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DRUKARNIA UNIWERSYTETU JAGIELLOŃSKIEGO W KRAKOWIE

On a class of topological spaces with a Scott sentence

by

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Abstract. By means of a topological game with two players we study the expressive power of the topological language $(L_{\omega_1\omega})_t$ for T_3 spaces. The main result of this paper is a partial characterization of homeomorphism type of the space by means of certain topological properties which are expressible in this language. In this way, we find a class of topological spaces with a Scott sentence which includes every countable ordinal with order topology.

§ 0. Introduction. The infinitary language $L_{\omega_1\omega}$ is obtained from the first order language $L_{\omega\omega}$ (in the classical sense) by adding the following formation rule: If Φ is a countable set of formulas, $\bigvee\Phi$ and $\bigwedge\Phi$ are formulas.

$(L_{\omega_1\omega})_t$ is the topological analog of the language $L_{\omega_1\omega}$. It is a formal language in the study of topological structures. $(L_{\omega_1\omega})_t$ is obtained from $L_{\omega_1\omega}$ by adding the symbol $\in$ and set variables $X, Y, \ldots$ The atomic formulas of $(L_{\omega_1\omega})_t$ are of the form $x = y$ and $x \in X$. The formation rules of $(L_{\omega_1\omega})_t$ are those of $L_{\omega_1\omega}$ and the following two rules:

(i) if a formula φ is positive in X, then $\forall X(x \in X \to \varphi)$ is a formula.

(ii) If a formula φ is negative in X, then $\exists X(x \in X \wedge \varphi)$ is a formula.

A formula φ is *positive* (*negative*) in X if each free occurrence of X in φ is within the scope of an even (odd) number of negation symbols. The set variables range over the class of open sets of the space and, intuitively, quantifications over sets in $(L_{\omega_1\omega})_t$ are quantifications over small enough neighborhoods of a point.

It is shown in [1] that in many cases it is possible to give a parallel treatment of classical and topological model theory.

Every space considered here is assumed to be T_3 (i.e. Hausdorff and regular). We denote T_3 spaces by $A, B, \ldots$ It is an immediate consequence of the Löwenheim–Skolem theorem for $(L_{\omega_1\omega})_t$ that, for every sentence φ of $(L_{\omega_1\omega})_t$, if φ is satisfied in a T_3 space then φ is satisfied in a countable metrizable space. This says that the class of countable metrizable spaces is, from the point of view of $(L_{\omega_1\omega})_t$, dense in the class of all T_3 spaces. Let A be a countable metrizable space. A sentence φ of $(L_{\omega_1\omega})_t$ is said to be a *Scott sentence* of A if $A \models \varphi$ and every countable metrizable space which satisfies φ is homeomorphic to A. In the present paper we find a class of countable metrizable spaces with a Scott sentence. It is not known whether there exists a countable metrizable space without such a sentence.

We study the set of accumulation points of a space by means of a certain topological game with two players. In this way, we partition the space into classes of points of the same type. The main result of this paper is a characterization of homeomorphism types, for a certain class of T_3 spaces which includes every countable ordinal with order topology, by means of certain topological properties which are expressible in $(L_{\omega_1\omega})_t$. Intuitively, a T_3 space A belongs to our class if for every $a \in A$ and every neighborhood U of a we can find a neighborhood U_0 of a with $U_0 \subset U$ in such a way that U_0 is sufficiently small and we can determine which types of points are in U_0. The main theorem permits us to characterize the $(L_{\omega_1\omega})_t$-theory of any space of our class. Then as a corollary we deduce that every countable ordinal with order topology has a Scott sentence.

The results of the present paper are an improvement of the results of [2]. A classification of the $(L_{\omega\omega})_t$-theories of T_3 spaces is given in [1].

§ 1. Accessible sets and types of convergence. If A^* is a subset of a space A and a is an accumulation point of A^*, we say that A^* *converges to* a and write $A^* \to a$. If A_1^*, A_2^* are subsets of A such that $A_1^* \to a$ for every $a \in A_2^*$, we write $A_1^* \to A_2^*$. In order to study that convergence, we introduce the notion of accessible set.

We are given a T_3 space A, a subset A^* of A and two players I and II. In the game $G(A^*, A)$ each player makes infinitely many moves. In his ith move player I first chooses an arbitrary finite sequence $a_1, \ldots, a_n$ of points in A and then in his ith move player II chooses a sequence of n neighborhoods U_1 of $a_1, \ldots, U_n$ of a_n in A. Let $U_1^i, \ldots, U_k^i$ be all the neighborhoods chosen by II in the moves $1, \ldots, i$. A^* is *covered* in move i if $A^* \subset U_1^i \cup \ldots \cup U_k^i$. Then I wins in the game $G(A^*, A)$ if A^* is covered in move i for some i. We say that the set A^* is *accessible* if I has a winning strategy in $G(A^*, A)$. Otherwise, we say that A^* is an *inaccessible* set.

For each ordinal ξ we introduce the notion of *ξ-accessible* set as follows. A^* is 0-accessible if $A^* = \emptyset$. If $\xi = \mu+1$ we say that A^* is ξ-accessible if, for some $n \in \omega$, there exist $a_1, \ldots, a_n \in A$ such that for all neighborhoods U_1 of $a_1, \ldots, U_n$ of a_n, $A^* - (U_1 \cup \ldots \cup U_n)$ is μ-accessible. If ξ is a limit ordinal, then A^* is ξ-accessible if A^* is μ-accessible for some $\mu < \xi$. The notion of ω-accessible set is the crucial notion we use in [2].

Remark. After the publication of [2], R. Telgársky has pointed out that the game we use to define the notion of accessible set is a refinement of the point-open game, which was introduced by F. Galvin and R. Telgársky and which has also been studied by other authors. This game is presented in [3].

Let A be a T_3 space. Let A^* be a subset of A. In the sequel, the following basic properties (i)–(vii) will be used without explicit mention.

(i) *If B^* is a subset of A^*, then A^* accessible (ξ-accessible) implies B^* accessible (ξ-accessible).*

(ii) *For every two ordinals ξ, η with $\xi < \eta$, A^* ξ-accessible implies A^* η-accessible.*

Note that if $\varkappa$ is a regular cardinal and the topology of A has a basis of cardinal $< \varkappa$, then A^* $(\varkappa+1)$-accessible implies A^* $\varkappa$-accessible. Hence we infer that if A^* is not $\varkappa$-accessible, player II has a winning strategy in the game $G(A^*, A)$. Therefore A^* accessible implies A^* $\varkappa$-accessible. Thus we obtain:

(iii) *A^* is accessible if and only if A^* is ξ-accessible for some ordinal ξ.*

The following are easy generalizations of the basic properties of ω-accessibility given in [2].

(iv) *$A^* = A_1^* \cup \ldots \cup A_n^*$ is accessible (ξ-accessible) if and only if $A_1^*, \ldots, A_n^*$ are accessible (ξ-accessible).*

(v) *If A^* is accessible (ξ-accessible), then the set of accumulation points of A^* is accessible (ξ-accessible).*

(vi) *If A^* is an infinite subset of A and no point of A is an accumulation point of A^*, then A^* is inaccessible.*

Note also:

(vii) If *$A^* \neq \emptyset$ and $A^* \to A^*$, then A^* is inaccessible.*

Suppose that A is a T_3 space and A^* is a subset of A. For every ordinal ξ, we define by transfinite induction the *ξ-derivative* of A^*, $(A^*)^\xi$, as follows:

$$(A^*)^0 = A^*,$$
$$(A^*)^{\xi+1} = \{a \in A \colon (A^*)^\xi \to a\},$$
$$(A^*)^\xi = \bigcap_{0<\mu<\xi} (A^*)^\mu \text{ if } \xi \text{ is a limit ordinal.}$$

We shall need the following result.

LEMMA 1.1. *Suppose that A^* is an accessible subset of A. Then, for every ordinal ξ, A^* is ξ-accessible if and only if $(A^*)^\xi = \emptyset$.*

Proof. We show that if $U_1, \ldots, U_n$ are open sets, then $(A^*)^\xi - (U_1 \cup \ldots \cup U_n) = \emptyset$ implies $A^* - (U_1 \cup \ldots \cup U_n)$ ξ-accessible, for each ξ. The case $\xi = 0$ is immediate. If $\xi = \mu+1$ and $(A^*)^\xi - (U_1 \cup \ldots \cup U_n) = \emptyset$, it is easy to infer that

$$(A^*)^\mu - (U_1 \cup \ldots \cup U_n)$$

is finite (otherwise, $(A^*)^\mu$ would not be accessible). Now assume that ξ is a limit ordinal and $(A^*)^\xi - (U_1 \cup \ldots \cup U_n) = \bigcap_{0<\mu<\xi} ((A^*)^\mu - (U_1 \cup \ldots \cup U_n)) = \emptyset$. Consider $\tilde{A} = (A^*)^1 - (U_1 \cup \ldots \cup U_n)$. Since $\tilde{A}$ is accessible and closed, $\tilde{A}$ is compact. Therefore $(A^*)^\mu - (U_1 \cup \ldots \cup U_n) = \emptyset$ for some $\mu < \xi$, and by the induction hypothesis. $A^* - (U_1 \cup \ldots \cup U_n)$ is μ-accessible.

On the other hand, for closed sets $U_1, \ldots, U_n$, one can check that if $(A^*)^\xi - (U_1 \cup \ldots \cup U_n) \neq \emptyset$ then $A^* - (U_1 \cup \ldots \cup U_n)$ is not ξ-accessible. ■

Suppose that A is a T_3 space, A^* is a subset of A and $a \in A$. If $A^* \to a$ we consider the following two *types of convergence*:

(a) $A^* \overset{1}{\to} a$ if there is a neighborhood U of a such that $A^* \cap U$ is accessible,

(b) $A^* \overset{0}{\to} a$ otherwise.

Assume that A is a T_3 space and $\mathscr{C}$ is a nonempty set of subsets of A. The game of infinitely many moves $G(\mathscr{C}, A)$ is defined in the same way as $G(A^*, A)$ for a subset A^* of A. We say that player I wins in $G(\mathscr{C}, A)$ if, for some natural number i, there exists an $A^* \in \mathscr{C}$ such that A^* is covered in move i. We say that $\mathscr{C}$ is *accessible* if I has a winning strategy in $G(\mathscr{C}, A)$; otherwise, $\mathscr{C}$ is *inaccessible*. If U is an open set (or a closed set) in A, we write $\mathscr{C} \restriction U = \{A^* \cap U\colon A^* \in \mathscr{C}\}$. We say that $\mathscr{C}$ *converges* to $a \in A$, $\mathscr{C} \to a$, if for every $A^* \in \mathscr{C}$ we have $A^* \to a$. If $\mathscr{C} \to a$ we consider the following two *types of convergence*:

(a) $\mathscr{C} \overset{1}{\to} a$ if there is a neighborhood U of a such that $\mathscr{C} \restriction U$ is accessible,

(b) $\mathscr{C} \overset{0}{\to} a$ otherwise.

We define by transfinite induction when a nonempty set $\mathscr{C}$ of subsets of A is ξ-*accessible*. $\mathscr{C}$ is 0-accessible if $\emptyset \in \mathscr{C}$. If $\xi = \mu+1$, we say that $\mathscr{C}$ is ξ-*accessible* if, for some $n \in \omega$, there exist $a_1, \ldots, a_n \in A$ such that for all neighborhoods U_1 of $a_1, \ldots, U_n$ of a_n, $\mathscr{C} \restriction A-(U_1 \cup \ldots \cup U_n)$ is μ-accessible. If ξ is a limit ordinal, then $\mathscr{C}$ is ξ-accessible if $\mathscr{C}$ is μ-accessible for some $\mu < \xi$.

Let us say that a subset A^* of A is $(L_{\omega_1\omega})_t$-*definable* (or, simply, *definable*) if there is a $(L_{\omega_1\omega})_t$-formula $\varphi(x)$ such that, for all $a \in A$, $A \models \varphi[a]$ iff $a \in A^*$. Note that if A^* is a $(L_{\omega_1\omega})_t$-definable subset of A and ξ is a countable ordinal, then the condition "A^* is ξ-accessible" is $(L_{\omega_1\omega})_t$-definable. If η is a limit ordinal and, for $\mu < \eta$, A_μ^* is a definable subset of A, let us say that $\{A_\mu^*\colon \mu < \eta\}$ is a *sequence* if $A_{\mu_1}^* \to A_{\mu_2}^*$ for $\mu_1 < \mu_2 < \eta$. Then, if ξ and η are countable, the condition "$\{A_\mu^*\colon \mu < \eta\}$ is ξ-accessible" is definable. Note that if η is not a limit ordinal and $A_{\mu_1}^* \to A_{\mu_2}^*$ for $\mu_1 < \mu_2 < \eta$, then $\{A_\mu^*\colon \mu < \eta\}$ is accessible (ξ-accessible) iff $A_{\eta-1}^*$ is accessible (ξ-accessible). In this paper we introduce a class of T_3 spaces in which the notion of accessibility can be treated as a $(L_{\omega_1\omega})_t$-definable notion. For any space A of this class there will exist a countable ordinal ξ such that the terms "accessible" and "ξ-accessible" will be equivalent for definable subsets and sequences of A. For example, in any space of a-finite type (in the sense of [2]), "accessible" and "ω-accessible" will be equivalent.

EXAMPLES. To characterize the $(L_{\omega_1\omega})_t$-theory of a countable ordinal Ω with order topology we consider, for every ordinal ξ, the definable set $\Omega^{(\xi)} = (\Omega)^\xi - (\Omega)^{\xi+1}$. We have $\Omega^{(0)}$ = the set of isolated points of Ω and, for $\xi > 0$, $\Omega^{(\xi)}$ = the set of accumulation points exactly of points of $\Omega^{(\mu)}$ for every $\mu < \xi$ (for example, if Ω is ω^ω then $\omega \in \Omega^{(1)}$, $\omega \cdot \omega \in \Omega^{(2)}, \ldots, \omega^n \in \Omega^{(n)} \ldots$ and $\Omega^{(\xi)} = \emptyset$ if $\xi \geqslant \omega$). Then we have to check whether the sets $\Omega^{(\xi)}$ and the sequences $\{\Omega^{(\mu)}\colon \mu < \eta\}$ are accessible or not. If $\xi > 0$ and $a \in \Omega^{(\xi)}$ we have to look how the sets $\Omega^{(\mu)}$ with $\mu < \xi$ and the sequences $\{\Omega^{(\mu)}\colon \mu < \eta\}$ with $\eta \leqslant \xi$ converge to a. In this case, these convergences are of type 1.

Now let us consider the spaces we presented in [2, Example 2.5]. We consider the ordinals $\omega+1$ and ω^ω, each with order topology, and for every $n \in \omega$ a homeomorphic copy $(\omega^\omega)_n$ of ω^ω. In $\omega+1$, replace each $n \in \omega$ by ω^ω. Let us denote by Ω_0 the resulting space and by e_0 the only point of $\omega+1$ not replaced. If $\eta \in \omega^\omega$ we denote the corresponding point in $(\omega^\omega)_n$ by $(\eta)_n$. We add a new point e_1 to the topological sum $\sum_{n\in\omega} (\omega^\omega)_n$ and take as a neighborhood basis of e_1 the sets of the form $\{e_1\} \cup \bigcup_{n\in\omega} \{\xi \in (\omega^\omega)_n\colon \xi > (\eta)_n\}$, where $\eta \in \omega^\omega$. Let us denote by Ω_1 the resulting space. For $i = 0, 1$ we consider the sets $\Omega_i^{(\xi)}$ (ξ ordinal) defined as before. Then $\Omega_i^{(\omega)} = \{e_i\}$ and $\Omega_i^{(\xi)} = \emptyset$ if $\xi > \omega$ $(i = 0, 1)$. We find that $\Omega_i^{(n)}$ is inaccessible and $\Omega_i^{(n)} \overset{0}{\to} e_i$ for every $n \in \omega$ $(i = 0, 1)$. However:

(a) $\{\Omega_0^{(n)}\colon n \geqslant 0\}$ is inaccessible and $\{\Omega_0^{(n)}\colon n \geqslant 0\} \overset{0}{\to} e_0$,

(b) $\{\Omega_1^{(n)}\colon n \geqslant 0\}$ is 1-accessible and therefore $\{\Omega_1^{(n)}\colon n \geqslant 0\} \overset{1}{\to} e_1$.

We can then infer that Ω_0 and Ω_1 are not $(L_{\omega_1\omega})_t$-equivalent. Now let us consider the topological sum $\omega^\omega + \Omega_1$ and let us denote this space by Ω_2. In this case, $\{\Omega_2^{(n)}\colon n \geqslant 0\}$ is inaccessible and $\{\Omega_2^{(n)}\colon n \geqslant 0\} \overset{1}{\to} e_1$.

§ 2. The notion of spectrum. Suppose that E is a nonempty set and $<\cdot$ is a binary transitive relation on E (possibly $<\cdot\, = \emptyset$). If the set $\{\alpha \in E\colon \alpha <\cdot\, \alpha\}$ is finite, we say that $(E, <\cdot)$ is *normal*.

Let $(E, <\cdot)$ be a normal relation. Suppose that $\alpha \in E$. We say that α is *comparable* in E if there is a $\beta \in E$ with $\alpha <\cdot\, \beta$ or $\beta <\cdot\, \alpha$. Let γ be a nonempty subset of E. We say that γ is a *chain* in E if, for some ordinal $\eta \geqslant 1$, we can write $\gamma = \{\alpha_\mu\colon \mu < \eta\}$ with $\alpha_{\mu_1} <\cdot\, \alpha_{\mu_2}$ and $\alpha_{\mu_1} \neq \alpha_{\mu_2}$ for $\mu_1 < \mu_2 < \eta$; then we say that η is the *length* of γ. Since $\{\alpha \in E\colon \alpha <\cdot\, \alpha\}$ is finite, it is easy to see that "η is the length of γ" is unambiguously defined. Note that even if $\alpha <\cdot\, \alpha$, the length of $\{\alpha\}$ is always one. Let γ be a chain in E. We say that γ is a *maximal chain* in E if, for every chain γ' in E, $\gamma \subset \gamma'$ implies $\gamma = \gamma'$. Note that if α is a noncomparable element of E then $\{\alpha\}$ is a maximal chain. We say that a chain $\gamma = \{\alpha_\mu\colon \mu < \eta\}$ is *open* if η is a limit ordinal; otherwise, we say that γ is *closed*.

Let $(E, <\cdot)$ be a normal relation. We write $\hat{E} = P\big(\bigcup_{\lambda=0,1} \{(\alpha, \lambda)\colon \alpha \in E\}\big)$ (where P denotes the power set operation). Let γ be a chain in E and $\hat{\alpha} \in \hat{E}$. We say that γ is a *chain* in $\hat{\alpha}$ if, for every $\beta \in \gamma$, $(\beta, \lambda) \in \hat{\alpha}$ for $\lambda = 0$ or $\lambda = 1$.

We say that $\langle (S_\xi, <_\xi)\colon \xi \text{ ordinal}\rangle$ is a *complex of types* if, for every ordinal ξ, $(S_\xi, <_\xi)$ is normal and S_ξ satisfies the following:

(i) $S_0 = \{*\}$.

(ii) If $\xi = \mu+1$, then S_ξ is a countable nonempty subset of the set $\{\langle \hat{\alpha}, \delta\rangle\colon \hat{\alpha} \in \hat{S}_\mu$ and δ is a set of pairs (γ, λ) such that $[(\gamma, \lambda) \in \delta$ for $\lambda = 0$ or $\lambda = 1$ iff γ is an open chain in $\hat{\alpha}]\}$.

(iii) If ξ is a limit ordinal, then S_ξ is a countable nonempty subset of the set $\{[\alpha_\mu]_{\mu<\xi}\colon \alpha_\mu \in S_\mu\}$.

Thus if we want to construct a complex of types we define $S_0 = \{*\}$ and take $<_0$ then we define S_1 and take $<_1$, and so on.

If $\langle (S_\xi, <_\xi)\colon \xi \text{ ordinal}\rangle$ is a complex of types, the members of S_ξ will be called *ξ-types*. Note that, for every natural number n, S_{n+1} does not depend on $<_n$.

Note that we denote chains by $\gamma, \gamma^1, \ldots$, and types by $\alpha, \beta, \ldots$

Let $S = \langle (S_\xi, <_\xi)\colon \xi \text{ ordinal}\rangle$ be a complex of types. For every ordinal ξ, we define the *ξ-type of a in A with respect to S*, $s_\xi(a, A)$, for every T_3 space A and $a \in A$. If $\alpha \in S_\xi$, we write $A_\alpha = \{a \in A\colon s_\xi(a, A) = \alpha\}$ and if γ is a chain in S_ξ, we write $A_\gamma = \{A_\alpha\colon \alpha \in \gamma\}$. We define $s_\xi(a, A)$ by transfinite induction on ξ as follows:

(i) $s_0(a, A) = *$.

(ii) If $\xi = \mu+1$, we consider $\hat{\alpha}_0 = \bigcup_{\lambda=0,1} \{(\beta, \lambda)\colon \beta \in S_\mu \text{ and } A_\beta \xrightarrow{\lambda} a\}$ and $\delta_0 = \bigcup_{\lambda=0,1} \{(\gamma, \lambda)\colon \gamma \text{ is an open chain in } \hat{\alpha}_0 \text{ and } A_\gamma \xrightarrow{\lambda} a\}$. Then $s_\xi(a, A) = \langle \hat{\alpha}_0, \delta_0\rangle$.

(iii) If ξ is a limit ordinal, then $s_\xi(a, A) = [s_\mu(a, A)]_{\mu\leqslant\xi}$.

To see an example, let Ω be a countable ordinal with order topology. Let ξ_0 be the least ordinal ξ such that $(\Omega)^\xi = \emptyset$. We will construct a complex of types $S^{(\xi_0)} = \langle (S_\xi^{(\xi_0)}, <_\xi)\colon \xi \text{ ordinal}\rangle$ in such a way that $S_\xi^{(\xi_0)}$ and $<_\xi$ have the following form:

$$\text{(a)} \qquad S_\xi^{(\xi_0)} = \begin{cases} \{\alpha_\xi^\varrho\colon \varrho < \xi\} \cup \{\beta_\xi\} & \text{if } \xi < \xi_0, \\ \{\alpha_\xi^\varrho\colon \varrho < \xi_0\} & \text{if } \xi \geqslant \xi_0. \end{cases}$$

(b) $\alpha <_\xi \beta$ iff $\alpha = \alpha_\xi^{\varrho_1}$, $\beta = \alpha_\xi^{\varrho_2}$ and $\varrho_1 < \varrho_2 < \xi, \xi_0$. Furthermore, the types α_ξ^ϱ, β_ξ will satisfy $\Omega_{\alpha_\xi^\varrho} = (\Omega)^\varrho - (\Omega)^{\varrho+1}$ for every $\varrho < \xi, \xi_0$ and $\Omega_{\beta_\xi} = (\Omega)^\xi$. We define the types α_ξ^ϱ, β_ξ by transfinite induction on ξ as follows. We put $\beta_0 = *$. Suppose that $\xi = \mu+1$. Let ϱ be an ordinal such that $\varrho < \xi, \xi_0$. To define α_ξ^ϱ we consider $\hat{\alpha}_\xi^\varrho = \{(\alpha_\mu^\theta, 1)\colon \theta < \varrho\}$ and put $\alpha_\xi^\varrho = \langle \hat{\alpha}_\xi^\varrho, (\gamma, 1)\colon \gamma \text{ is an open chain in } \hat{\alpha}_\xi^\varrho\}\rangle$. If $\xi < \xi_0$, we set $\beta_\xi = \langle \hat{\alpha}_\xi^\mu \cup \{(\beta_\mu, 1)\}, \{(\gamma, 1)\colon \gamma \text{ is an open chain in } \hat{\alpha}_\xi^\mu\}\rangle$. Now suppose that ξ is a limit ordinal. If $\varrho < \xi, \xi_0$ we put $\alpha_\xi^\varrho = [\alpha_\mu]_{\mu<\xi}$ where $\alpha_\mu = \beta_\mu$ if $\mu \leqslant \varrho$ and $\alpha_\mu = \alpha_\mu^\varrho$ if $\mu > \varrho$. If $\xi < \xi_0$ we set $\beta_\xi = [\beta_\mu]_{\mu<\xi}$.

If $S = \langle (S_\xi, <_\xi)\colon \xi \text{ ordinal}\rangle$ is a complex of types, A is a T_3 space and $a \in A$, then it is possible that $s_\xi(a, A) \notin S_\xi$. To see this, consider, for any ordinal $\xi_0 \geqslant 1$, the complex of types $S = S^{(\xi_0)} = \langle (S_\xi^{(\xi_0)}, <_\xi)\colon \xi \text{ ordinal}\rangle$ defined before. Let R be the space of real numbers with the usual topology. Then, for every $x \in R$, $s_1(x, R) = \langle \{(*, 0)\}, \emptyset\rangle \notin S_1^{(\xi_0)}$.

Let $S = \langle (S_\xi, <_\xi)\colon \xi \text{ ordinal}\rangle$ be a complex of types. Let A be a T_3 space. A type $\alpha \in S_\xi$ is *satisfiable* in A if $s_\xi(a, A) = \alpha$ for some $a \in A$. We say that A *satisfies* S if for every ordinal ξ the following conditions hold:

(i) $s_\xi(a, A) \in S_\xi$ for all $a \in A$.

(ii) For all comparable types α, β in S_ξ which are satisfiable in A, $\alpha <_\xi \beta$ iff $A_\alpha \to A_\beta$.

In what follows we assume that every complex of types $S = \langle (S_\xi, <_\xi)\colon \xi \text{ ordinal}\rangle$ has an associate function which assigns to each comparable type α in S_ξ and to each $\eta > \xi$ a comparable type $\alpha(\eta)$ in S_η in such a way that if α, β are comparable types in S_ξ and $\eta > \xi$ then $\alpha <_\xi \beta$ iff $\alpha(\eta) <_\eta \beta(\eta)$. If γ is an open chain in S_ξ and $\eta > \xi$, we write $\gamma(\eta) = \{\alpha(\eta)\colon \alpha \in \gamma\}$.

Let $S = \langle (S_\xi, <_\xi)\colon \xi \text{ ordinal}\rangle$ be a complex of types. We say that a T_3 space A *is associated with* S if A satisfies S and for every ordinal ξ the following conditions hold:

(i) For every $\alpha \in S_\xi$ there is an $a \in A$ such that $s_\xi(a, A) = \alpha$.

(ii) For every ξ-type α comparable in S_ξ and every ordinal $\eta > \xi$ we have $A_\alpha = A_{\alpha(\eta)}$.

Now we define the central notion of this section. Let $S = \langle (S_\xi, <_\xi)\colon \xi \text{ ordinal}\rangle$ be a complex of types. We say that S is a *spectrum* if the following four conditions hold:

(i) For every ordinal ξ, every chain in S_ξ is contained in a maximal chain in S_ξ.

(ii) For every ordinal ξ, the set of maximal chains in S_ξ is finite.

(iii) There exists a countable ordinal η such that every member of S_η is comparable in S_η.

(iv) There exists a countable metrizable space associated with S.

The least ordinal η satisfying (iii) will be denoted by $\mu(S)$.

Let Ω be a countable ordinal greater than ω with order topology, and ξ_0 the least ordinal ξ such that $(\Omega)^\xi = \emptyset$. Consider the complex of types

$$S^{(\xi_0)} = \langle (S_\xi^{(\xi_0)}, <_\xi)\colon \xi \text{ ordinal}\rangle$$

defined before. Note that if α is comparable in $S_\xi^{(\xi_0)}$, we have $\alpha = \alpha_\xi^\varrho$ for some $\varrho < \xi, \xi_0$; then, for every $\eta > \xi$, we put $\alpha(\eta) = \alpha_\eta^\varrho$. Now it is easy to check that $S^{(\xi_0)}$ is a spectrum, Ω is associated with $S^{(\xi_0)}$ and $\xi_0 = \mu(S^{(\xi_0)})$. Furthermore, every ordinal less than Ω with order topology satisfies $S^{(\xi_0)}$. We see that in $S_{\xi_0}^{(\xi_0)}$ there is just one maximal chain. If we consider the spaces Ω_0 and Ω_1 presented in § 1, we infer that the topological sum $\Omega_0 + \Omega_1$ is associated with a spectrum S in such a way that $\mu(S) = \omega+1$ and in $S_{\omega+1}$ there are just two maximal chains. We leave it to the reader to find for each n a spectrum S such that in $S_{\mu(S)}$ there are n maximal chains.

Now suppose that $S = \langle (S_\xi, <_\xi)\colon \xi \text{ ordinal}\rangle$ is a spectrum, $\hat{\alpha} \in \hat{S}_\xi$ and γ is a chain in $\hat{\alpha}$. We say that γ is a *λ-chain* in $\hat{\alpha}$ if $(\beta, \lambda) \in \hat{\alpha}$ for all $\beta \in \gamma$ $(\lambda = 0, 1)$. And γ is a *λ-maximal chain* in $\hat{\alpha}$ if moreover for every λ-chain γ' in $\hat{\alpha}$, $\gamma \subset \gamma'$ implies $\gamma = \gamma'$ $(\lambda = 0, 1)$. The following lemma follows easily from the definition of spectrum.

Lemma 2.1. *Let $S = \langle (S_\xi, <_\xi); \xi \text{ ordinal}\rangle$ be a spectrum. For any $\langle \hat{\alpha}, \delta\rangle \in S_{\xi+1}$ we have:*

(a) *Every λ-chain in $\mathfrak{A}$ is contained in a λ-maximal chain in $\mathfrak{A}$ ($\lambda = 0, 1$).*
(b) *The set of λ-maximal chains in $\mathfrak{A}$ is finite ($\lambda = 0, 1$).*

Let $S = \langle (S_\xi, <_\xi)\colon \xi \text{ ordinal}\rangle$ be a spectrum. Suppose that A satisfies S, U is an open set in A and α, β are comparable types in S_ξ which are satisfiable in A. It should be noted that if $A_\alpha \cap U$ is accessible and $\alpha <_\xi \beta$, then $A_\beta \cap U$ is accessible.

Lemma 2.2. *Let $S = \langle (S_\xi, <_\xi)\colon \xi$ ordinal$\rangle$ be a spectrum. Suppose that A satisfies S, $a \in A$, $\xi = \mu+1$ and $\mathfrak{A} = \bigcup_{\lambda=0,1} \{(\beta, \lambda)\colon \beta \in S_\mu$ and $A_\beta \overset{\lambda}{\to} a\}$. Then $s_\xi(a, A)$ is determined by $\mathfrak{A}$ and the convergences of the open 0-maximal chains in $\mathfrak{A}$.*

Proof. We have to keep in mind that if γ is an open chain in $\mathfrak{A}$, we have:

(i) If γ is not a 0-chain in $\mathfrak{A}$, then $A_\gamma \overset{1}{\to} a$.

(ii) If γ is a 0-chain in $\mathfrak{A}$ and there is a $\beta \in S_\mu$ with $\alpha <_\mu \beta$ for all $\alpha \in \gamma$ and $A_\beta \overset{0}{\to} a$, then $A_\gamma \overset{0}{\to} a$.

The desired conclusion now follows from (i) and (ii). ∎

In the next lemma, whose proof is immediate, we give a basic property of spectra.

Lemma 2.3. *Let $S = \langle (S_\xi, <_\xi)\colon \xi$ ordinal$\rangle$ be a spectrum. For every $P \subset S_\xi$ there exists a $P_0 \subset P$ with P_0 finite such that for each $\alpha \in P - P_0$ there is a $\beta \in P_0$ with $\beta <_\xi \alpha$.*

Let $S = \langle (S_\xi, <_\xi)\colon \xi \text{ ordinal}\rangle$ be a spectrum. Suppose that A satisfies S, $a \in A$, $P \subset S_\xi$ and P_0 is a finite subset of P given by Lemma 2.3. In the sequel we shall make use of the following properties without explicit mention:

(i) If A_α is accessible ($A_\alpha \overset{1}{\to} a$) for every $\alpha \in P_0$, then $\bigcup_{\alpha\in P} A_\alpha$ is accessible ($\bigcup_{\alpha\in P} A_\alpha \overset{1}{\to} a$).

(ii) If $\bigcup_{\alpha\in P} A_\alpha \to a$, then $A_\alpha \to a$ for some $\alpha \in P_0$.

(iii) For every $a \in A$ there is a neighborhood U of a such that, for each $\alpha \in S_\xi$, $A_\alpha \not\to a$ implies $A_\alpha \cap (U - \{a\}) = \emptyset$.

From (iii) we infer that if A satisfies a spectrum S, $a \in A$ and $\xi = \mu+1$, then a is an isolated point in A iff $s_\xi(a, A) = \langle \emptyset, \emptyset\rangle$.

§ 3. The main theorem. Let $S = \langle (S_\xi, <_\xi)\colon \xi \text{ ordinal}\rangle$ be a spectrum. For $\xi < \eta$ and $\alpha \in S_\eta$, we define the ξ-type $(\alpha)_\xi^S$ in such a way that, proceeding by transfinite induction on ξ, one can prove

Lemma 3.1. *If A is a T_3 space which satisfies S and $a \in A$, then*

$$s_\xi(a, A) = (s_\eta(a, A))_\xi^S .$$

We define $(\alpha)_\xi^S$ by transfinite induction. We put $(\alpha)_0^S = *$. If ξ is limit, then $(\alpha)_\xi^S = [(\alpha)_\mu^S]_{\mu<\xi}$. If $\xi = \mu+1$ we consider two cases. If η is a limit ordinal and $\alpha = [\alpha_\varrho]_{\varrho<\eta}$, then $(\alpha)_\xi^S = \alpha_\xi$. Suppose that $\eta = \eta'+1$. If $\alpha = \langle \emptyset, \emptyset\rangle$, we put $(\alpha)_\xi^S = \langle \emptyset, \emptyset\rangle$. If $\alpha = \langle \mathfrak{A}, \delta\rangle$ with $\mathfrak{A} = \{(\alpha_i, \lambda_i)\colon i \in I\}$ and $I \neq \emptyset$ we consider

$$\mathfrak{A}^* = \{(\beta, \lambda_\beta)\colon \text{(a) the set } J \text{ of all } i \in I \text{ such that } (\alpha_i)_\mu^S = \beta \text{ is nonempty, and (b) } \lambda_\beta = 0 \text{ if there is an } i \in J \text{ with } \lambda_i = 0,\ \lambda_\beta = I \text{ otherwise}\},$$

$$\delta^* = \{(\gamma, \lambda)\colon \gamma \text{ is an open chain in } \mathfrak{A}^* \text{ and } (\gamma(\eta'), \lambda) \in \delta\}.$$

Then $(\alpha)_\xi^S = \langle \mathfrak{A}^*, \delta^*\rangle$.

To show Lemma 3.1 it suffices to prove, for each ξ, the following two conditions: (a) $s_\xi(a, A) = (s_\eta(a, A))_\xi^S$ for all $a \in A$ and $\eta > \xi$; and then (b) if α is comparable in S_ξ, then $s_\xi(a, A) = \alpha$ implies $s_\eta(a, A) = \alpha(\eta)$ for all $a \in A$ and $\eta > \xi$. Use Lemma 2.3 in the nontrivial case.

Now assume that A satisfies a spectrum $S = \langle (S_\xi, <_\xi)\colon \xi \text{ ordinal}\rangle$ and $a \in A$. Note that if $\alpha_0 = s_\xi(a, A)$ is comparable in S_ξ then, for every $\alpha \in S_\xi$, $A_\alpha \overset{\lambda}{\to} a$ implies $A_\alpha \overset{\lambda}{\to} A_{\alpha_0}$ ($\lambda = 0, 1$). Thus if $\xi_0 = \mu(S)$ and $\alpha_0 = s_{\xi_0}(a, A)$ then, for every $\alpha \in S_{\xi_0}$, $A_\alpha \overset{\lambda}{\to} a$ implies $A_\alpha \overset{\lambda}{\to} A_{\alpha_0}$ ($\lambda = 0, 1$). In what follows, we shall make use of this fact without explicit mention.

Proceeding by transfinite induction on ξ it is easy to show the following lemma.

Lemma 3.2. *Suppose that S is a spectrum, A is a T_3 space satisfying S, $a \in A$ and U is an open neighborhood of a with the relative topology of A. Then $s_\xi(a, A) = s_\xi(a, U)$ for every ordinal ξ, and U satisfies S.*

Lemma 3.3. *Let $S = \langle (S_\xi, <_\xi)\colon \xi$ ordinal$\rangle$ be a spectrum, $\xi_0 = \mu(S)$ and $\xi^* = \max\{\eta\colon \eta$ is the length of some maximal chain in $S_{\xi_0}\}$. Suppose that A is a T_3 space which satisfies S, $\alpha \in S_{\xi_0}$ and A_α accessible. Then A_α is ξ^*-accessible.*

Proof. By using Lemma 2.3 and the hypothesis that $\xi_0 = \mu(S)$ it is easy to prove by transfinite induction that for each $\xi \geqslant 1$ there is a subset P of S_{ξ_0} with $(A_\alpha)^\xi = \bigcup_{\beta\in P} A_\beta$. Now, for every $\xi \geqslant 1$, we can obtain:

(+) If $(A_\alpha)^\xi \neq \emptyset$ then for every $a \in (A_\alpha)^\xi$ there exists a chain $\{\alpha_\mu\colon \mu < \xi\}$ in S_{ξ_0} with $\alpha_0 = \alpha$ and $A_{\alpha_\mu} \to a$ for each $\mu < \xi$.

We can prove (+) by transfinite induction on $\xi \geqslant 1$. The condition is trivial if $\xi = 1$. If $1 < \xi = \mu+1$, consider $P \subset S_{\xi_0}$ such that $(A_\alpha)^\mu = \bigcup_{\beta\in P} A_\beta$, and then make use of Lemma 2.3. If ξ is a limit ordinal and $a \in (A_\alpha)^\xi$, use Lemma 2.1 for $\lambda = 1$ and $\langle \mathfrak{A}, \delta\rangle = s_{\xi_0+1}(a, A)$.

Let us set $\tilde{\xi}$ = the least ordinal ξ such that A_α is ξ-accessible. Assume $\tilde{\xi} > 1$. From Lemma 1.1 we obtain $(A_\alpha)^{\tilde{\xi}-1} \neq \emptyset$. By (+) there exists a chain of length $\tilde{\xi}$ in S_{ξ_0}, whence $\tilde{\xi} \leqslant \xi^*$. Therefore A_α is ξ^*-accessible. ∎

Suppose that $S = \langle (S_\xi, <_\xi)\colon \xi \text{ ordinal}\rangle$ is a spectrum. If A satisfies S we define, for every ordinal ξ, the function $E_\xi^A\colon S_\xi \to \omega \cup \{\infty\}$ by $E_\xi^A(\alpha)$ = the number of $a \in A$ with $s_\xi(a, A) = \alpha$. Assume that A and B satisfy S. We say that A and B are *S-equivalent*, $A \equiv_S B$, if for any ξ the following two conditions hold:

(a) If $\alpha \in S_\xi$, then $E_\xi^A(\alpha) = E_\xi^B(\alpha)$, and A_α is accessible if and only if B_α is accessible,

(b) If γ is an open chain in S_ξ, then A_γ is accessible if and only if B_γ is accessible.

From now on we work with countable metrizable spaces. We shall tacitly use the well-known fact that the topology of any countable T_3 space has a clopen basis.

Let $S = \langle (S_\xi, <_\xi): \xi \text{ ordinal}\rangle$ be a spectrum, $\xi_0 = \mu(S)$ and $\xi^* = \max\{\eta: \eta$ is the length of some maximal chain in $S_{\xi_0}\}$. Suppose that A is a countable metrizable space satisfying S and ξ is an ordinal. From Lemmas 2.3, 3.1 and 3.3 we can verify the following:

(*) (a) If $\alpha \in S_\xi$, then A_α accessible implies A_α ξ^*-accessible.

(b) If γ is an open chain in S_ξ, then A_γ accessible implies A_γ ξ^*-accessible.

Now suppose that we modify the definitions of "$A^* \overset{\lambda}{\to} a$" and "$\mathscr{C} \overset{\lambda}{\to} a$" (see § 1) by setting "$\xi^*$-accessible" instead of "accessible" and denote by $s_\xi^{(\xi^*)}(a, A)$ the corresponding ξ-type of a in A which respect to these new definitions. In the same way as we have worked with the notion of $s_\xi(a, A)$, we can also work with the notion of $s_\xi^{(\xi^*)}(a, A)$. Proceeding by transfinite induction on ξ, it is easy to check by Lemma 3.2 and (*) (a), (b) (and by using the fact that the topology of A has a clopen basis) that for every $\alpha \in S_\xi$ and $a \in A$ we have

(**) $$s_\xi(a, A) = \alpha \quad \text{iff} \quad s_\xi^{(\xi^*)}(a, A) = \alpha .$$

Then it is not difficult to prove the following result. Consider Lemma 2.2 in order to construct the sentence φ_A.

LEMMA 3.4. *Suppose that A is a countable metrizable space which satisfies a spectrum S. Then we can find a sentence φ_A in $(L_{\omega_1\omega})_t$ such that, for every countable metrizable space B, $B \models \varphi_A$ if and only if (B satisfies S and $A \equiv_S B$).*

It is also possible to prove (*), (**) and Lemma 3.4 for uncountable T_3 spaces. Our main result is

THEOREM 1. *Let S be a spectrum. Suppose that A, B are countable metrizable spaces with $A \equiv_S B$. Then A and B are homeomorphic.*

We can show Theorem 1 by using a back and forth argument. Put $\xi_0 = \mu(S)$. We define the symmetric relation R between T_3 spaces with a finite (possibly empty) set of distinguished points by

$(A', a_1 \dots a_m) R (B', b_1 \dots b_m)$ iff (a) A' and B' are (possibly empty) countable metrizable spaces satisfying S and $a_i \neq a_j$, $b_i \neq b_j$ $(i \neq j)$; (b) $A' \equiv_S B'$; and (c) $s_{\xi_0}(a_i, A') = s_{\xi_0}(b_i, B')$ $(i = 1, \dots, m)$.

Note that if A, B satisfy the assumptions of Theorem 1, then ARB holds. We need to show that R satisfies the two back and forth properties, that is, the properties (1) and (2) of [2, Theorem 2.2]. To carry out the proof of the nontrivial back and forth property, we give a criterion for choosing small neighborhoods of a point.

Assume that A is a countable metrizable space which satisfies a spectrum $S = \langle (S_\xi, <_\xi): \xi \text{ ordinal}\rangle$, a is an accumulation point of A and $\xi_0 = \mu(S)$. Then consider $\langle \hat{\beta}, \delta\rangle = s_{\xi_0+1}(a, A)$ and $\Gamma = \bigcup_{\lambda=0,1} \{\gamma;\ \gamma$ is a λ-maximal chain in $\hat{\beta}\}$. Note that, for $\alpha \in \gamma \in \Gamma$, if A_γ is inaccessible (A_α is inaccessible), then there is a neighborhood U of a such that $A_\gamma \restriction (A-U)$ is inaccessible ($A_\alpha \cap (A-U)$ is inaccessible). For each $\gamma = \{\alpha_\mu\colon \mu < \eta\} \in \Gamma$ we take a neighborhood U_γ of a by distinguishing the following five cases:

Case (1). γ is a closed 1-chain in $\hat{\beta}$. Put U_γ such that $A_{\alpha_0} \cap U_\gamma$ is accessible. If there is a neighborhood U of a with $A_{\alpha_{\eta-1}} \cap (A-U)$ infinite, then $U_\gamma \subset U$.

Case (2). γ is a closed 0-chain in $\hat{\beta}$. Consider U_γ such that $A_{\alpha_{\eta-1}} \cap (A-U_\gamma)$ is inaccessible.

Case (3). γ is an open 1-chain in $\hat{\beta}$. Take U_γ with $A_{\alpha_0} \cap U_\gamma$ accessible. If there is a neighborhood U of a with $A_{\alpha_\mu} \cap (A-U) \neq \emptyset$ for all $\mu < \eta$, then $U_\gamma \subset U$.

Case (4). γ is an open 0-chain in $\hat{\beta}$ with $A_\gamma \overset{1}{\to} a$. Consider U_γ such that $A_\gamma \restriction U_\gamma$ is accessible. If there is a neighborhood U of a such that $A_{\alpha_\mu} \cap (A-U)$ is inaccessible for all $\mu < \eta$ (respectively $A_{\alpha_\mu} \cap (A-U) \neq \emptyset$ for all $\mu < \eta$), then $U_\gamma \subset U$.

Case (5). γ is an open 0-chain in $\hat{\beta}$ with $A_\gamma \overset{0}{\to} a$. Take U_γ such that $A_\gamma \restriction (A-U_\gamma)$ is inaccessible.

Then if U is an open set such that $a \in U \subset \bigcap_{\gamma \in \Gamma} U_\gamma$ and, for every $\beta \in S_{\xi_0}$, $A_\beta \nrightarrow a$ implies $A_\beta \cap (U - \{a\}) = \emptyset$, we say that U is a *good neighborhood* of a. Note that, by Lemmas 2.1 and 2.3, we can always find a good neighborhood of a point, and if U is a good neighborhood of a we have:

(a) If $\alpha \in S_{\xi_0}$ and $A_\alpha \overset{1}{\to} a$, then $A_\alpha \cap U$ is accessible.

(b) If γ is an open chain in S_{ξ_0} and $A_\gamma \overset{1}{\to} a$, then $A_\gamma \restriction U$ is accessible.

Now assume that $S = \langle (S_\xi, <_\xi): \xi \text{ ordinal}\rangle$ is a spectrum, $\xi_0 = \mu(S)$, A and B are countable metrizable spaces with $A \equiv_S B$, $a \in A$, $b \in B$ and $\langle \hat{\alpha}, \delta\rangle = s_{\xi_0+1}(a, A) = s_{\xi_0+1}(b, B)$. Under these assumptions we show the following two lemmas.

LEMMA 3.5. *Suppose that γ is an open chain in $\hat{\alpha}$. Then:*

(a) *If there is a neighborhood U of a such that $A_\alpha \cap (A-U) \neq \emptyset$ for every $\alpha \in \gamma$, then there is a neighborhood V of b such that $B_\alpha \cap (B-V) \neq \emptyset$ for every $\alpha \in \gamma$.*

(b) *If there is a neighborhood U of a such that $A_\alpha \cap (A-U)$ is inaccessible for every $\alpha \in \gamma$, then there is a neighborhood V of b such that $B_\alpha \cap (B-V)$ is inaccessible for every $\alpha \in \gamma$.*

Proof. The lemma is trivial if A_γ is inaccessible. Assume then that A_γ is accessible. To show (a), note that if $A_\alpha \cap (A-U) \neq \emptyset$ for each $\alpha \in \gamma$ and U is open then there is an $a' \in A-U$ with $A_\gamma \to a'$. To show (b), suppose that U is an open neighborhood of a with $A_\alpha \cap (A-U)$ inaccessible for each $\alpha \in \gamma$. Since A_γ is accessible, one can check that there is an $a' \in A-U$ such that if $\langle \hat{\alpha}_1, \delta_1\rangle = s_{\xi_0+1}(a', A)$ then γ is a 0-chain in $\hat{\alpha}_1$. ■

Note that, by using Lemma 3.1 and the fact that $\xi_0 = \mu(S)$, we can infer that, for $\alpha, \beta \in S_{\xi_0}$, $A_\alpha \overset{\lambda}{\to} A_\beta$ iff $B_\alpha \overset{\lambda}{\to} B_\beta$ $(\lambda = 0, 1)$.

LEMMA 3.6. *Consider $\Sigma = \{\gamma: \gamma$ is an open 0-maximal chain in $\hat{\alpha}$ such that for every neighborhood U of a there is an $\alpha \in \gamma$ with $A_\alpha \cap (A-U)$ accessible$\}$. Let U^0, V^0 be clopen good neighborhoods of a, b respectively. Then we can find clopen sets U, V with $a \in U \subset U^0$, $b \in V \subset V^0$ in such a way that for each $\gamma \in \Sigma$ we have:*

(a) *For every $\alpha \in \gamma$, $A_\alpha \cap (A-U)$ is accessible iff $B_\alpha \cap (B-V)$ is accessible.*

(b) *For every open chain $\gamma' \subset \gamma$, $A_{\gamma'} \restriction (A-U)$ is accessible iff $B_{\gamma'} \restriction (B-V)$ is accessible.*

Proof. Put $S^* = \{\beta \in S_{\xi_0}: A_\beta \to a\} = \{\beta \in S_{\xi_0}: B_\beta \to b\}$. Suppose

$$\Sigma = \{\gamma_1, \ldots, \gamma_m\}$$

and $\{\gamma_1, \ldots, \gamma_n\} = \{\gamma_i \in \Sigma$: there is an $\alpha_i' \in \gamma_i$ such that, for every $\beta \in S^*$, $A_{\alpha_i'} \to A_\beta$ implies $A_{\alpha_i'} \overset{1}{\to} A_\beta\}$. By Lemma 2.1 we can find, for $i = 1, \ldots, n$, an $\alpha_i \in \gamma_i$ with $\alpha_i' <_{\xi_0} \alpha_i$ such that, for every $\beta \in S^* - \gamma_i$, $A_{\alpha_i} \to A_\beta$ implies $A_\alpha \to A_\beta$ for all $\alpha \in \gamma_i$. Since $\alpha_i' <_{\xi_0} \alpha_i$ we deduce that, for every $\beta \in S^*$, $A_{\alpha_i} \to A_\beta$ implies $A_{\alpha_i} \overset{1}{\to} A_\beta$. We may assume that $A_{\alpha_i} \cap (A - U^0)$ and $B_{\alpha_i} \cap (B - V^0)$ are accessible $(i = 1, \ldots, n)$. Note that if $A_{\alpha_i} \to A_{\alpha_i}$ we would infer that $A_{\alpha_i} \overset{0}{\to} A_{\alpha_i}$, which is impossible because we have $\alpha_i' <_{\xi_0} \alpha_i$. Thus $A_{\alpha_i} \not\to A_{\alpha_i}$. From the fact that γ_i is a 0-maximal chain in $\hat{\alpha}$ and by the way in which α_i is chosen we infer that, for $i \neq j$, if $A_{\alpha_i} \to A_{\alpha_j}$ then $\alpha_j \in \gamma_i$. Thus we may assume that $A_{\alpha_i} \not\to A_{\alpha_j}$ for $i \neq j$.

Let U^1 be a clopen neighborhood of a such that $U^1 \subset U^0$ and $A_{\alpha_i} \cap (U^0 - U^1)$ is inaccessible $(i = 1, \ldots, n)$. Let $\{a_r: r \in \omega\}$ be an enumeration of all c with $c \in U^0 - U^1$ and $A_{\alpha_i} \to c$ for some $i \in \{1, \ldots, n\}$. Let $\{a_r': r \in \omega\}$ be an enumeration of all c with $c \in U^0 - U^1$ and $A_{\alpha_i} \not\to c$ $(i = 1, \ldots, n)$. For each $k \in \omega$ we take clopen sets U_k, U_k', U_k'' as follows. At step k, we consider $\bar{a}$ = the first element c in the enumeration $\{a_r: r \in \omega\}$ with $c \notin \bigcup_{i<k} (U_i \cup U_i' \cup U_i'')$ (if such an element does not exist, we put $U_k = U_k' = \emptyset$). Let $\bar{U}$ be a clopen good neighborhood of $\bar{a}$ such that $\bar{U} \subset U^0 - U^1$ and $\bar{U} \cap (\bigcup_{i<k} (U_i \cup U_i' \cup U_i'')) = \emptyset$. Since $A_{\alpha_i} \cap \bar{U}$ is accessible $(1 \leqslant i \leqslant n)$ and $A_{\alpha_i} \not\to A_{\alpha_j}$ $(i \neq j)$, we can take a clopen set U_k with $U_k \subset \bar{U}$ such that, for $1 \leqslant i \leqslant n$, if $A_{\alpha_i} \to \bar{a}$ then $(A_{\alpha_i})^1 \cap \bar{U} \subset U_k$ and $A_{\alpha_i} \cap (\bar{U} - U_k)$ is finite and nonempty Put $U_k' = \bar{U} - U_k$. Now consider a' = the first element c in $\{a_r': r \in \omega\}$ with $c \notin \bigcup_{i<k} (U_i \cup U_i' \cup U_i'') \cup \bar{U}$ (if such an element does not exist, we put $U_k'' = \emptyset$). We take a clopen good neighborhood U_k'' of a' such that $U_k'' \subset U^0 - U^1$ and $U_0, \ldots, U_k$, $U_0', \ldots, U_k'$, $U_0'', \ldots, U_k''$ are pairwise disjoint. Consider $\tilde{U} = U^1 \cup \cup (\bigcup_{k \in \omega} U_k)$. Proceeding in the same way we construct the corresponding neighborhood $\tilde{V}$ of b. Then, for $1 \leqslant i \leqslant n$ and $\alpha \in \gamma_i$, $A_\alpha \cap (A - \tilde{U})$ is accessible iff $B_\alpha \cap (B - \tilde{V})$ is accessible iff $\alpha_i <_{\xi_0} \alpha$.

Now let us consider, for $i = n+1, \ldots, m$, an $\alpha_i \in \gamma_i$ such that $A_{\alpha_i} \cap (A - \tilde{U})$ and $B_{\alpha_i} \cap (B - \tilde{V})$ are accessible, a $\beta_i \in S^*$ with $A_{\alpha_i} \overset{0}{\to} A_{\beta_i}$ and $a_i \in \tilde{U}$, $b_i \in \tilde{V}$ with $s_{\xi_0}(a_i, A) = s_{\xi_0}(b_i, B) = \beta_i$. Take clopen good neighborhoods U_i, V_i of a_i, b_i respectively in such a way that $a \notin U_i \subset \tilde{U}$ and $b \notin V_i \subset \tilde{V}$. Put $U = \tilde{U} - \bigcup_{i=n+1}^{m} U_i$ and $V = \tilde{V} - \bigcup_{i=n+1}^{m} V_i$. ■

We can now refine the argument employed in [2, Theorem 2.2] and obtain in this way Theorem 1. The following result is an immediate consequence of Theorem 1 and Lemma 3.4. Corresponding results for uncountable T_3 spaces (concerning $(L_{\omega_1\omega})_t$-equivalence) can also be obtained.

THEOREM 2. *If A is a countable metrizable space which satisfies a spectrum, then A has a Scott sentence in $(L_{\omega_1\omega})_t$.*

As a consequence of Theorem 2 we obtain

THEOREM 3. *Every countable ordinal with order topology has a Scott sentence in $(L_{\omega_1\omega})_t$.*

Finally, we construct an ω-topological tree A with just one minimal element $\bar{a}$ in such a way that if S is a complex of types satisfied by A then for every neighborhood U of $\bar{a}$ we can find an ω-type α in S such that $A_\alpha \not\to \bar{a}$ and, nevertheless, there is an $a \neq \bar{a}$ of ω-type α with $a \in U$ (for any complex $S = \langle (S_\xi, <_\xi): \xi$ ordinal$\rangle$, $S_{\xi+1}$ depends only on S_ξ if $\xi < \omega$). Clearly such a space can not satisfy any spectrum. If $a \in A$ we denote by $N(a)$ the set of immediate successors of a. First, we define the n-type α_n by induction on $n \geqslant 1$: $\alpha_1 = \langle \emptyset, \emptyset \rangle$, $\alpha_{n+1} = \langle \{(\alpha_1, 1), \ldots, (\alpha_n, 1)\}, \emptyset \rangle$. For each $n \geqslant 1$ we consider an ω-topological tree $A^{(n)}$ with just one minimal element a_n. Fix $n \geqslant 1$. To define $A^{(n)}$ we consider countable infinite sets (pairwise disjoint) of the form $A_a^{\alpha_n}$, A_a^k $(k \geqslant 1)$. We suppose that if b belongs to a set of the form $A_a^{\alpha_n}$ then the n-type of b is α_n. We put $N(a_n) = A_{a_n}^{\alpha_n} \cup A_{a_n}^1$ and, for $k \geqslant 1$ and $a \in A_b^k$, $N(a) = A_a^{\alpha_n} \cup A_a^{k+1}$. Now consider the topological sum $\sum_{n \geqslant 1} A^{(n)}$ and a new point $\bar{a}$, and put $N(\bar{a}) = \{a_n: n \geqslant 1\}$.

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