) = x \in U_k \). Thus \( U_k \) is not open, a contradiction.

We show that \( U_A^k \) covers \( X - f(A) \). Observe that
\[
f(A) = \bigcup_{k=1}^{n} U_A^k \cap A \cup \bigcup_{k=1}^{n} U_A^k.
\]

Suppose that \( z \in Z \) for some component \( Z \) of \( X - f(A) \). If \( Y_Z \subseteq U_k \) for some \( k \) then \( z \in U_A^k \) and if \( Y_Z \) is not contained in any member of \( U \) and \( z \neq q_Z \), then \( z \in U_A^k \) for each \( k \) such that \( a_Z \subseteq U_k \) (resp. \( b_Z \subseteq U_k \)) provided \( z \in [f(A)(a_Z), q_Z] \) (resp. \( z \in [r_Z, f(A)(b_Z)) \)). If \( z = q_Z \) and \( Y_Z \) is not contained in any member of \( U \), then \( z \in U_A^n \).

4. Metrizability of special T-sets.

4.1. Lemma. If \( X \) is a locally connected continuum and \( E \) is a metrizable T-subset of \( X \), then \( X - E \) has only countably many components.

Proof. Let \( \rho \) be a metric on \( E \). Suppose that \( X - E \) has uncountably many components. For some positive integer \( m \), we can find a sequence \( W_1, W_2, \ldots \) of distinct components of \( X - E \) such that \( bd(W_k) = \{ a_k, b_k \} \) and \( m \leq |a_k| \). Hence \( \lim(a_k, b_k) = m \); so \( a \neq b \). Let \( W_1, x \in S \) be a net finer than
\( W_{\sigma} \in \{1, 2, \ldots\}, \) such that \( W = \lim W_{\sigma} \) exists. Note that \( a, b \in W. \) This contradicts Lemma 3.1.

4.2. Lemma. If \( X \) is a locally connected continuum and \( E \) is a metrizable \( T \)-set in \( X \) such that each component of \( X - E \) is homeomorphic to \( [0, 1], \) then \( X \) is metrizable.

Proof. By Lemma 4.1, the set \( X - E \) has countably many components. One can find a countable basis for a Hausdorff topology on \( X \) weaker than the original topology of \( X \) for points of \( E, \) define basic neighborhoods similarly to the definition of the sets \( U_{\sigma} \) in Lemma 3.7. Since \( X \) is compact, the new topology of \( X \) coincides with the original one.

Recall that a continuum \( X \) is said to be hereditarily locally connected provided each subcontinuum of \( X \) is locally connected. It is not difficult to show that each hereditarily locally connected continuum is arcwise connected (24). Corollary 4, p. 125).

If \( X \) is a continuum such that, for each open covering \( U \) of \( X, \) every family \( H \) of pairwise disjoint subcontinua of \( X, \) none of which is contained in a member of \( U, \) is finite, then we say that \( X \) is a finitely Suslinian continuum. A continuum \( X \) is said to be rim-finite if each point of \( X \) has arbitrarily small open neighborhoods with finite boundary. Recall that every rim-finite continuum is finitely Suslinian and every finitely Suslinian continuum is hereditarily locally connected (24). Note that a continuum \( X \) is finitely Suslinian if and only if, for any two disjoint closed sets \( F, G \subset X, \) each family \( H \) of pairwise disjoint subcontinua of \( X \) with the property that \( Y \cap F \neq \emptyset \neq Y \cap G \) for \( Y \in H \) is finite. Observe also that if a Hausdorff space \( Z \) is the image of a finitely Suslinian continuum \( X \) under a continuous and monotonic mapping, then \( Z \) is also a finitely Suslinian continuum.

4.3. Lemma. If \( X \) is a locally connected continuum and \( A \) is a zero-dimensional \( T \)-set in \( X \) such that each component of \( X - A \) is homeomorphic to \( [0, 1], \) then \( X \) is rim-finite, and so finitely Suslinian.

Proof. Let \( x \in X \). If \( x \in X - A \) then \( x \) has arbitrarily small neighborhoods with two-point boundary. Suppose that \( x \in A \) and let \( U \) be any open neighborhood of \( x. \) Since \( A \) is zero-dimensional, there is an open set \( V \) such that \( x \in V \subset U \) and \( \text{bd}(V) \cap A = \emptyset. \) By Lemma 3.1, \( \text{bd}(V) \) intersects only finitely many components of \( X - A. \) Let \( W \) be the component of \( V \) so that \( x \in \text{bd}(W). \) Since \( X \) is locally connected, \( W \) is open. Moreover, \( \text{bd}(W) \cap A = \emptyset \) and \( \text{bd}(W) \) intersects only finitely many components of \( X - A. \) Since \( W \) is connected and the components of \( X - A \) are homeomorphic to \( [0, 1], \) \( \text{bd}(W) \) is finite.

Remark. The result of Lemma 4.3 was used in the proof of Lemma 2 in (20), p. 85, without being proved there. See also (15), p. 346, for a weaker result.

4.4. Lemma. Let \( X \) be a hereditarily locally connected continuum, \( A \) a closed subset of \( X, \) \( W \) a component of \( X - A, \) \( x \) a point of \( W, \) \( y \) a point of \( \text{bd}(W), \) and \( U \) an open set such that \( y \in U. \) Then there exist a point \( z \in A \cap U \) and an arc \( I \subset X \) with endpoints \( z, x \) such that \( I \cap A = \{z\}. \)

Proof. We may assume that \( X = A \cup W \) and \( A = \text{bd}(W), \) Note that for any two points \( p, q \in W \) there is a subcontinuum \( Y \) of \( X \) such that \( p, q \in Y \subset W \) (see for example [6], Theorem 3.7). Thus for each \( p \in W \) there is an arc \( I_p \subset W \) with endpoints \( x, p. \) Let \( V \) be a connected open set such that \( y \in V \subset V \subset U. \) Since \( y \in \text{bd}(W), \) there is a point \( q \) such that \( q \in V \cap W. \) Let \( J \) be any arc in \( V \) with endpoints \( q, y. \) Observe that there is an arc \( I = I_p \cup J \) with endpoints \( x, y, \) Moreover, \( I \cap A = I_p \cap A \subset V \cap A \subset U \cap A. \) Finally, there is an arc \( I \subset V \) with endpoints \( z, x \) such that \( I \cap A = \{z\}. \)

4.5. Theorem. Let \( X \) be a hereditarily locally connected continuum. If \( E \) is a metrizable closed subset of \( X, \) then the family

\[ H = \{W : W \text{ is a component of } X - E \text{ and } |\text{bd}(W)| \geq 2\}\]

is countable.

Proof. Suppose that \( H \) is uncountable. By Lemma 4.4, for each component \( W \) of \( X - E \) such that \( |\text{bd}(W)| > 1, \) there is an arc \( J_w \subset W \) with endpoints \( a_w, b_w \) such that \( J_w \cap E = \{a_w, b_w\}. \) Let \( q \) be a metric on \( E. \) There are distinct components \( W_1, W_2, \ldots \) of \( X - E \) such that \( |\text{bd}(W_i)| > 1, \) and for some positive integer \( m, \)

\[ g(a_w, b_w), \] but the limits \( \lim a_w = a, \lim b_w = b \text{ exist}; \] so \( g(a, b) > 1/m. \)

Let \( U, V \) be open neighborhoods of \( a, b, \) respectively, such that \( U \cap V = \emptyset. \) We assume that \( J_w \cap U \neq \emptyset \neq J_w \cap V, \) for \( n = 1, 2, \ldots. \) Hence, for each \( i, \) there is a subarc \( I_{1/n} \) of \( J_w \) such that \( I_{1/n} \subset W_1 \) and \( I_{1/n} \cap U \neq \emptyset \neq I_{1/n} \cap V. \) Since \( W_1, W_2, \ldots \) are pairwise disjoint open subsets of \( X, \) it follows that \( I_{1/n} \cap U = \emptyset \) for \( k = 1, 2, \ldots. \) [17], Theorem 4, p. 246, \( X \) is not hereditarily locally connected, a contradiction.

Remark. If \( X \) is a dendron which contains exactly one point \( x \) so that \( X - \{x\} \) has more than two components (i.e., \( X \) is a "fan") and, moreover, \( X - \{x\} \) has uncountably many components, then \( E = \{x\} \times [0, 1] \) is a closed metrizable subset of a locally connected continuum \( Y = X \times [0, 1] \) such that \( Y - E \) has uncountably many components. However, if we assume in addition that \( E \) is zero-dimensional, then Theorem 4.5 remains true for locally connected continua (it suffices to modify the proof of Lemma 4.1).

4.6 Lemma. Let \( Z \) be a finitely Suslinian continuum, \( J \) a separable arc in \( Z \) with end-points \( J_0, J_1, < \text{ the natural ordering of } J \text{ from } J_0 \text{ to } J_1, \) and \( W \) a countable subset of \( J. \) Put

\[ T = \{t \in J : \text{there are pairwise disjoint arcs } J_{t_n} \text{ in } Z, n = 1, 2, \ldots, \} \text{ with end-points } a_n, b_n \text{ such that } J \cap J_{t_n} = [a_n, b_n) \text{ and } a_n < t < b_n \} \text{.} \]

Then:

(a) if \( Z - J \) has exactly one component \( W_0 \) and \( \text{bd}(W_0) = J, \) then \( Z - T \) is countable;
(b) if \( Z - J \) has exactly one component \( W_0, \) then \( \text{bd}(W_0) - T \) is countable;
Suppose that there is a point \( x \neq x_n \in \bigcap_{n \geq 0} \text{bd}(W_n) \). Let \( F, G \) be open sets such that \( F \cap G = \emptyset \), \( F \subseteq C \) and \( x \in G \). By Lemma 4.4, there is an arc \( M_n \subseteq W_n \) with end-points \( x_n, y_n \), satisfying \( M_n \cap J = \{x_n, y_n\} \) and \( y_n \in G \) for \( n = 0, 1, \ldots \) Since \( x \neq x_n \) and \( \lim x_n = x_0 \), it follows that, for each \( n \), the set \( M_n \cap \{x_0, x_0, \ldots, \} \) is finite, and so \( M_n \cap W_n \subseteq \emptyset \) for some integer \( m \). Now, it is easy to find pairwise disjoint arcs \( l_m, l_m, \ldots, l_1 < x_0 < \ldots \), such that \( l_m \subseteq M_m \) and \( F \cap l_m \neq \emptyset \) \( \cap l_m \), for \( m = 1, 2, \ldots \) Thus \( Z \) is not finally Suslinian, a contradiction.

For each \( s \in S \), put

\[
R_s = \{ t \in J : t \text{ is an end-point of some component of } J - \text{bd}(W_s) \}.
\]

so \( R_s \) is countable. Put \( R = \{t \in J_n : \bigcup_{s \in J_n} R_s \} \); hence \( R \) is also countable. We now show that \( J - T \subseteq R \), which will finish the proof of (a).

Let \( y \in J - R \). Since \( S_n \) are finite and \( \text{bd}(W) = J \) for every \( n \), thus \( S' = \{x \in S' : y \notin \text{bd}(W) \} \) is infinite. By König's Lemma (see Lemma 3.3 above), there are \( x_0, x_1, \ldots \in S' \) such that \( x_0 \in S_0 \) and \( x_0 < x_1 < \ldots \) for \( n = 0, 1, \ldots \). Since \( y \notin R \), there are \( x_0, x_1, \ldots \in S' \) such that \( x_0 < x_1 < \ldots \). Arguing as in the above proof that \( Z \) is an arc, it is easy to find arcs \( l_0, l_0 \subseteq W_n \) with end-points \( x_0, x_0, \ldots \), respectively. There is an arc \( J_0 \subseteq W_0 \subseteq W_1 \) with end-points \( x_0, x_0, \ldots \). There is an integer \( n_1 \) such that \( x_0 < n_1 ) \), and \( x_0, x_1, \ldots \in \text{bd}(W) \); so \( J_0 \subseteq W_n \subseteq \emptyset \). Apply the above argument to find an arc \( J_1 \subseteq W_n \subseteq \{x_1, \ldots \} \) with end-points \( x_1, x_1, \ldots \) such that \( x_1 < x_1 < \ldots \). Note that \( J_0 \cup J_1 \subseteq \emptyset \). Proceeding by induction we find a sequence \( J_0, J_1, \ldots \) of arcs which is required to show that \( y \in T \).

(b) We may assume that \( \text{bd}(W) \) is uncountable. Let \( A \) denote the set of all condensation points of \( \text{bd}(W) \) (see [7]); so \( A \subseteq \text{bd}(W) \cap C \) is a closed set without isolated points and \( \text{bd}(W) - A \) is countable. Put \( B = \{ F : F \text{ is a component of } J - A \} \) and let \( G \) be the decomposition of \( Z \) into points and sets which belong to \( B \). Note that \( G \) is an upper semi-continuous and monotone. Let \( g : Z \to Z/G = Y \) be the quotient map and \( I = g(J) \); so \( I \) is homeomorphic to \([0, 1]\). \( Y \) is a finally Suslinian continuum, and \( g(W) \) is the unique component of \( Y - I \). Moreover, \( \text{bd}(g(W)) = I \).

Put \( S = \{ t : g^{-1}(t) \text{ is nondegenerate} \} \); note that \( g^{-1}(S) \cap \text{bd}(W) = (J \cup B) \cap \text{bd}(W) \) is countable. Put

\[
T = \{ t \in J : \text{there are pairwise disjoint arcs } l_n \subseteq Y, n = 1, 2, \ldots \}.
\]

with end-points \( a_n, b_n \) such that \( l_n = [a_n, b_n] \) and \( a_n < t < b_n \),

where \( < \) denotes also a natural ordering of \( J \). By (a), \( S' = J - T \) is countable (one can also prove that \( S \subseteq S' \)). Observe that \( \text{bd}(W) - T \subseteq \text{bd}(W) \cap g^{-1}(S \cup S') \). Finally, \( \text{bd}(W) \cap g^{-1}(S \cup S') \) is countable.

For each \( k \in \{0, 1, \ldots \} \), let \( [k, d_k] \) denote the smallest subset of \( J \) such
that $bd(W) = \{a_n, b_n\}$, $a_n < b_n$, and put

$$T_n = \{t \in J_n: \text{there are a pair of disjoint arcs } J_n \text{ in } J \cup W_n, n = 1, 2, \ldots, \text{ with endpoints } a_n, b_n \text{ such that } J \cap J_n = \{a_n, b_n\} \text{ and } a_n < t < b_n\}.$$  

$$T' = \{t \in J: \text{there are a pair of disjoint arcs } J_n \text{ in } J_n, n = 1, 2, \ldots, \text{ with endpoints } a_n, b_n \text{ such that } J \cap J_n = \{a_n, b_n\} \text{ and } a_n < t < b_n, \text{ and if } n \neq m \text{ then } J_n \text{ and } J_m \text{ do not intersect the same component of } Z - J\}.$$  

Note that $T = T' \cup \bigcup_{n \in \mathbb{N}} T_n$. By (b), each $T_n$ is a $G_\delta$-set. Moreover,  

$$T' = \bigcap_{n \in \mathbb{N}} \bigcup_{k \in \mathbb{N}} \{a_n, b_n\},$$

and so $T'$ is a $G_\delta$-set. Thus $T$ is a $G_\delta$-set.

(d) Let $T'$, $T_n$ and $[a_n, b_n]$ for $k = 0, 1, \ldots$ be as in the proof of (c). Suppose that for any $c, d \in J - J_n$, $c < d$, there is a component $W_n$ of $Z - J_n$ such that

$$bd(W_n) \cap [c, d] \neq \emptyset \neq bd(W)(c, d).$$

Then for any $n, q \in [a_n, b_n] \cap [c, d], d \neq \emptyset \neq [a_n, b_n] - [c, d]$, there is a component $W_n$ of $Z - J_n$ such that $bd(W_n) \cap [c, d] \neq \emptyset \neq [a_n, b_n] - [c, d]$.

First, let us assume that

$$(*) \text{ if } p_n, q_n \in [0, 1], p_n < q_n, n = 0, 1, \ldots, \text{ for all } c, d \in J, c < d, \text{ there is a nonnegative integer } n \text{ such that } [p_n, q_n] \cap [c, d] \neq \emptyset \neq [p_n, q_n] - [c, d].$$

Suppose that $c, d \in A$, $c < d$. Then, for some $n$, $[p_n, q_n] \cap [c, d] \neq \emptyset$, then $[p_n, q_n] \subseteq [c, d]$ (because $c, d \notin [p_n, q_n]$), and $[p_n, q_n] - [c, d] = \emptyset$, a contradiction. Put $A_n = \bigcup_{n \in \mathbb{N}} [a_n, b_n]$. Then, for any $a_n < b_n$, $bd(W) \cap [a_n, b_n] = \emptyset$. Suppose that for some integer

$I \geq 0$ we have already constructed countable sets $A_{I+1}, \ldots, A_I$, such that

$$B_1 = bd(W_{I+1}) \cup \ldots \cup bd(W_{I+1}) \cup A_I \cup A_{I+1}$$

is a closed set (we put $W_{I+1} = \emptyset$). Let $C_n$ denote the family of all components of $\bigcup_{n \in \mathbb{N}} B_{I+1}$. If $D \subseteq I', \forall i \in C_n$ then, by $(*)$, the set $A_{I+1} = D \cup \bigcup_{i \in I', \forall i \in C_n}$ contains

at most one point. Put $A_{I+1} = \bigcup_{i \in I', \forall i \in C_n} A_{I+1}$. Then, by $(*)$, $A_{I+1}$ is closed and $A_{I+1}$ is countable and $B_{I+1}$ is closed (because $A_{I+1} \subseteq I' \cup A_{I+1}$).

Put $B = \bigcup B_i$ and note that $T = T' \cup \bigcup_{n \in \mathbb{N}} [a_n, b_n]$, $k = 0, 1, \ldots$, are countable. Since

$$B = \bigcup_{n \in \mathbb{N}} (E_n \cup A_n) = \bigcup_{n \in \mathbb{N}} T_n \quad \text{and} \quad T' = T' \cup \bigcup_{n \in \mathbb{N}} T_n,$$

it follows that $J - T$ is also countable.

(e) Assume that $T = \emptyset$. Let $f: J \rightarrow [0, 1] \times [0, 1] \times [0, 1]$ be any homeomorphism. Let $R$ denote the (for a moment) set of real numbers. First, we construct a subcontinuum $Y$ of the plane $R^2$.

Let $e_1, e_2, \ldots$ be the sequence of points of $f(U) - \{(0, 0), (1, 0), 0\}$ such that each point appears in the sequence infinitely many times and $0 \leq e_1 < e_2 < \ldots < e_3 + 2^{-3\alpha+3} < e_4 + 2^{-3\alpha+3} < e_5 + 2^{-3\alpha+3} < \ldots$. Put

$$Y = \{0, 1\} \times [0, 1] \bigcup (x, y) \in R^2: (x - e_0 3) + y^2 = 2^{-3\alpha+3} \quad \text{and} \quad y > 0\}.$$

$$S = \{t \in J, t \in [0, 1] \times [0, 1]: \text{there are pairwise disjoint arcs } J_n \text{ in } Y, \quad n = 1, 2, \ldots, \text{ with endpoints } (a_n, 0), (b_n, 0), \text{ where} \quad [0, 1] \times [0, 1] \bigcup (a_n, b_n) \text{ and } a_n < t < b_n, \quad S' = \bigcup_{n \in \mathbb{N}} (a_n - 2^{-3\alpha+3}, b_n + 2^{-3\alpha+3}) \times [0, 1].$$

Note that $f(U) - \{(0, 0), (1, 0), 0\} \subseteq S' - [0, 1] \times [0, 1] - S'$ is uncountable (because $\sum_0^1 \bigcup_0^1 2^{-3\alpha+3} = 1 < 1$).

We may assume that $T \cap Y = \emptyset$. Let $G$ be the decomposition of $Y = J \times Y$ into points and the sets $\{t, f(t)\}, t \in T$, so $G$ is upper semi-continuous. Put $X = (Z \times Y) / G$ and let $g: Z / Y \rightarrow X$ denote the quotient map. Observe that $X$ is a finitely Suslinian continuum and $g(J) = g((0, 1) \times [0, 1])$ is an arc in $X$. Put

$$T = \{t \in g(J): \text{there are pairwise disjoint arcs } J_n \text{ in } X, n = 1, 2, \ldots, \text{ with endpoints } (a_n, b_n) \text{ such that } g(J) \cap J_n = \{(a_n, b_n)\} \text{ and } a_n < t < b_n\}.$$  

Note that $T' = g(T) \cup g(S) = g(S)$; $g(J) - T$ is uncountable. Use (d) to find $e', e'' \in g(J) - g(J), g(J) - T = [e', e'']$ such that $bd(V) \in [e', e''], \text{ or } bd(V) = Y - g(J)$ for some component

$$V \text{ of } X - g(J). \text{ Observe that } e' \notin g(U) \text{ and } e'' \notin g(U). \text{ Finally, let } e, d \in J \text{ be the unique points such that } g(e) = e' \text{ and } g(d) = d'$$  

4.7. **Lemma** [7], § 24, Section VII, Theorem 3, p. 252. Let $X$ be a separable metric space and $A: t \in [0, 1]$ be a family of closed subsets of $X$ such that $A_i \subseteq A_i$ for $t < s$. Then the equality $A_t = \bigcup_{t \in [0, 1]} A_t$ holds for every $t \in [0, 1]$ except for a countable set of indices.

4.8. **Lemma.** Let $I$ be an arc and $C$ a subset of $I$ such that the end-points of $I$ belong to $C$, $C$ is compact, separable and zero-dimensional, and $I - C$ has uncountably many components. Let $R_C$ be a relation in $I$ defined as follows: $x_R y$ provided $x = y$ or $[x, y] - C \times [x, y]$ has countably many components. Then $R_C$ is an equivalence relation whose classes are subarcs of $I$ (possibly degenerate) with end-points in $C$. Thus the decomposition $C^I$ of $I$ defined by $R_C$ is upper semi-continuous and the quotient space $J = I / C^I$ is homeomorphic to $[0, 1]$. Let $g_C: I \rightarrow J$ be the quotient map and put $x^*$
\[ P = \{ x \in J : |g_c^{-1}(x) \cap C | > 2 \}, \quad Q = \{ x \in J : |g_c^{-1}(x) \cap C | = 2 \}. \] Then \( P \) is countable and \( \{ x, y \} \cap Q \) is uncountable for any \( x, y \in J, x \neq y \).

**Proof.** Since all decreasing and all increasing sequences of points of \( C \) (in a natural ordering of \( J \)) are countable, it follows that \( \mathcal{R} \) is indeed an equivalence relation whose classes are arcs with end-points in \( C \). Note that \( J = g_c(C) \) is separable and nondegenerate; so \( J \) is homeomorphic to \([0, 1]\). For each \( x \in P \), the set \( g_c^{-1}(x) - b d(g_c^{-1}(x)) \cap C \) is open in \( C \) and nonempty. Since these sets are pairwise disjoint and \( C \) is separable, it follows that \( P \) is countable.

Let \( \prec \) denote natural orderings on \( I \) and \( J \) such that \( g_c \) is an increasing map. Suppose that \( x, y \in J, x < y \), and \( \{ x, y \} \cap Q \) is countable. Let \( u \) (resp. \( v \)) be the first (resp. last) point of \( I \) such that \( g_c(u) = x \) (resp. \( g_c(v) = y \)). Since \( \{ x, y \} \cap Q \) is countable and \( P \) is countable, it follows that \( \{ u, v \} - (C \cap \{ u, v \}) \) has countably many components. Thus \( u \mathcal{R} v \), and so \( g_c(u) = g_c(v) \), a contradiction.

**4.9. Theorem.** Let \( X \) be a locally connected continuum without cut points such that:

(a) if \( E \subset E' \subset X \) and \( E' \) is separable, then \( E \) is also separable, and
(b) if \( E' \) is a continuous and monotone image of \( X \) and \( E \) is a separable continuum in \( E' \), then \( E \) is metrizable.

Suppose that \( A \) is a closed metrizable subset of \( X \) and \( A' \) is a separable \( T \)-set in \( X \) such that \( A \subset A' \). Then there exists a metrizable \( T \)-set \( B \) in \( X \) such that \( A \subset B \subset A' \). More precisely, each separable \( T \)-subset of \( X \) which is minimal with respect to the property "contains \( A \)" is metrizable.

**Proof.** Let \( B \) be any separable and minimal \( T \)-subset of \( X \) so that \( A \subset B \subset A' \) (Lemma 2.4 and assumption (a), \( B \) does exist). Because of Lemma 2.1, we may assume that each component of \( X-B \) is homeomorphic to \([0, 1]\).

Let \( G \) be the decomposition of \( X \) into the components of \( B \) and points. Note that \( G \) is upper semi-continuous and monotone (see [20], Lemma 2, p. 85). Let \( Y \) be the quotient space \( X/G \) and \( f : Y \to X \) be the quotient map. The set \( f(B) \) is zero-dimensional and each component of \( Y-f(B) \) is homeomorphic to \([0, 1]\). By Lemma 4.3, \( Y \) is finite Suslinian. Hence \( Y \) is arcwise connected.

Let \( Y_1 \) be a nondegenerate cyclic element of \( Y \) and let \( r : Y \to Y_1 \) denote the unique monotone retraction from \( Y \) onto \( Y_1 \). Since \( X \) has no cut points, it follows that \( f(\{ y \}) \) is nondegenerate, then \( f(B) \) is a separable zero-dimensional \( T \)-subset of \( Y_1 \) such that \( Y_1 - f(B) \) is homeomorphic to \([0, 1]\). Let \( \{ \gamma_0, \gamma_1, \ldots \} \) be a countable dense subset of \( f(B) \). For each positive integer \( n \) choose an arc \( I_n \subset Y_1 \) from \( \gamma_0 \) to \( \gamma_n \). Put \( Y_n = \bigcup_{i=1}^{n} I_n \subset Y_1 \); so \( Y_n \) is a subcontinuum of \( Y_1 \) such that \( f(B) \subset Y_n \). By (b), \( Y_n \) is metrizable if and only if for each positive integer \( n \) the set \( L_n - f(B) \) has countably many components.

Suppose that \( Y_n \) is not metrizable. Hence there is an \( n_0 \) such that \( L_{n_0} - f(B) \) has uncountably many components. Since \( f(B) \) is metrizable, \( L_{n_0} - f(B) \) has only countably many components. Therefore there is an arc \( I \) contained in \( I_{n_0} \) with end-points in \( f(B) \) such that \( J \cap f(B) = I \) and \( f(B) \setminus f(\{ y \}) \) still has uncountably many components. Put \( C = f(B) \cap I \); so \( C \) is separable and compact and zero-dimensional. Let \( G_I \) be the decomposition of \( Y_1 \) into points and the classes of the decomposition \( G_{I_1} \) of the arc \( I \) (see Lemma 4.8). Note that \( G_I \) is upper semi-continuous and monotone. Let \( Z \) denote the quotient space \( Z = Y_1/G_I \) and \( g : Y_1 \to Z \) the quotient map. Note that \( g \) is monotone and \( g|_I = g_c \) (Lemma 4.8). Moreover, \( Z \) is finitely Suslinian.

Let \( I = g(j) \); so \( J \) is homeomorphic to \([0, 1]\). Let \( j_0, j_1 \) denote the end-points of \( I \), put \( J_0 = J - [j_0, j_1] \) and let \( \prec \) be the natural ordering of \( J \) from \( j_0 \) to \( j_1 \). Let \( P = \{ I \cap \{ y \} : |g^{-1}(I) \cap C | > 2 \}, \quad Q = \{ I \cap \{ y \} : |g^{-1}(I) \cap C | = 2 \} \). Note that for each \( g \in G \) there is a unique component \( W_g \subset X-B \) so that \( f(W_g) = g \). Let \( H \) be the decomposition of \( X \) into points and the sets \( W_g, g \in G \). By Lemma 3.6, \( H \) is upper semi-continuous. Put \( X_0 = X/H \) and let \( h : X \to X_0 \) be the quotient map; so \( h \) is monotone. Let \( k : X_0 \to Z = h(B) = Y_1/G_I \) be the mapping defined by the formula: \( k(x) = x \) provided \( g(x) \in C \). It is easy to see that \( k \) is well-defined, continuous (see [3], Theorem 6.3.2, p. 125) and monotone. Put \( X' = f^{-1}(I) \) and \( k(X') \). We show that \( X' \) is metrizable.

For each \( g \in P \) put \( X_g = f^{-1}(\gamma(g)) \); so \( X_g \) is a subcontinuum of \( X' \). Note that each \( X_g \) is a union of some component of \( X' \) and of countably many components of \( X-B \). By (b), \( X_0 \) is separable for each \( g \in P \). Observe that \( X_0 \) is separable and metrizable (by (b)). Let \( q \) denote a metric on \( X_0 \).

Observe that \( k(X_g) = J \) and define \( k' : X_g \to J \) as \( k' = k|X_g \); so \( k' \) is continuous. Note that \( k' \) is also monotone and write \( M_i = (k')^{-1}(i) \) for each \( i \in J \). For each positive integer \( n \) put \( R_n = \{ i \in J : \text{diam} M_i > 1/n \} \). Put \( R = \bigcup_{n=1}^{\infty} R_n \). Note that \( R \) are closed.

Let \( T = \{ I : \text{there are pairwise disjoint arcs } J_n \subset Z, n = 1, 2, \ldots, \text{ with end-points } a_n, b_n \text{ such that } J_n \cap J_m = \emptyset \} \). Suppose that \( t \in T \), \( J_n \subset Z \). Let \( J_n = [a_n, b_n] \subset Z \). Let \( a_n < t < b_n \). Let (a, b) denote the first (resp. last) point of \( I \) which is mapped by \( g \) onto \( I \) (so \( a \neq b \)) and put \( I_n = \{ y \in I : y < a \} \). Note that \( g^{-1}(I_n) = \bigcup_{y \in I_n} I \). We show that \( g^{-1}(I_n) \cap I \neq \emptyset \), for \( I_n \), \( I \) are closed and disjoint, \( Y \) is not finitely Suslinian, a contradiction.
This shows that \( T \cap Q = \emptyset \). Now, we prove that \( R \cap Q \) is countable.

For each \( t \in T \) put \( N_t = (k_t^{-1})(l_t, u_t) \). \( N_t \) is \((k_t^{-1})(l_t, u_t) \). So \( M_t = N_t \cap M_t \times M_t \). Take any point \( t \in Q \). Then \( h(t) \) consists of a single point \( m_t \in M_t \) such that \( m_t \) cuts \( X_t \) between \( M_t \) and \( M_t \). Write \( g^{-1}(t) \cap C = (a_t, b_t) \), where \( a_t < b_t \), and put \( h^{-1}(a_t) = M_t \), \( h^{-1}(b_t) = M_t \). Note that \( M_t = M_t \cup M_t \) and \( M_t = M_t \times M_t \). Moreover, \( \bigcup_{t \in T} N_t \times M_t = \{ m_t \} \) and \( \bigcup_{t \in T} N_t = \{ m_t \} \).

Therefore if \( M_t \neq \{ m_t \} \) then \( \bigcup_{t \in T} N_t \neq N_t \), and if \( M_t \neq \{ m_t \} \) then \( \bigcup_{t \in T} N_t \neq N_t \).

Thus if \( t \in R \cap Q \), then either \( \bigcup_{t \in T} N_t \neq N_t \) or \( \bigcup_{t \in T} N_t \neq N_t \). By Lemma 4.7, \( R \cap Q \) is countable.

Since \( I \cap f(t)(A) = \emptyset \), it follows that \( J \cap g(t)(A) = \emptyset \). Therefore there are only finitely many components of \( Z-J \) which intersect the cloated set \( g(t)(A) \). Moreover, the boundary of each such component is nowhere dense in \( J \). Indeed, if \( W \) is a component of \( Z-J \) so that \( bd(W) \) contains a nondegenerate subarc \( J' \) of \( J \), then \( J' \) is countable (see Lemma 4.6(b)), \( J' \cap Q = \emptyset \), and \( J' \cap Q \) is countable (Lemma 4.8), a contradiction.

Thus we can find a nondegenerate subarc \( K \) of \( J \) with end-points \( k_0 \) and \( k_1 \), where \( j_0 < k_0 < k_1 < j_1 \), such that \( bd(W) \cap K = \emptyset \) for each component \( W \) of \( Z-J \) so that \( g(t)(A) \cap W \neq \emptyset \).

Note that \( K = (R \cap T) \) is a Borel set in \( K \), because \( R \) is an \( F_\sigma \)-set and \( T \) is a \( G_\delta \)-set (Lemma 4.6(c)). Since \( Q \cap R = Q \cap T \), and \( K \cap Q \) is countable (Lemma 4.8), it follows that the set \( K = (R \cap T) \) is countable.

There is therefore a subset \( K' \) of \( K = (R \cap T) \) which is homeomorphic to the Cantor set (see for example [7], §37, Section I, Theorem 3 (of Alexandrov–Hausdorff), p. 447). We may assume that \( k_0, k_1 \in K' \).

Let \( H' \) be the decomposition of \( Z \) into points and the sets \( W \) for each component \( W \) of \( K-K' \). Then \( H' \) is upper semi-continuous and monotone. Put \( Z_t = Z[H' \) and let \( l: Z \to Z_t \) denote the quotient map. Then \( Z_t \) is a finitely Suslinian continuum. Put \( l_t = l(K) = l(K') \); so \( l_t \) is a separable (nondegenerate) arc in \( Z_t \). Let \( l \) denote the natural ordering of \( L \) such that \( l_t \) is an increasing map and put \( l_t = l(l_t) \), \( l_t = l(l_t) \); so \( l_t \) is the end-points of \( L \) and \( b_t < a_t \).

\( T' = \{ t \in L : l(t) \) is a pairwise disjoint arcs \( J_t \) in \( Z_t \}, \)

with end-points \( a_t, b_t \) such that

\[ L \cap J_t = \{ a_t, b_t \} \] .

Since \( T \cap K' = \emptyset \), it follows that \( T' = \emptyset \).

Put \( U_t = \{ t \in L : l(t) \) is a nondegenerate \}; so \( U_t \) is countable. Observe that \( P \subset R \). Therefore \( l(P \cap K) \subset l(R \cap K) = U_t \). Put

\[ U_t = \{ t \in L : l(t) \) is a component \( W \) of \( Z_t \) and \( bd(W) \) is countable\} .

Because of Lemma 4.6(b), the boundary of each component of \( Z_t \) is countable (since \( T' = \emptyset \)). By Theorem 4.5, \( U_t \) is countable. Now, put \( U = U_t \cup U_2 \); so \( U \) is countable.

By Lemma 4.6(c), there are \( c, d \in L-(U \cup \{ l_0, l_1 \} \), \( c < d \), such that either \( bd(W) = [c, d] \) or \( bd(W) \cap C = \emptyset \) for each component \( W \) of \( Z_t \). Moreover, if \( W \) is a component of \( Z_t-L \) such that \( [c, d] \cap bd(W) \neq \emptyset \), then either \( bd(W) = [c, d] \) or \( bd(W) = [d, c] \) (because \( c, d \in U_2 \)). Put

\[ V = \{ c, d \}, d_t \cup \{ W : W \) is a component of \( Z_t \) such that \( bd(W) = [c, d] \} \]

and note that \( V \) is a component of \( Y_t \).

Since \( c, d \in U_t \), each of the sets \( l_t^{-1}(c) \) and \( l_t^{-1}(d) \) consists exactly of one point which does not belong to \( P \). Therefore each of the sets \( g^{-1}l_t^{-1}(c) \cap C \) and \( g^{-1}l_t^{-1}(d) \cap C \) consists of either one or two points (which obviously belong to \( P \)). Put \( e_t = \min (g^{-1}l_t^{-1}(c) \cap C \), \( d_t = \min (g^{-1}l_t^{-1}(d) \cap C \); so \( c_t < d_t \). Recall that \( l_t(R \cap K) = U_t \). Hence \( c_t, d_t \notin g^{-1}l_t^{-1}(R) \), and so \( f_t^{-1}(c_t) = (c_t) \) and \( f_t^{-1}(d_t) = (d_t) \) for some points \( c_t, d_t \in B \).

We show that \( Y_t = g^{-1}l_t^{-1}(c_t) \) is a component of \( Y_t \) \( \{ c_t, d_t \} \), since \( c_t, d_t \) are monotonous mappings, it follows that \( Y_t \) is a component of \( Y_t \) \( \{ c_t, d_t \} \).

It suffices to prove that \( bd(Y_t) = \{ c_t, d_t \} \). Observe that

\[ V_t = \{ c_t, d_t \}, d_t \cup \{ W : W \) is a component of \( Y_t \) \( g^{-1}l_t^{-1}(c_t, d_t) \}

such that \( bd(W) \subset \{ c_t, d_t \} \) (because \( c_t, d_t \notin g^{-1}l_t^{-1}(c_t, d_t) \). Since \( Y_t \) is locally connected, it follows that \( Y_t \) \( \{ c_t, d_t \} \), and so \( bd(Y_t) = \{ c_t, d_t \} \).

Now, we prove that \( Y_t = f_t^{-1}l_t^{-1}(c_t, d_t) \) is a component of \( X_\) \( \{ c_t, d_t \} \). Since \( r_t \) is monotonous mapping, it follows that \( Y_t \) is a component of \( X_\) \( \{ c_t, d_t \} \).

Since \( f_t^{-1}(c_t) = (c_t) \) and \( f_t^{-1}(d_t) = (d_t) \), it suffices to show that \( r^{-1}(c_t) = (c_t) \) and \( r^{-1}(d_t) = (d_t) \). We check that \( r^{-1}(c_t) = (c_t) \); the proof of the second equality is analogous.

Suppose that \( r^{-1}(c_t) \) is nondegenerate. Hence \( c_t \) is a cut point of \( Y_t \). Let \( W \) be any component of \( Y_t \) \( \{ c_t \} \). Since \( f_t \) is monotonous, \( f_t^{-1}(W) \) is a component of \( X_\) \( \{ c_t \} \). Thus \( X \) has cut points, a contradiction.

Put \( D = B-Y_t \). Recall that \( bd(W) \cap K = \emptyset \) for each component \( W \) of \( Z-J \) such that \( g(t)(A) \cap W \neq \emptyset \) and, moreover, \( K \cap g(t)(A) = \emptyset \). Therefore \( l_t^{-1}(c_t) \cap V = \emptyset \), and so \( A \subset D \) (because \( A \subset B \)). Since \( l_t^{-1}(c_t) \cap V = \emptyset \) it follows that \( D \neq B \). Moreover, \( B-D \subset V_t \) and \( V_t \cap (B-D) = \emptyset \), and so \( W \) is also a component of \( X-B \). Thus \( D \) is a \( T \)-set in \( Z \) such that \( A \subset D \subset B \). This contradicts the minimality of \( B \).

We have finished the proof that \( Y_t \) is metrizable. By Theorem 4.5, \( Y_t \) has countably many components. By (b) (see also Lemma 4.2), \( Y_t \) is metrizable.

We have shown that each cyclic element of \( Y \) is metrizable.
4.10. COROLLARY. If $X$ is a locally connected continuum without cut-points which is a continuous image of some compact linearly ordered topological space, and $E$ is a closed metrizable subset of $X$, then there is a metrizable $T$-set $E'$ in $X$ such that $E \subset E'$.

Proof. See Chapter 6, the proof of the implication (iv) $\rightarrow$ (v) of Theorem 1.1.

5. Approximation by finite dendrons. Let $X$ be a continuum and $J$ a family of finite dendrons which approximates $X$. Recall that $X$ is locally connected (this follows from a theorem of L. E. Ward, [26], Theorem 1, p. 370).

If $a, b \in J$ then there is a unique arc $[a, b]$ contained in $X$ with end-points $a, b$ such that $[a, b] \subset T$ for each set $T \in J$ with $a, b \in T$. Write

$$J'(a) = \bigcup_{k=1}^{n} [a, b_k]; \quad b_k \in \bigcup J, \quad k = 1, \ldots, n, \quad a, \ldots, n, \quad n = 1, 2, \ldots$$

for each $a \in \bigcup J$, and

$$J' = \bigcup [J'(a); \quad a \in \bigcup J] = \{T; \quad T \text{ is a finite dendron and}$$

$$T \subset T' \text{ for some } T' \in J\}.$$}

It can easily be shown that $J \subset J'$, $J \cup J' = J \cup J'$, $[x, y]$ is a dendron for all $x, x \in \bigcup J$, and each of the families $J'(a)$ and $J'$ consists of finite dendrons and approximates $X$.

Now, we recall some definitions and facts from [26]. $X$ is still a continuum approximated by a family $J$ of finite dendrons. If $T$ and $T'$ are dendrons and $T \subset T'$ then there is a unique monotone retraction $r: T' \rightarrow T$. Hence $J$ is an inverse system of dendrons, where the ordering of $J$ is inclusion and bonding maps are monotone retractions. Write $T_j = \lim\{T_j\}$, so $T_j$ is a dendron. If $(x_t) \in T$, then $(x_t)$ is a convergent net in $X$. Moreover, the function $g_j; T_j \rightarrow X$ defined by $g_j((x_t)) = \lim x_t$ is a continuous surjection.

If $T \in J$ then we can treat $T$ as a subdendron of $T_j$:

$$T = \{(x_a) \in T_j; \quad x_a = x_t \text{ if } T \subset S\}.$$}

Put $T_\infty = \bigcup J$, $T_\infty \subset T_j$, and denote that $T_j - T_\infty$ is a subset of the set of all endpoints of $T_j$.

5.1. LEMMA. Let $X$ be a continuum which is approximated by a family $J$ of finite dendrons, and let $A$ be a subset of $X$ such that $\bigcup J \subset A$. Then there is a family $J'$ of finite dendrons such that $J'$ approximates $X$ and $\bigcup J' = A$. In particular, $X$ can be strongly approximated by finite dendrons.

Proof. Let $a$ be a fixed point of $\bigcup J$. For each $x \in A - \bigcup J$ choose $(x_t) \in T$ so that $g_j((x_t)) = x$. Note that the image $g_j([a, (x_t)])$ of the subarc $[a, (x_t)]$ of $T$.
is an arc in \( X \) — indeed, \( g_{J_{k(a),(b)}} \) is one-to-one and continuous. Let

\[
J' = \left( \bigcup_{k=1}^{n} L_k \right) \quad \text{either } L_k = \{a, d_k\} \text{ for some } d_k \in J \text{ or } L_k = g_{J_{k(a),(b)}}(x_k^2, x_k^3) \]

for some \( x_k^2 \in A - \bigcup J; \ k = 1, \ldots, n; \ n = 1, 2, \ldots \)

and observe that \( J' \) is a family of finite dendrons such that \( J'(a) \subset J' \) and \( \bigcup J' = A \); so \( \bigcup J' \) is dense in \( X \). Obviously \( J' \) is directed by inclusion.

Let \( U = \{U_1, \ldots, U_l\} \) be an open covering of \( X \) and let \( Y = \{V_1, \ldots, V_l\} \) be a shrink of \( U \), i.e., \( Y \) is an open covering of \( X \) so that \( \bigcap V_i = U_i \) for \( i = 1, \ldots, l \) (see [4], Theorem 1.5.18, p. 67). Let \( T \in J'(a) \) be such that, for each \( t \in T \), each component of \( T - T_r \) is contained in some member of \( V \). We show that if \( S \subset T' \), then each component \( S' \) of \( S - T_r \) is contained in \( V_i \subset U_i \) for some \( i \). This will finish the proof.

Let \( s_1, \ldots, s_m \) denote all end-points of \( S \). There is a directed set \( B \) and dendrons \( S_k \in J'(a) \), \( b \in B \), such that \( S - \{s_1, \ldots, s_m\} \subset \bigcup_{b \in B} S_b \subset S \) and \( S_b \subset S \), if \( b < c \). For each \( b \in B \) let \( S_b \subset \subset \), denote the component of \( S_b - T_r \) so that \( S_b \subset S' \). Hence \( \bigcup_{b \in B} S_b \subset S' \), if \( b < c \). Therefore, there is an index \( b_0 \in \{1, \ldots, l\} \) such that \( S_{b_0} \subset V_{b_0} \), for \( b \in B \).

Since \( S' \subset \bigcup_{b \in B} S_b \subset S' \), it follows that \( S' \subset V_{b_0} \).

5.2. Lemma. Let \( X \) be a locally connected continuum. The following conditions are equivalent:

(i) \( X \) can be strongly approximated by finite dendrons,

(ii) each cyclic element of \( X \) can be approximated by finite dendrons.

Proof. We show that (ii) implies (i). The implication (i) \( \Rightarrow \) (ii) will not be used in this paper (it can be derived from Theorem 1.1).

For each cyclic element \( Y \) of \( X \) let \( J_Y \) denote a family of finite dendrons such that \( J_Y \) strongly approximates \( Y \) (Lemma 5.1) and \( J_Y = J' \). Let \( a \) be a fixed point of \( X \). For each point \( b \in X \) let \( C(b, b) \) be the cyclic chain from \( a \) to \( b \), i.e.,

\[
C(a, b) = \{a, b\} \cup \{x \in X: x \text{ separates } a \text{ and } b\} \cup C,
\]

where \( C \) is the union of the family \( H \) of nondegenerate cyclic elements of \( X \) defined as follows: \( Y \in H \) if and only if \( Y \) contains exactly two points \( x_0 \) and \( y_0 \) from the set \( \{a, b\} \cup \{x: x \text{ separates } a \text{ and } b\} \).

Put

\[
[a, b] = \{a, b\} \cup \{x \in X: x \text{ separates } a \text{ and } b\} \cup \bigcup_{y \in Y} [x_0, y_0],
\]

and note that \( [a, b] \) is an arc (see [2], p. 254). Observe that

\[
J = \{ \bigcup_{k=1}^{n} [a_k, b_k]: b_k \in X, k = 1, \ldots, n, n = 1, 2, \ldots \}
\]

is a directed family of finite dendrons and \( \bigcup J = X \). Now, use [2], Lemma 2, p. 256, to finish the proof that \( J \) approximates \( X \).

5.3. Lemma. Let \( X \) be a locally connected continuum and let \( a, b \in X \) be such that \( X \) is a cyclic chain from \( a \) to \( b \). Suppose that

(i) each cyclic element of \( X \) which does not contain \( a \) or \( b \) can be approximated by finite dendrons, and

(ii) either \( [a] \) is a degenerate cyclic element of \( X \) or the unique cyclic element \( Y \) of \( X \) which contains \( a \) can be approximated by a family \( J \) of finite dendrons such that \( \bigcup J = Y \) or \( a \).

Then \( X \) can be approximated by a family \( K \) of finite dendrons such that \( \bigcup K = X - \{a\} \).

Proof. The family \( K \) can be constructed as in the proof of Lemma 5.2.

5.4. Lemma. Let \( X \) be a metrizable locally connected continuum. If \( a \) is a non-cut point of \( X \), then there exists a family \( J \) of finite dendrons such that \( J \) approximates \( X \) and \( \bigcup J = X - \{a\} \).

Proof. If \( X - \{a\} \) has property \( S \) (observe that if \( a \) is a local separating point, then \( U - \{a\} \) has only finitely many components for each connected open neighborhood \( U \) of \( a \)). By [1], Theorem 1, p. 1103 and Theorem 5, p. 1104, then \( X - \{a\} \) is partitionable (for the definition see [1] or [27]). Now, it is easy to apply the methods of L. E. Ward, [27], pp. 286-287, to find an increasing sequence \( D_1, D_2, \ldots \) of finite dendrites such that \( \bigcup_{i=1}^{\infty} D_i \approx X \) and \( a \notin \bigcup_{i=1}^{\infty} D_i \). By Lemma 5.1, there is a family \( J \) of finite dendrites such that \( J \) approximates \( X \) and \( \bigcup J = X - \{a\} \).

5.5. Lemma. Let \( X \) be a continuum, \( J \) a family of finite dendrons which strongly approximates \( X \) and \( A \) a \( T \)-set in \( X \) such that each component of \( X - A \) is homeomorphic to \( \{0\} \). Then there is a family \( J' \) of finite dendrons such that

(i) \( J' \) strongly approximates \( X \),

(ii) for each component \( Y \) of \( X - A \) there is a point \( y \in \bigcap_{i=1}^{\infty} D_i \) such that the arc \( \{a, y\} \subset Y \),

(iii) if \( Y \) is a component of \( X - A \) and \( a \in \text{bd}(Y) \) is a point such that, for each \( y \in Y \), \( [a, y] \subset Y \), then also, for each \( y \in \bigcap_{i=1}^{\infty} D_i \), \( [a, y] \subset Y \),

(iv) \( [a, b] \subset [a, b] \), for all \( a, b \in A \).

Proof. Let \( H \) be the family of all components \( Y \) of \( X - A \) such that there are \( x, y \in Y \) with the property \( \{a, x\} \subset Y \) and \( \{a, y\} \subset Y \) for some \( x \neq y \), where \( \text{bd}(Y) = \{a, y\} \). Hence there is a unique point \( x \in Y \) such that either

\[
(1) \{a, x\} \subset Y \text{ and } \{a, y\} \subset Y \text{ for } y \in X - \{a, x\},
\]

or

\[
(2) \{a, x\} \subset Y \text{ and } \{a, y\} \subset Y \text{ for } y \in X - \{a, x\}.
\]

Define \( d_x \) to be \( d_x \) if (1) holds, and \( d_x \) if (2) holds. Let \( a \in A \) be fixed and put

\[
J' = \{ \bigcup_{k=1}^{n} L_k: \text{ either } L_k = [c, d_k] \text{ for some } c \in A - \bigcup H \}\text{ or } L_k = [b, d_k] \cup [c, x_k] \text{ for some } \]

\[
x_k \in Y; k = 1, \ldots, n; n = 1, 2, \ldots ;
\]

for some
It can easily be shown that \( J' \) is a family of finite dendrons which approximates \( X \) (use Lemma 3.2), \( J' = X_0 \) and \( J' \) fulfills (ii)-(iv).

5.6. **Lemma.** Let \( X \) be a locally connected continuum without cut points and let \( A \) be a \( T \)-set in \( X \). Suppose that

(i) \( X_0 \) can be strongly approximated by a family \( J \) of finite dendrons such that for each component \( Z \) of \( X_0 - f_A(A) \) there is a point \( d_z \in \bd(Z) \) so that, for each \( z \in Z \), the arc \([d_z, s_z]\) is contained in \( Z \cup \{d_z\} \),

(ii) If \( Y \) is a component of \( X - A \), then \( Y \) can be approximated by a family \( J^\prime \) of finite dendrons such that \( \cup J^\prime \subset Y - b_2 \), where \( Z \) is the unique component of \( X_0 - f_A(A) \) such that \( f_A(Y) \in Z \), and \( b_2 \in \bd(Y) \) is the unique point of \( A \cap \bd(Z) - \{d_z\} \).

Then \( X \) can be strongly approximated by a family \( K \) of finite dendrons such that \( f_A(p, q, d) = (f_A(p), f_A(q), d) \), for all \( p, q \in A \).

**Proof.** For each component \( Z \) of \( X - f_A(A) \) let \( Y_Z \) denote the unique component of \( X - A \) so that \( f_A(Y_Z) \subset Z \) and let \( \bd(Z) = \{d_z, e_z\} \), \( \bd(Y_Z) = \{d_z, e_z\} \), so \( f_A(d_z) = d_z \), and \( f_A(e_z) = e_z \). Let \( H \) be the family of all components \( Z \) of \( X_0 - f_A(A) \) such that \([d_z, e_z] \cap Z \neq \emptyset \). Now, put \( J_Z = J^\prime \) if \( Z \notin H \); for \( Z \in H \) let \( J_Z \) be any family of finite dendrons which strongly approximates \( Y_Z \) (see Lemma 5.1).

Let \( a \in A \). Observe that if \( Z \) is a component of \( X - f_A(A) \), then either \( e_z \notin \{f_A(d), e_z\} \) or \( d_z \notin \{f_A(d), e_z\} \), for every \( e_z \in X \) put:

(1) \( L(x) := \{A \cap \bigcap_{e \in A} ([f_A(d), e_z]) \cup \{[d_z, e_z]\} \subset Z \in \{f_A(d), f_A(e_z)\} \) and \( Z \) is a component of \( X_0 - f_A(A) \).

(2) \( L(x) \) is a cut \( L(x) = L(d_e) \cup \{[d_z, e_z]\} \) if \( x \in Y_2 \) for a component \( Z \) of \( X_0 - f_A(A) \) so that \( e_z \notin \{f_A(d), e_z\} \) and \( d_z \notin \{f_A(d), e_z\} \).

(3) \( L(x) \) is a cut \( L(d_e) \cup \{[d_z, e_z]\} \) if \( x \in Y_2 \) for a component \( Z \) of \( X_0 - f_A(A) \) so that \( d_z \notin \{f_A(d), e_z\} \), note (this case \( Z \notin H \)).

We show that the sets \( L(x) \), \( x \in X \), are arcs.

First consider the case \( x \in A \). Suppose that there is a point \( y \in \bd(Z) \cap L(x) \). If \( y \neq A \), then there is a component \( Y \) of \( X - A \) such that \( y \in Y \), \( Y = Y_Z \) for some component \( Z \) of \( X - f_A(A) \). Observe that \( Z \in \{f_A(d), f_A(e_z)\} \), \( b_2 \in A \). Therefore \( L(x) \cap Y/Z \subset \{[d_z, e_z]\} \) and \( X_0 - f_A(A) \). Notice therefore \( L(x) \cap Y/Z \subset \{[d_z, e_z]\} \) is a neighbourhood of \( y \) disjoint from \( L(x) \), a contradiction which shows that \( y \neq A \). Note that

\[ \bigcap_{e \in A} ([f_A(d), e_z]) \cap A = L(x) \cap A \]

is closed and \( f_A(y) \in \{f_A(d), f_A(e_z)\} \), so \( y \notin L(x) \cap A \), again a contradiction. Thus \( L(x) \) is compact. Put

\[ L = \{A \cap \bigcap_{e \in A} ([f_A(d), e_z]) \cup \{[d_z, e_z]\} \] is a component of \( X_0 - f_A(A) \) and \( Z \in \{f_A(d), f_A(e_z)\} \).

By Lemma 2.3, \( L \) is a continuum such that

(4) if either \( M = \{y\} \) and \( y \in L \cap A - \{a, x\} \), or \( M \) is a component of \( X - A \) so that \( M \cap L \), then \( L = L(M) \) is a union of two mutually separated connected sets \( P, Q \) such that \( x \in P \) and \( x \notin Q \).

Observe that \( L(x) \subset L \) and \( L(x) \cap A = L \cap A \). Suppose that \( L(x) \) is not connected, i.e. there are two closed disjoint and nonempty sets \( F, G \) such that \( L(x) = F \cup G \). Put

\[ F' = F \cup \{[a_2, a_2]\} \in L(x) \text{ and } [a_2, b_2] \in X - G \neq \emptyset \],

\[ G' = G \cup \{[a_2, b_2]\} \in L(x) \text{ and } [a_2, b_2] \in X - G \neq \emptyset \],

and note that \( F', G' \) are closed, disjoint and \( L = F' \cup G' \), a contradiction. By (4), it follows that each point of \( L(x) - \{a, x\} \) is a cut point of \( L(x) \), and so \( L(x) \) is an arc (cf. Theorem 2-25).

If \( x \notin A \), then \( L(x) \) is a union of two arcs with exactly one common point which is an end-point of each of them. We have thus shown that \( L(x) \) is an arc for each \( x \in X \).

Let \( x, x' \in X \). We show that \( L(x) \cap L(x') \) is connected. Note that if \( x \notin A \) and \( x \notin A \) can be considered to satisfy the case \( x = y \in E \). In this case, \( f_A(L(x)) = \{f_A(d), f_A(e_z)\} \) and \( f_A(L(x')) = \{f_A(d), f_A(x')\} \). There is a \( z \in X_0 \) such that

\[ f_A(d), f_A(e_z) \in \{f_A(d), f_A(x')\} \]

Obviously, \( z \in X_0 \). Let \( z \in X_0 \) be the unique point of \( f_A(x') \cap A \); note that \( L(x) \cap L(x') = L(z) \). Therefore \( L(x) \) is a finite dendron for any \( x, x \in X_0 \), \( n = 1, 2, 3 \)... Put

\[ K = \{L(x_1) \cup \ldots \cup L(x_n) \} \]

Then \( K \) is a family of finite dendrons such that \( K \cap X \) is directed by inclusion. Moreover, the equality \( f_A(L(x)) = f_A(L(x')) \) for \( x \in A \) implies that \( f_A(p, q, d) = f_A(p, q, e_z) \), for all \( p, q \in A \) (because \( f_A(p, q, d) = f_A(p, q, e_z) \) for \( x \in E \)).

Let \( U = \{U_1, \ldots, U_n\} \) be an open covering of \( X \). For each component \( Z \) of \( X_0 - f_A(A) \) choose \( p_2, q_2, e \in Z \) such that \( d_2 < p_2 < q_2 < e_2 \) in a natural ordering of \( Z \). Let \( U' = \{U_1', \ldots, U_n'\} \) be an open covering of \( X \) constructed in Lemma 3.7. Put

\[ D = \{Z \subset X_0 - f_A(A) \}

is not contained in any member of \( U \);

so \( D \) is finite (Lemma 3.2). For each \( Z \in D \) let \( T_Z^\prime \subset T_Z \) be such that, for each \( T_Z \in \mathcal{T}_Z \) and each component \( T_Z^\prime \subset T_Z - f_A(A) \) is contained in some member of the open covering \( \{U_1, \ldots, U_n\} \cup \{Y_2^\prime\} \subset T_Z \).

Let \( S \subset T \) be such that for each \( S \in T \) and each component \( S' \subset S \), there is a \( k \) such that \( S' \subset U_k \). We may assume that \( S \subset T \) is not empty for each \( S \in T \), and that

\[ S = \bigcup_{i=1}^{n} \{f_A(d), f_A(e_z)\} \cup \bigcup_{i=1}^{n} \{f_A(d), q_2z_i\} \]
for some $a_1, \ldots, a_k \in A$ and some components $Z_1, \ldots, Z_n$ of $X_A - f_A(A)$ such that $z \in Z_i$. Put

$$T_0 = \bigcup_{i=1}^n L(n) \cup L(a_k) \cup \bigcup \{L(x) : x \text{ is an end-point \ of } T_0 \text{ for some } Z \in D \}$$

so $T_0 \in \mathcal{K}$. Let $T'$ be a component of $T - T_0$ for some $T \in \mathcal{K}$, and let $t$ be the unique point such that $T' = \{t\}$; so $t \in T \cap T_0$. We show that $T'$ is contained in some member of $U$.

First, suppose that $t \not\in A$. Hence $t \in Y_Z$ for some component $Z = X_A - f_A(A)$. If $Z \in H$ then $a \not\in R$ for each $R \not\in H$. Moreover, $b \not\in D$, and so $t \in T'$, $Z \not\in T_0$. Thus $x \in T_0$. If $Z \not\in H$ then $Z \not\in T_0$. Therefore $T'$ is contained in some component $T''$ of $T''$ (note that $T_0 \cap T$ is a dendron belonging to $T_0$), and there is an $i \in \{1, \ldots, n\}$ such that $T'' \subseteq T_0$. Hence $T' \subseteq T_0$. Now, suppose that $t \in A$. We may assume that $T' \subseteq T_0$. If $T'' \subseteq T_0$, then $t \not\in A$. Hence $T' \subseteq T_0$. We show that $T' \subseteq U_i$. Note that $U_i \cap T = f_{i+1}(U_{i+1}) \cap A$ and $T' \subseteq A = f_{i+1}(S') \cap A$. Suppose that $x \not\in U_i$. Observe that $U_i \cap T = f_{i+1}(U_{i+1}) \cap A$, and $x \not\in U_i \cap T$. If $x \not\in S'$, then $x \not\in U_i \cap T$. If $x \in T$, then $x \not\in U_i \cap T$. Hence $T' \subseteq U_i$. Now, suppose that $t \not\in A$. We may assume that $T' \subseteq T_0$. If $t \not\in A$, then $T' \subseteq T_0$. We show that $T' \subseteq U_i$. Note that $U_i \cap T = f_{i+1}(U_{i+1}) \cap A$ and $T' \subseteq A = f_{i+1}(S') \cap A$. Suppose that $x \not\in U_i$. Observe that $U_i \cap T = f_{i+1}(U_{i+1}) \cap A$, and $x \not\in U_i \cap T$. If $x \not\in S'$, then $x \not\in U_i \cap T$. If $x \in T$, then $x \not\in U_i \cap T$. Hence $T' \subseteq U_i$.

5.7. Lemma. Let $X$ be a locally connected continuum without cut points, and let $A, A_1, A_2, \ldots$ be T-sets in $X$ such that $A_1 \subset A_2 \subset \ldots$ and $Y$ is a component of $X - A$. Suppose that, for each $n$, there is a family $\mathcal{A}_n$ of finite dendrons which strongly approximates $X_A$, such that each component $Z$ of $X_A - f_A(A)$ there is a $d_{Z} \in \mathcal{A}_Z$. Then $Z$ is represented in $Z \cup \{d_{Z}\}$.

Put

$$L_n(p, q) = f_{i+1}(L(p, q)) \cup \{Y : Y \text{ is a component of } X - A_n, f_{i+1}(Y) = [f_{i+1}(p), f_{i+1}(q)]_{i+1}\}
$$

for each $n \in \{1, 2, \ldots\}$ and all $p, q \in A$.
Let $p_b = (x_i, y_i, z_i) \in \mathbb{R}^3$ for $i = 1, 2, \ldots, n$. Then $p_b$ belongs to the set $S_b$ if and only if $x_i < y_i < z_i$ for all $i = 1, 2, \ldots, n$. The set $S_b$ is open and connected.

If $b = (x, y, z) \in \mathbb{R}^3$, then $b$ belongs to the set $S_b$ if and only if $x < y < z$. The set $S_b$ is open and connected.

The set $S_b$ is defined as the intersection of $S_b$ and $S_b$ with the $b$-plane, where $b$ is a fixed point in $\mathbb{R}^3$. The set $S_b$ is open and connected.

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The set $S_b$ is defined as the intersection of $S_b$ and $S_b$ with the $b$-plane, where $b$ is a fixed point in $\mathbb{R}^3$. The set $S_b$ is open and connected.
(*) if $U$ is a component of $Y - A$, if $V$ is the unique component of $Z - A$ such that $f_{A,1}(U) \subset V$ and if $W$ is the unique component of $Z_{1} - g(A)$ such that $g'(V) \subset W$, then $W$ is the unique component of $Z_{1} - g(A)$ such that both $g_{A,1}(U) \subset W$ and $h_{A,1}(U) \subset W$.

Put $J = \{ (T) : T \in J_{1} \}$; so $J$ is a family of finite dendrons which strongly approximates $Z_{1}$ and, by (I), for each component $N$ of $Z_{1} - g(A)$ there is a $b_{N} \in b_{n}(N)$ so that, for each $T \in J$, the arc $[b_{N}, z_{T}]$ is contained in $N \cup [b_{N}]$.

Let $N$ be a component of $Z_{1} - g(A)$. Let $M_{N}$ be the unique component of $Z_{1} - A$ so that $g(M_{N}) = N$, and let $a_{N}$ be the unique point of $A \cap g^{-1}(N)$. Hence $a_{N} \in b_{n}(M_{N})$, let $b_{N}$ be the second point of $b_{n}(M_{N})$. Note that $M_{N}$ is a cyclic chain from $a_{N}$ to $b_{N}$. Let $C$ be a nondegenerate cyclic element of $M_{N}$. Put

$$A_{1} = C \cap f_{A,1}(A_{1}) .$$

Note that if $U$ is the unique component of $Y - A$, such that $f_{A,1}(U) \subset M_{N}$, then there is a unique cyclic element $C'$ of $U$ such that $f_{A,1}(C' \cap A_{1}) = A_{1}$. By (C), $A_{1}$ is metrizable. Since $A_{1}$ is a T-set in $C$ and each component of $A_{C} - A_{1}$ is homeomorphic to $[0, 1]$, in such an interval, it follows that $C$ is a metrizable locally connected continuum. By [26], Theorem 2, p. 373, $C$ can be approximated by finite dendrons. Moreover, if $b_{N} \in C$, then $b_{N}$ is a non-cut point of $C$; so, $C$ can be approximated by a family $I$ of finite dendrons such that $| I | = C - [b_{N}]$ (Lemma 5.4). By Lemma 5.3, $M_{N}$ can be approximated by a family $J_{N}$ of finite dendrons such that $| J_{N} | = M_{N} - [b_{N}]$.

By Lemma 5.6, $Z$ can be strongly approximated by a family $K$ of finite dendrons such that $g(p_k, q_k) = [g(p), g(q)]$ for all $p, q \in A$. By Lemma 5.5, there is a family $K'$ of finite dendrons which strongly approximates $Z$ such that $| K' | = [p, q]$ for all $p, q \in A$. By Lemma 5.6.5, $L_{1} = f_{A,1}(A_{1})$ (Lemma 5.6).

Set $b_{N} \in A_{N}$ be a fixed point. We shall show that $L_{1}(b) \subset N(b)$. Observe that $g'(f_{A,1}(a), f_{A,1}(b))_{m_{N}} = [g_{A,1}(a), g_{A,1}(b)]_{m_{N}}$. It follows that

$$h^{-1}g'(f_{A,1}(a), f_{A,1}(b))_{m_{N}} = h^{-1}g_{A,1}(a), g_{A,1}(b))_{m_{N}} = h^{-1}g_{A,1}(a), g_{A,1}(b))_{m_{N}}$$

Therefore $L_{1}(b) \cap A_{1} = L_{1}(b) \cap A_{1}$. Take any point $y \in L_{1}(b) \cap A_{1}$. We show that $y \in L_{1}(b)$.

Let $U$ be the component of $Y - A$ such that $y \in U$, let $V$ be the unique component of $Z_{1} - g(A)$ such that $f_{A,1}(U) \subset V$, and let $W$ be the unique component of $Z_{1} - g(A)$ such that $g'(V) \subset W$. Thus $W$ is the unique component of $Z_{1} - g(A)$ such that $g_{A,1}(U) \subset W$.

Since $L_{1}(b) \cap A_{1} = L_{1}(b) \cap A_{1}$, $U$ is a component of $Y - A$, and $U \cap L_{1}(b) \cap A_{1} \neq \emptyset$, it follows that $b_{N}(U) \cap L_{1}(b) \cap A_{1} \neq \emptyset$. Suppose that $b_{N}(U) \cap L_{1}(b) \cap A_{1} \neq \emptyset$. Suppose that $c \neq (a, b)$. By Lemma 2.3, $L_{1}(b) - [c]$ has exactly two components, $P$ so that $a \in P$ and $b \in Q$. Note the nonempty set $U \cap L_{1}(b)$ is a union of some family of components of $L_{1}(b) - [c]$ (since $d \notin L_{1}(b)$) but, $b \notin U$, a contradiction. Suppose that $c \in (a, b]$. Hence $L_{1}(b) - [c]$ is connected, so $L_{1}(b) \cap U \subset U \subset [c]$, again a contradiction (because $L_{1}(b)$ is nondegenerate, and so $c \neq (a, b]$). We have thereby proved that $b_{N}(U) \subset L_{1}(b)$.

Since $L_{1}(b)$ is a continuum, $b_{N}(U) \subset L_{1}(b)$, and $y \in U \cap L_{1}(b) \cap A_{1} \neq \emptyset$, it follows that $L_{1}(b) \cap U$ is a continuum. Since $W$ is homeomorphic to $[0, 1]$, $W = g_{A,1}(U)$, and $b_{N}(W) = g_{A,1}(b_{N}(U)) = g_{A,1}(b_{N}(U))$, it follows that $g_{A,1}(L_{1}(b) \cap U) = W$. Observe that, by $(*)$.

$$W = g_{A,1}(L_{1}(b)) = g_{A,1}(f_{A,1}(a), f_{A,1}(b))_{m_{N}} = h_{A,1}(f_{A,1}(a), f_{A,1}(b))_{m_{N}}$$

and $h^{-1}(W) \subset f_{A,1}(a), f_{A,1}(b))_{m_{N}}$. By $(*)$, $h^{-1}(W)$ is the unique component of $A_{1} - f_{A,1}(a), f_{A,1}(b))_{m_{N}}$ such that $f_{A,1}(a), f_{A,1}(b))_{m_{N}} \subset h^{-1}(W)$. Therefore $y \in U \subset L_{1}(b)$.

The property that we have pointed out a mistake in the proof of the main result of [15] (see Introduction, above). Recently we have proved that each hereditarily locally connected continuum is a continuous image of an arc, thus the proof in [15] cannot be repaired.

References


On irreducibility and indecomposability of continua

by

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Abstract. Kuratowski (1927) showed that in metric continua their points of indecomposability are always points of irreducibility. The aim of this paper is to exhibit a general form of those Hausdorff continua for which the result of Kuratowski does not hold.

1. Introduction and preliminaries. In this paper $X$ will always be a Hausdorff continuum, shortly a $\mathcal{F}_X$-continuum, i.e. a connected and compact topological space which satisfies the $\mathcal{F}_X$-axiom of separability. A point $a$ of $X$ is said to be a point of indecomposability of $X$ if there is no decomposition of $X$ into two proper subcontinua which both contain $a$, i.e. for every two subcontinua $K_1$ and $K_2$ of $X$

(1.1) $a \in K_1 \cap K_2$ and $K_1 \cup K_2 = X$ imply $K_1 = X$ or $K_2 = X$.

A point $a$ of $X$ is said to be a point of irreducibility of $X$ if there is $b \in X$ such that no proper subcontinuum of $X$ contains both $a$ and $b$, i.e. for every subcontinuum $K$ of $X$

(1.2) $a \in K$ and $b \in K$ imply $K = X$.

$X$ is then said to be irreducible between $a$ and $b$.

Directly by the above two definitions, every point of irreducibility is a point of indecomposability, and the converse assertion:

(1.3) Every point of indecomposability is a point of irreducibility

has been proved for metric continua in [10] (Théorème XIX, p. 270). In connection with some fixed point theorems [4], [12], [13] and [15], the assertion (1.3) has been proved for hereditarily decomposable $\mathcal{F}_X$-continua in [14] (Theorem 1, p. 52, where in fact no axiom of separability is used). In [2], a $\mathcal{F}_X$-continuum has been constructed which is indecomposable but not irreducible, i.e. its every point is a point of indecomposability but no point is a point of irreducibility, so that the above assertion (1.3) is not true for an arbitrary $\mathcal{F}_X$-continuum $X$.

In the present paper, we shall characterize those $\mathcal{F}_X$-continua $X$ which have this singularity, i.e. such $X$ that

(1.4) There exists a point of indecomposability of $X$ which is not a point of irreducibility of $X$;