Marczewski sets, measure and the Baire property

by

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Abstract. This paper is a study of Marczewski sets. This study will examine how Marczewski sets are related to other types of sets, such as universally measurable sets, and sets with Baire property in the restricted sense. It will also look at theorems about Marczewski sets that parallel theorems about these other types of sets. The paper is divided into three sections. The first is an introduction. The second deals with the construction of sets which are hereditarily Marczewski. The third section deals with Marczewski sets and Borel measurable functions. A necessary and sufficient condition for Marczewski sets to be preserved by Borel measurable functions is given here.

I. Introduction. In this paper all spaces considered are Polish spaces (i.e. complete, separable and metric), and will usually be referred to as X or Y. This paper is concerned with classes of sets, in particular Marczewski sets. Many fundamental theorems, relationships and examples associated with the classes of sets studied here can be found in [Ku66], [Sz35] and [BrCo82]. All arguments will only rely on ZFC unless otherwise noted and C will be used to denote the cardinality of the continuum.

In [Si35] Sierpiński defined, using a continuity condition, a class of functions that is closed under pointwise convergence and composition. A function f is in this class of Sierpiński if every perfect set P contains a perfect set Q such that $f|Q$ is continuous. In [Sz35], a study of Sierpiński's class of functions, Marczewski labeled Sierpiński's class of functions, as functions having *property* (s). Marczewski defined a set M to have *property* (s) if every perfect set P contains a perfect subset Q such that either $Q \subseteq M$ or $Q \cap M = \emptyset$. Sets possessing property (s) are referred to as Marczewski sets. In this same paper Marczewski showed that a function f has property (s) if and only if for each open set $\mathcal{O}$, $f^{-1}(\mathcal{O})$ has property (s).

This theorem of Marczewski is similar to other theorems that link continuity properties of functions to set properties of inverse images of open sets. A theorem of Baire states that a function is B measurable of class 1 if and only if it is pointwise discontinuous when restricted to any perfect set, [Ku66 p. 419]. Another theorem states that a function f has the Baire property in the wide sense (i.e. inverse images of open sets have the Baire property in the wide sense, defined below), if and only if there exists a set F of first category such that $f|(X-F)$ is continuous, [Ku66 p. 400]. Finally, a function f has the Baire property in the restricted sense (i.e. inverse images of open sets have the Baire property in the restricted sense, defined below) if and only if every perfect set P contains a set F of first category relative to P such that $f|(P-F)$ is continuous, [Ku66 p. 403].

A set is *totally imperfect* if it contains no homeomorphic copy of the Cantor set. A set M is *Bernstein* relative to a perfect set P if both $P \cap M$ and $P-M$ are totally imperfect, in other words both M and $P-M$ intersect every perfect subset of P. In [BrCo82] a through study of totally imperfect sets is made.

A set has *property* B_w (i.e. the Baire property in the wide sense) if it is the symmetric difference of an open set and a set of first category. A set M has *property* B_r (i.e. the Baire property in the restricted sense) if for each perfect set P, M has property B_w relative to P. A set is AFC (i.e. always of first category) if it is of first category relative to every perfect set.

A set has *property* U (i.e. universally measurable) if it is measurable in the completion of every Borel measure on the space. In other terms a set M has property U if for each Borel measure μ there exist Borel sets B_1 and B_2 such that $B_1 \subseteq M \subseteq B_2$ and $\mu(B_1) = \mu(B_2)$. A set has *property* U_0 (i.e. universal null) if it has measure zero in the completion of each continuous Borel measure.

One way to describe a set M that has property (s) (i.e. Marczewski) is that the set M is not Bernstein relative to any perfect set. A set M has *property* (s^0) if every perfect set P contains a perfect subset Q such that $Q \cap M = \emptyset$. A set will have property (s^0) if and only if it has property (s) and is totally imperfect. The sets with property (s) form a σ-algebra and the sets with property (s^0) form a σ-ideal (i.e. hereditary and closed under countable unions) in the sets with property (s). Other details about sets with property (s) can be found in [Sz35].

II. Examples. This section develops methods of constructing sets with property (s^0). Theorem 2.2 will be useful in section three when dealing with B measurable functions. Theorem 2.6 can be applied to other types of sets in constructing examples with positive dimension.

THEOREM 2.1. *All sets of cardinality less than the continuum have property* (s^0).

Proof. This directly follows from the fact that all perfect sets can be divided into a collection of continuum many disjoint uncountable closed sets.

THEOREM 2.2. *If* $\mathscr{D} = \{D_\alpha\colon \alpha < c\}$ is a collection of disjoint uncountable Borel sets then there exists a set M with property (s^0) that intersects each member of $\mathscr{D}$.

Proof. Let $\mathscr{P}$ denote the collection of all perfect subsets of X. Let $\mathscr{E} = \{P \in \mathscr{P}\colon$ for each $D_\alpha \in \mathscr{D}$, $|P \cap D_\alpha| < C\} = \{E_\alpha\colon \alpha < c\}$. Now choose $x_\alpha \in D_\alpha - (\bigcup_{\beta<\alpha} E_\beta)$. This can be done since the intersection of any member of $\mathscr{D}$ with any member of $\mathscr{E}$ is at most a countable set and all uncountable Borel sets contain a perfect subset [Ku66, p. 479]. Let $M = \{x_\alpha\colon \alpha < c\}$. Certainly $M \cap D_\alpha = \{x_\alpha\}$ is nonempty for each $\alpha < c$. Now suppose P is a perfect set. If $P = E_\alpha$ then $M \cap P \subseteq \{x_\beta\colon \beta \leqslant \alpha\}$ so $|M \cap P| < C$ and therefore must contain a perfect subset which misses M. If $P \notin \mathscr{E}$ then $|P \cap D_\alpha| = C$ for some $\alpha < c$. Now $(P \cap D_\alpha) - M = (P \cap D_\alpha) - \{x_\alpha\}$ is an uncountable Borel set, which must contain a perfect subset that misses M, so P contains a perfect subset that misses M. Therefore M has property (s^0).

Note that if $|\mathscr{D}| < C$ Theorem 2.2 would be a trivial consequence of Theorem 2.1 This also gives a useful method of constructing sets with property (s^0) having cardinality C. An example of such a set is given in [Mi84].

If $M \subseteq X \times Y$, let $\mathrm{proj}_X M = \{x\colon (x, y) \in M$ for some $y \in Y\}$, $\mathrm{proj}_Y M = \{y\colon (x, y) \in M$ for some $x \in X\}$, $G(x, M) = M \cap (\{x\} \times Y)$, and $G(M, y) = M \cap (X \times \{y\})$. In [Sz35] Marczewski showed that property (s) and property (s^0) are preserved under Cartesian products. Property U_0 is also preserved under Cartesian products but it is not known whether or not AFC sets are preserved. If $M \subseteq X$ is first category (Lebesgue measure zero) then $M \times Y$ will be first category (Lebesgue measure zero). Theorem 2.3 is a strengthening of Marczewski's result dealing with property (s^0) and Cartesian products. A similar result does hold for property U_0 but not for property (s). The proof of Theorem 2.3 is straightforward and omitted here.

THEOREM 2.3. *If* $M \subseteq X \times Y$, $\mathrm{proj}_X M$ *has property* (s^0) *and* $G(x, M)$ *has property* (s^0) *for each* $x \in X$ *then* M *has property* (s^0).

Two sets with property (s) (property U, property B_r) can be considered equivalent if their symmetric difference has property (s^0) (property U_0, AFC). In [Si34] Sierpiński showed (using CH) that there exists a collection with cardinality 2^c of nonequivalent sets with property B_r. Sierpiński also noted that without using CH this collection has cardinality at least $2^{\aleph_1}$. In [Sz55] Marczewski showed (using CH) that there exists a collection with cardinality 2^c of nonequivalent sets with property U. In [GrRy80] Grzegorek and Ryll–Nardzewski showed (without CH) that this collection can have cardinality at least $2^{\aleph_1}$. The following theorem is a similar result dealing with property (s).

THEOREM 2.4. *In* $R \times R$ *there exists a collection with cardinality* 2^c *of nonequivalent sets with property* (s).

Proof. Let $S \subseteq R$ have property (s^0) and cardinality C. Now for each S' and S'' subsets of S, the sets $S' \times R$ and $S'' \times R$ have property (s) and the set $(S' \times R) \cap (S'' \times R)$ does not have property (s^0) whenever $S' \neq S''$. Therefore

$$|\{S' \times R\colon S' \subseteq S\}| = 2^c.$$

The final theorem of this section allows the construction of a "worst" example of a set with property (s^0). It will also apply (assuming CH) to several other smallness properties. A set M has *property* λ (i.e. is rarified) if every countable subset of M is G_δ relative to M. A set M has *property* λ' if for each countable set $A \subseteq X$ the set

$M \cup A$ has property λ. A set M has *property* σ if every F_σ subset of M is also a G_δ subset of M. A set has *property* S if it intersects every Lebesgue measure zero set in a countable set. A set M has *property* $C(\mathrm{rel}\,\delta)$ (δ is a metric) if for every sequence $\{a_n\}$ of positive numbers, there exists a sequence $\{x_n\}$ of elements of M such that $M \subseteq \bigcup_{i=1}^{\infty} N_\delta(x_i, a_i)$ where $N_\delta(x, a) = \{y \in X\colon \delta(x, y) < a\}$. More on these sets can be found in [BrCo82].

In [MaSz37] Mazurkiewicz and Marczewski showed that if a set has either property σ or $C(\mathrm{rel}\,\delta)$ it has dimension zero. Note that since property S implies property σ, property S also implies dimension zero. They also gave a theorem, based on a theorem of Hilgers [Hi37], that showed (assuming CH) the existence of sets with property U_0 or property λ that have positive dimension.

THEOREM 2.5 (Mazurkiewicz and Marczewski). *If property* (?) *satisfies the conditions that*

(1) *there exists a linear set with property* (?) *and cardinality* C, *and*

(2) *property* (?) *is preserved under one-to-one functions with continuous inverses, then there exists a set of dimension* n *in* R^{n+1} *which also has property* (?).

Note here that this theorem cannot be applied to property λ' or AFC sets since they do not satisfy Condition (2), see [Si45] and [Lu33] respectively. However Theorem 2.5 is easily applied to property (s^0). By using methods developed in showing Theorem 2.6 it can be seen that condition (2) of Theorem 2.5 could be weakened to (2′); property (?) is preserved under inverses of one-to-one projections.

A set M is *C-dense in the set* B if every open subset of B contains continuum many points of M. The following theorem is a variation of the one by Mazurkiewicz and Marczewski.

THEOREM 2.6. *If property* (?) *satisfies the conditions that*

(1) *there exists a linear set with property* (?) *and cardinality* C,

(2) *property* (?) *is preserved under inverses of one-to-one projections and*

(3) *property* (?) *is preserved under countable unions,*

then there exists a set with property (?) *that is C-dense in every nondegenerate closed connected subset of* $R \times R$.

Remarks. If a set is C-dense in every nondegenerate closed connected subset of $R \times R$, it will have dimension $\geqslant 1$. Also note that Theorem 2.6 can be applied to property (s^0) and assuming CH it can be applied to property λ' [Si37], [Si45] and property U_0. However it cannot be applied to property λ since property λ is not preserved under countable unions [Ro39]. This is a "worst" case for sets with property (s^0) since a set with property (s^0) cannot intersect every uncountable closed subset of $R \times R$. Since both property U_0 and property λ' imply property (s^0) this will also be a "worst" case for them.

The proof of theorem 2.6 will use the following two lemmas which are stated without proofs.

LEMMA 2.7. *If* H *is a nondegenerate closed connected subset of* $R \times R$, $(x, y) \in H$ *and* $\mathcal{O}$ *is an open set containing* (x, y) *then either* $\mathrm{proj}_X(H \cap \mathcal{O})$ *or* $\mathrm{proj}_Y(H \cap \mathcal{O})$ *will contain an interval.*

LEMMA 2.8. *If* M *is C-dense in* R *then* $M = \bigcup_{\alpha < c} M$ *where* $M_\alpha \cap M_\beta = \emptyset$ *for* $\alpha \neq \beta$ *and* M_α *is C-dense in* R *for each* $\alpha < c$.

Proof of Theorem 2.6. Since there exists a linear set with property (?) and cardinality C, and property (?) is preserved under countable unions there exists a set M with property (?) that is C-dense in R. Let $M = \bigcup_{\alpha < c} M_\alpha$ as in Lemma 2.8. Let $\mathscr{H} = \{H_\alpha\colon \alpha < c\}$ be the collection of all nondegenerated closed connected subsets of $R \times R$. For each $\alpha < c$ and each $x \in M_\alpha$ let $A_{\alpha x}$ be a countable dense subset of $(\{x\} \times R) \cap H_\alpha$ unioned with $\{(x, 0)\}$. For each $\alpha < c$ and each $y \in M_\alpha$ let $B_{\alpha y}$ be a countable dense subset of $(R \times \{y\}) \cap H_\alpha$ unioned with $\{(0, y)\}$. Let $S_1 = \bigcup_{\alpha < c} \bigcup_{x \in M_\alpha} A_{\alpha x}$ and $S_2 = \bigcup_{\alpha < c} \bigcup_{y \in M_\alpha} B_{\alpha y}$ and $S = S_1 \cup S_2$. Since $\mathrm{proj}_X S_1 = M$ and $G(x, S_1)$ is countable for each $x \in M$ it is easy to see that S_1 is the countable union of sets with property (?) so S_1 will have property (?). Similarly S_2 will have property (?) and therefore S will have property (?). Now suppose that $H_\alpha \cap \mathcal{O}$, $p \in H_\alpha$ and $\mathcal{O}$ is an open set containing p. By Lemma 2.7 either $\mathrm{proj}_X(H_\alpha \cap \mathcal{O})$ or $\mathrm{proj}_Y(H_\alpha \cap \mathcal{O})$ will contain an interval. Without loss of generality assume $\mathrm{proj}_X(H_\alpha \cap \mathcal{O})$ contains the interval (a, b). Now since M_α is C-dense in R, (a, b) will contain continuum many elements of M_α. Now $A_{\alpha x}$ will intersect $H_\alpha \cap \mathcal{O}$ at least once for each $x \in M_\alpha \cap (a, b)$, since $A_{\alpha x}$ is dense in $(\{x\} \times R) \cap H_\alpha$ So $\mathcal{O} \cap H_\alpha$ will contain continuum many points of S_1. Therefore S is C-dense in every nondegenerate closed connected subset of $R \times R$ and has property (?).

COROLLARY 2.9. *There exists* (*assuming* CH) *a set with property* λ' *that does not have property* σ.

Proof. By theorem 2.6 there will exist a set in $R \times R$ with property λ' and positive dimension, but all sets with property σ have dimension zero.

III. B **measurable functions.** A function $f\colon X \to Y$ is B *measurable* if $f^{-1}(\mathcal{O})$ is a Borel set for each open set $\mathcal{O} \subseteq Y$. A B measurable function $f\colon X \to Y$ is *bimeasurable* if $f(B)$ is a Borel set for each Borel set $B \subseteq X$. A one-to-one function $f\colon X \to Y$ is a generalized homeomorphism if both f and f^{-1} are B measurable. If f is a function let $U(f) = \{y\colon f^{-1}(y)$ is uncountable$\}$.

By using a result of Lusin [Ku66 p. 498 Corollary 5] it can be shown that if $f\colon X \to Y$ is B measurable and $U(f)$ is countable then there exist a sequence of Borel sets $B_0, B_1, B_2, \ldots$ such that $X = \bigcup_{i=1}^{\infty} B_i$, $f^{-1}(U(f)) = B_0$ and for each positive integer i, $f|B_i$ is a one-to-one B measurable function. Since a one-to-one B measurable function defined on a Borel set is a generalized homeomorphism [Ku66

p. 489], it can easily be shown that a sufficient condition for a B measurable function o be bimeasurable is that $U(f)$ be countable.

In [Pu66] Purvis showed this sufficient condition for a B measurable function to be bimeasurable is also a necessary condition. In [Da70] and [Da71] Darst showed (using CH) that a B measurable function f is bimeasurable if and only if $f(M)$ has property U (property U_0) for each $M \subseteq X$ with property U (property U_0). In [Gr81] Grzegorek removed the use of CH from Darst's argument.

Theorem 3.1 can be found in [Sz35] but is stated here, without proof, for completeness.

THEOREM 3.1. *If $f\colon X \to Y$ is a one-to-one B measurable function and $M \subseteq X$ has property (s) then $f(M)$ has property (s).*

THEOREM 3.2. *If $f\colon X \to Y$ is a one-to-one B measurable function and $M \subseteq X$ has property (s^0) then $f(M)$ has property (s^0).*

Proof. This follows directly from Theorem 3.1 and the fact that a set has property (s^0) if and only if has property (s) and is totally imperfect.

THEOREM 3.3. *A B measurable function $f\colon X \to Y$ is bimeasurable if and only if for each $M \subseteq X$ with property (s), $f(M)$ has property (s).*

Theorem 3.4 will be stated now, and then both Theorem 3.3 and Theorem 3.4 will be proven simultaneously.

THEOREM 3.4. *A B measurable function $f\colon X \to Y$ is bimeasurable if and only if for each $M \subseteq X$ with property (s^0) $f(M)$ has property (s^0).*

Proof of Theorems 3.3 and 3.4. If $f\colon X \to Y$ is bimeasurable then, $U(f)$ is countable. Now if $M \subseteq X$ has property (s) (property (s^0)) then, by Theorem 3.1 (Theorem 3.2) and the previously cited theorem of Lusin $f(M)$ has property (s) (property s^0). So f preserves both property (s) and property (s^0). If f is not bimeasurable then, $U(f)$ is uncountable. Now $U(f)$ is an uncountable analytic set [Ku66 p. 496] so it will contain a perfect set P and a set P' such that both P' and $P-P'$ are totally imperfect. Now P' does not have property (s). But $\mathscr{D} = \{D\mathfrak{y}\colon f^{-1}(y) = D\mathfrak{y}$ for some $y \in P'\}$ is a collection of disjoint uncountable Borel sets so there exists a set S with property (s^0) that intersects each member of $\mathscr{D}$, by Theorem 2.2. Now $f(S) = P'$ does not have property (s) so f does not preserve either property (s) or property (s^0).

The final theorem presented here came from examining $\{E \subseteq R\colon f^{-1}(E)$ has property (s) for each B measurable function $f\colon R \to R\}$. Which is somewhat similar to Darst's examination of $\{E \subseteq R\colon f^{-1}(E)$ is Lebesgue measurable for each Lebesgue measurable function $f\colon R \to R\}$, although the result was quite different.

THEOREM 3.5. *If $f\colon X \to Y$ is B measurable and $M \subseteq Y$ has property (s) then $f^{-1}(M)$ has property (s).*

Proof. Suppose P is a perfect subset of X and note that $f(P)$ is an analytic set. The theorem can now be reduced to two cases. For the first case suppose $f(P)$ is countable and $y \in U(f|P)$. Now $f^{-1}(y) \cap P$ is an uncountable Borel set which will

either be a subset of $f^{-1}(M)$ or miss $f^{-1}(M)$. For the second case assume $f(P)$ is uncountable and D is a perfect subset of $f(P)$ such that either $D \subseteq M$ or $D \cap M = \emptyset$. Note that $f^{-1}(D) \cap P$ is an uncountable Borel set and the remainder of the theorem easily follows.

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