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DEPARTMENT OF MATHEMATICS
DELFT UNIVERSITY OF TECHNOLOGY
P. O. Box 356
2600 AJ Delft
The Netherland

Residuality of the set of embeddings into Nagata's n-dimensional universal spaces

by

Elżbieta Pol (Warszawa)

Abstract. We prove that the set of homeomorphic embeddings of an n-dimensional metrizable space X of weight $\tau \geqslant \aleph_0$ into the universal n-dimensional Nagata's space $K_n(\tau) \subset S(\tau)^{\aleph_0}$, $S(\tau)$ being the standard τ-star-space, is residual in the function space of all continuous mappings of X into $S(\tau)^{\aleph_0}$. This answers in a strong form a question posed by K. Kuratowski (see [N2], p. 260). The proof is based on a classical Baire-category method.

1. Introduction. The aim of this paper is to extend some classical embedding results for n-dimensional separable metrizable spaces to nonseparable spaces. More specifically, we show that, given an n-dimensional metrizable space X of weight $\tau \geqslant \aleph_0$, the embeddings of X into Nagata's universal space $K_n(\tau)$ (a generalization of the classical Nöbeling's universal space; see [E], Theorem 1.11.5) form a residual set in the space of all mappings of X into the universal metrizable space $S(\tau)^{\aleph_0}$, where $S(\tau)$ is the star-space of weight τ.

This result answers (in a strong form) a question in [N2], which J. Nagata attributes to K. Kuratowski; an answer to the original question follows also from [P1], where some refinements of Nagata's embedding theorems for n-dimensional and countable-dimensional metrizable spaces ([N3], Theorems VI. 5 and [N1], Theorem 9) are given.

In this paper embedding theorems are obtained by the classical Baire-category method, while the embedding problems dealt with in the paper [P1] do not admit such an approach. In particular, the set of all embeddings of a countable-dimensional metric space of weight $\tau \geqslant \aleph_0$ into $K_\infty(\tau) = \bigcup_{n=1}^{\infty} K_n(\tau)$, which is dense in $C(X, S(\tau)^{\aleph_0})$ by [P], Corollary 2.2, may not be residual in $C(X, S(\tau)^{\aleph_0})$ (see Remark 3.7).

2. Notation and definitions. Our terminology follows [E] and [N3]. By dimension we understand the covering dimension dim. The term function and a symbol $f\colon X \to Y$ always denotes a continuous function. By I we denote the unit interval $[0, 1]$, by Q — the set of rationals in I, by N — the set of integers and by I^ω — the Hilbert cube. A family $\mathscr{A}$ of subsets of a metric space (X, ϱ) is δ-discrete, if $\varrho(A, B) \geqslant \delta$ for every distinct $A, B \in \mathscr{A}$, where

$$\varrho(A, B) = \min\{\varrho(a, b)\colon a \in A, b \in B\}.$$

2.1. Given an arbitrary space X and a metric space Y with a fixed bounded metric ϱ, we denote by $C(X, Y)$ the space of all continuous mappings from X into Y endowed with the metric $d(f, g) = \sup\{\varrho(f(x), f(y)): x, y \in X\}$. If ϱ is a complete metric in Y, then d is a complete metric in $C(X, Y)$.

2.2. A subset A of a space X is *residual* if its complement $X \setminus A$ is a first category set (i.e. $X \setminus A$ is the union of countably many nowhere dense sets). A countable intersection of residual subsets of X is residual in X. Note that if X is a complete metric space then by the Baire theorem every residual subset of X is dense in X (hence A is residual if and only if A contains a dense G_δ-subset of X).

2.3. Let τ be a cardinal number $\geqslant \aleph_0$. By $S(\tau)$ we denote the τ-star-space, i.e. the set $S(A)$ obtained by identifying all zeros in the set $\bigcup\{I_\alpha: \alpha \in A\}$, where $I_\alpha = I$ for $\alpha \in A$, and $|A| = \tau$, equipped with the complete metric

$$\sigma(x, y) = \begin{cases} |x-y| & \text{if } x, y \text{ belong to the same interval } I_\alpha, \\ x+y & \text{if } x, y \text{ belong to distinct intervals.} \end{cases}$$

A countable power $S(\tau)^{\aleph_0}$ of $S(\tau)$ is the universal space for all metrizable spaces of weight τ (see [K]). By $K_n(\tau)$ ($K_\infty(\tau)$) we denote the subspace of $S(\tau)^{\aleph_0}$ consisting of all points in $S(\tau)^{\aleph_0}$ which have at most n (only finitely many) rational coordinates distinct from 0. By Nagata's embedding theorems (see [N1], Theorem 9 and [N3], Theorem VI. 5) a metrizable space X of weight $\tau \geqslant \aleph_0$ has dimension $\leqslant n$ (is countable dimensional) if and only if X is homeomorphic to a subset of $K_n(\tau)$ ($K_\infty(\tau)$). We fix a complete metric ϱ in $S(\tau)^{\aleph_0}$ by putting

$$\varrho(\{x_i\}_{i=1}^{\infty}, \{y_i\}_{i=1}^{\infty}) = [\sum_{i=1}^{\infty} 1/2^i \cdot \sigma(x_i, y_i)^2]^{1/2}.$$

2.4. Given a cardinal number $\tau \geqslant \aleph_0$, we denote by $B(\tau)$ the generalized Baire space of weight τ, i.e. the countable power $D(\tau)^{\aleph_0}$ of the discrete space of weight τ. The space $B(\tau) \times I^\omega$ is a universal space for the class of strongly metrizable spaces of weight τ (see [M]). Recall that a space X is *strongly metrizable* if it has a base which is the union of countably many star-finite open coverings of X. Let N_n^ω (N_∞^ω) be the subspace of the Hilbert cube I^ω consisting of all points which have at most n (only finitely many) rational coordinates. As proved by Nagata (see [N1] and [N3]), a strongly metrizable space X of weight $\tau \geqslant \aleph_0$ has dimension $\leqslant n$ (is countable-dimensional) if and only if X is homeomorphic to a subset of the product $B(\tau) \times N_n^\omega$ ($B(\tau) \times N_\infty^\omega$).

3. The results. The main results of this paper are the following two theorems.

3.1. THEOREM. *If X is a metrizable n-dimensional space of weight $\tau \geqslant \aleph_0$, then the set*

$$\mathcal{H} = \{h \in C(X, S(\tau)^{\aleph_0}): h \text{ is a homeomorphic embedding and } \overline{h(X)} \subset K_n(\tau)\}$$

is residual in $C(X, S(\tau)^{\aleph_0})$.

3.2. THEOREM *If X is a strongly metrizable space of weight $\tau \geqslant \aleph_0$, then the set*

$$\mathcal{H} = \{h \in C(X, B(\tau) \times I^\omega): h \text{ is a homeomorphic embedding and } \overline{h(X)} \subset B(\tau) \times N_n^\omega\}$$

is residual in $C(X, B(\tau) \times I^\omega)$.

The theorems are immediate consequences of the following four propositions.

3.3. PROPOSITION. *If X is a normal n-dimensional space, then for any cardinal number τ the set*

$$\mathcal{H}_1 = \{h \in C(X, S(\tau)^{\aleph_0}): \overline{h(X)} \subset K_n(\tau)\}$$

is residual in $C(X, S(\tau)^{\aleph_0})$.

3.4. PROPOSITION [T]. *If X is a metrizable space of weight $\tau \geqslant \aleph_0$, then the set*

$$\mathcal{H}_2 = \{h \in C(H, S(\tau)^{\aleph_0}): h \text{ is a homeomorphic embedding}\}$$

is residual in $C(X, S(\tau)^{\aleph_0})$.

Note that the analogue of Proposition 3.4 for $C(X, S(\tau)^{\aleph_0})$ equipped with the limitation topology was proved by H. Toruńczyk (see [T], Lemma 3.8, by [T], Remark after Theorem 5.1, $S(\tau)^{\aleph_0}$ is homeomorphic to a Hilbert space).

3.5. PROPOSITION. *If X is a normal n-dimensional space, then for every cardinal number τ the set*

$$\mathcal{H}_3 = \{h \in C(X, B(\tau) \times I^\omega): \overline{f(X)} \subset B(\tau) \times N_n^\omega\}$$

is residual in $C(X, B(\tau) \times I^\omega)$.

3.6. PROPOSITION. *If X is a strongly metrizable space of weight $\tau \geqslant \aleph_0$, then the set*

$$\mathcal{H}_4 = \{h \in C(X, B(\tau) \times I^\omega): h \text{ is a homeomorphic embedding}\}$$

is residual in $C(X, B(\tau) \times I^\omega)$.

3.7. Remark. In [P1] we proved that, given a sequence $X_1, X_2, \ldots$ of 0-dimensional subspaces of a metrizable space X of weight $\tau \geqslant \aleph_0$, the set $\mathcal{F}$ of all homeomorphic embeddings $h: X \to S(\tau)^{\aleph_0}$ such that $h(X_n) \subset K_{n-1}(\tau)$ for every $n \in N$ is dense in $C(X, S(\tau)^{\aleph_0})$. This result applies to n-dimensional space as well as to countable-dimensional ones. Note that the set $\mathcal{F}$ need not be residual, even if $X_i = \emptyset$ for $i \geqslant 2$, since the set of all embeddings of I into I^ω such that $h(P) \subset P^{\aleph_0}$, where P is the set of irrationals, is of the first category (see [P1], Remark 5.2). In the case when $X = \bigcup_{i=1}^{\infty} X_i$ the set $\mathcal{F}$ also need not be residual, even if X is compact. This follows from the following result obtained recently by the author:

Let X be a complete metric separable space. Then the set of all embeddings of X into Nagata's universal space N_∞^ω is residual in $C(X, I^\omega)$ if and only if X is strongly countable-dimensional, i.e. X is the union of countably many closed finite-dimensional subsets.

The proof can be found in [P2].

3.8. Remark. By the completion theorem (see [E], Theorem 4.1.20) every metrizable space has a completion of the same weight and dimension. Note that, by Theorem 3.1, for every n-dimensional metrizable space X of weight $\tau \geqslant \aleph_0$ the set of all embeddings h of X into $S(\tau)^{\aleph_0}$ such that the complete space $\overline{h(X)}$ is n-dimensional, is residual in $C(X, S(\tau)^{\aleph_0})$.

4. The proofs.

4.1. Proof of Proposition 3.3. Define

$$\mathscr{K} = \{K = \{q_1, \ldots, q_{n+1}\}\colon q_i \in Q\setminus\{0\} \text{ for } i = 1, \ldots, n+1\} \text{ and}$$

$$\mathscr{J} = \{J = \{i_1, \ldots, i_{n+1}\}\colon i_j \in N \text{ and } i_j \neq i_k \text{ for } j \neq k, \text{ where } 1 \leqslant j, k \leqslant n+1\};$$

note that the sets $\mathscr{K}$ and $\mathscr{J}$ are countable.

For $K = \{q_1, \ldots, q_{n+1}\} \in \mathscr{K}$ and $J = \{i_1, \ldots, i_{n+1}\} \in \mathscr{J}$ let

$$F(K, J) = \{\{x_i\}_{i=1}^{\infty} \in S(\tau)^{\aleph_0}\colon \sigma(x_{i_j}, 0) = q_j \text{ for } j = 1, \ldots, n+1\}$$

and

$$\mathscr{F}(K, J) = \{f \in C(X, S(\tau)^{\aleph_0})\colon \varrho(f(X), F(K, J)) > 0\}.$$

We have

(1) $S(\tau)^{\aleph_0}\setminus K_n(\tau) = \bigcup\{F(K, J)\colon K \in \mathscr{K},\ J \in \mathscr{J}\}$.

It is easy to see that

(2) every set $\mathscr{F}(K, J)$ is open in $C(X, S(\tau)^{\aleph_0})$.

We will show that, for every $K = \{q_1, \ldots, q_{n+1}\} \in \mathscr{K}$ and $J = \{i_1, \ldots, i_{n+1}\} \in \mathscr{J}$,

(3) the set $\mathscr{F}(K, J)$ is dense in $C(X, S(\tau)^{\aleph_0})$; i.e.

(4) for every function $f' = \{f_i\}_{i=1}^{\infty}\colon X \to S(\tau)^{\aleph_0}$ and every $\varepsilon > 0$ there exists a function $g' = \{g_i\}_{i=1}^{\infty}\colon X \to S(\tau)^{\aleph_0}$ such that $d(g', f') < \varepsilon$ and

$$\varrho(g'(X), F(K, J)) > 0.$$

We can assume without loss of generality that $\{i_1, \ldots, i_{n+1}\} = \{1, \ldots, n+1\} = J_0$. Let $S(\tau)^{\aleph_0} = \mathop{\mathrm{P}}_{i=1}^{\infty} S_i(A_i)$, where $S_i(A_i) = S(A_i)$ and $A_i = A$ for $i = 1, 2, \ldots$, where $|A| = \tau$. Let $\pi\colon S(\tau)^{\aleph_0} \to S(\tau)^{n+1}$ be the projection, where $S(\tau)^{n+1} = \mathop{\mathrm{P}}_{i=1}^{n+1} S_i(A_i)$, and let $\tilde{\varrho}$ be a metric in $S(\tau)^{n+1}$ defined by

$$\tilde{\varrho}(\{x_i\}_{i=1}^{n+1}, \{y_i\}_{i=1}^{n+1}) = [\sum_{i=1}^{n+1} 1/2^i \sigma(x_i, y_i)^2]^{1/2}.$$

Put $F = \{x = \{x_i\}_{i=1}^{n+1} \in S(\tau)^{n+1}\colon \sigma(x_i, 0) = q_i \text{ for } i = 1, \ldots, n+1\}$; we have $F(K, J_0) = \pi^{-1}(F)$. To prove (4) it suffices to show that

(5) for every $\varepsilon > 0$ and $f = \{f_i\}_{i=1}^{n+1}\colon X \to S(\tau)^{n+1}$ there exists a function $g = \{g_i\}_{i=1}^{n+1}\colon X \to S(\tau)^{n+1}$ such that $\tilde{\varrho}(g(X), F) > 0$ and $\sigma(f_i(x), g_i(x)) < \varepsilon$ for every $x \in X$ and $i = 1, \ldots, n+1$.

Indeed, if a function $g = \{g_i\}_{i=1}^{n+1}$ satisfying (5) is defined, then we put $g' = \{g_i\}_{i=1}^{\infty}$, where $g_i = f_i$ for $i \geqslant n+2$; since $\pi(g'(X)) = g(X)$ and $F(K, J_0) = \pi^{-1}(F)$, the condition (4) is satisfied.

Put $T = A^{n+1}$ and for $t = \{\alpha_1, \ldots, \alpha_{n+1}\} \in T$ let $p_t = \{x_1, \ldots, x_{n+1}\} \in S(\tau)^{n+1}$, where $x_i = q_i \in J_{\alpha_i}$. We have $F = \bigcup\{p_t\colon t \in T\}$. Take $\eta > 0$ such that

$$\eta < \min\{q_1, \ldots, q_{n+1}, \varepsilon/2\}.$$

For

$$t = \{\alpha_1, \ldots, \alpha_{n+1}\} \in T$$

put

$$U_t = \{\{x_1, \ldots, x_{n+1}\} \in S(\tau)^{n+1}\colon x_i \in J_{\alpha_i} \text{ and } |x_i - q_i| < \eta \text{ for } i = 1, \ldots, n+1\},$$

$$K_t = \overline{U}_t \quad\text{and}\quad S_t = K_t\setminus U_t.$$

Then $\{U_t\}_{t\in T}$ is a discrete family of open neighborhoods of the points p_t and the family $\{f^{-1}(K_t)\}_{t\in T} \cup \{f^{-1}(S(\tau)^{n+1})\setminus \bigcup_{t\in T} U_t\}$ is a locally finite closed covering of X. Observe that every set K_t is an $(n+1)$-dimensional cube. Suppose that (i) $q_i + \eta \leqslant 1$ for every $i = 1, \ldots, n+1$. Then for every $t \in T$ the set S_t is the union of all faces of the cube K_t, hence it is homeomorphic to the n-dimensional sphere. Since $f^{-1}(K_t)$ is a closed subset of a normal space X, we have $\dim f^{-1}(K_t) \leqslant \dim X \leqslant n$. By a theorem on the extension of mappings to spheres (see [E], Theorem 3.2.10), for every $t \in T$ there exists a function $g_t = \{g_{ti}\}_{i=1}^{n+1}\colon f^{-1}(K_t) \to S_t$ such that $g_t|f^{-1}(S_t) = f|f^{-1}(S_t)$. Now suppose that (ii) $q_i + \eta > 1$ for some $i \in \{1, \ldots, n+1\}$.

Then, for every $t \in T$, S_t is the union of $\leqslant 2n-1$ faces of the $(n+1)$-cube K_t; hence, there exists a continuous retraction $r_t\colon K_t \to S_t$. In this case we put $g_t = r_t \circ f|f^{-1}(K_t) \to S_t$ for every $t \in T$. In both cases (i) and (ii), for every $t \in T$ we have

$$\tilde{\varrho}(p_t, g_t(f^{-1}(K_t))) \geqslant \tilde{\varrho}(p_t, S_t) \geqslant \frac{1}{2^{n+1}} \cdot \eta \quad\text{and}$$

$$\sigma(g_{ti}(x), f_i(x)) \leqslant 2\eta < \varepsilon \quad \text{for every } x \in X \text{ and } i = 1, \ldots, n+1.$$

Let $g\colon X \to S(\tau)^{n+1}$ be the combination of the functions g_t, $t \in T$ and the function $f|f^{-1}(S(\tau)^{n+1}\setminus\bigcup\{U_t\colon t \in T\})$. Since these functions are compatible and defined on a locally finite family of closed sets, the function g is continuous. From the construction it follows that $\tilde{\varrho}(g(X), F) > 0$ and $\tilde{\varrho}(g_i(x), f_i(x)) < \varepsilon$ for every $x \in X$ and $i = 1, \ldots, n+1$. This ends the proof of (5). If

$$f \in \bigcap\{\mathscr{F}(K, J)\colon K \in \mathscr{K} \text{ and } J \in \mathscr{J}\},$$

then $\overline{f(X)} \cap F(K, J) = \emptyset$ for every $K \in \mathscr{K}$ and $J \in \mathscr{J}$; hence by (1) we have $\overline{f(X)} \subset K_n(\tau)$. Thus

(6) $\mathscr{H}_1 \supset \bigcap \{\mathscr{K}(K, J)\colon K \in \mathscr{K} \text{ and } J \in \mathscr{J}\}$,

and so by (2) and (3) the set $\mathscr{H}_1$ is residual in $C(X, S(\tau)^{\aleph_0})$.

4.2. Proof of Proposition 3.4. Let X be a metrizable space of weight $\tau \geqslant \aleph_0$. Take a σ-discrete base of X, which is the union of discrete families

$$\mathscr{W}_m = \{W_{m\alpha}\colon \alpha \in A_m\}$$

for $m = 1, 2, \ldots$ We can assume without loss of generality that there exist open sets $V_{m\alpha}$, $\alpha \in A_m$, $m = 1, 2, \ldots$, such that $F_{m\alpha} = \overline{V_{m\alpha}} \subset W_{m\alpha}$ and for every neighborhood $U(x)$ of any point $x \in X$ there exist m and $\alpha \in A_m$ for which $x \in F_{m\alpha} \subset W_{m\alpha} \subset U(x)$. We can assume that $A_m = A$ for $m = 1, 2, \ldots$, where $|A| = \tau$. Put $W_m = \bigcup \{W_{m\alpha}\colon \alpha \in A_m\}$ and $F_m = \bigcup \{F_{m\alpha}\colon \alpha \in A_m\}$; W_m and F_m are open and closed sets, respectively, such that $F_m \subset U_m$. Let $S(\tau)^{\aleph_0} = \mathop{\mathbf{P}}\limits_{i=1}^{\infty} S(A_i)$. For $m \in N$ put

$$\mathscr{F}_m = \{f \in C(X, S(\tau)^{\aleph_0})\colon \varrho(f(X \setminus W_m), f(F_m)) > 0\}$$

and

$$\mathscr{G}_m = \{f \in C(X, S(\tau)^{\aleph_0})\colon \text{the family } \{f(W_{m\alpha})\}_{\alpha \in A_m} \text{ is } \delta\text{-discrete for some } \delta > 0\}.$$

It is easy to see that

(7) the sets $\mathscr{F}_m$ and $\mathscr{G}_m$ are open in $C(X, S(\tau)^{\aleph_0})$.

We will show that

(8) the sets $\mathscr{F}_m$ and $\mathscr{G}_m$ are dense in $C(X, S(\tau)^{\aleph_0})$.

Take an arbitrary $f = \{f_m\}_{m=1}^{\infty} \in C(X, S(\tau)^{\aleph_0})$ and arbitrary $\varepsilon > 0$.

Let $k \in N$ be such that $2^{-k} < \varepsilon$. Let $g_k\colon X \to S(\tau)$ be any continuous function such that $g_k(X \setminus W_m) = 0$ and $g_k(F_m) = 1 \in I_\alpha$ for some $\alpha \in A_k$ and let $h_k\colon X \to S(\tau)$ be any continuous function such that $h_k(W_{m\alpha}) = 1 \in I_\alpha$ (hence the family $\{h_k(W_{m\alpha})\colon \alpha \in A_m\}$ is 2-discrete in $S(\tau)$). Put $g_m = h_m = f_m$ for $m \neq k$ and let $g = \{g_m\}_{m=1}^{\infty}$ and $h = \{h_m\}_{m=1}^{\infty}$. Then $g \in \mathscr{F}_m$, $h \in \mathscr{G}_m$, $d(g, f) < \varepsilon$ and $d(h, f) < \varepsilon$. This ends the proof of (8).

We will show that

(9) $\bigcap\limits_{m=1}^{\infty} (\mathscr{F}_m \cap \mathscr{G}_m) \subset \mathscr{H}_2$.

Let $f \in \bigcap\limits_{m=1}^{\infty} (\mathscr{F}_m \cap \mathscr{G}_m)$. We claim that

(10) for every $x \in X$ and every closed subset F of X such that $x \notin F$ we have $f(x) \notin \overline{f(F)}$.

There exist $m \in N$ and $\alpha \in A_m$ such that $x \in F_{m\alpha} \subset W_{m\alpha} \subset X \setminus F$. Let $F' = F \cap (X \setminus W_m)$ and $F'' = F \cap \bigcup \{W_{m\beta}\colon \beta \in A_m \text{ and } \beta \neq \alpha\}$; then $F = F' \cup F''$. Since $f \in \mathscr{F}_m$, we have $\varrho(f(F_m), f(X \setminus W_m)) > 0$, and hence $\varrho(f(x), f(F')) > 0$. Since $f \in \mathscr{G}_m$, the family $\{f(W_{m\alpha})\}_{\alpha \in A_m}$ is δ-discrete for some $\delta > 0$ and hence $\varrho(f(x), f(F'')) \geqslant \delta > 0$. Thus $\varrho(f(x), f(F)) > 0$, proving the claim (10).

Condition (10) implies that f is a homeomorphic embedding, i.e. that $f \in \mathscr{H}_2$. From (7), (8) and (9) it follows that $\mathscr{H}_2$ is residual in $C(X, S(\tau)^{\aleph_0})$.

4.3. Proof of Proposition 3.5. Since the space $C(X, B(\tau) \times I^\omega)$ is homeomorphic with $C(X, B(\tau)) \times C(X, I^\omega)$ in a natural way, with the set $\mathscr{H}_3$ corresponding to a set containing the set $C(X, B(\tau)) \times \mathscr{H}'_3$, where

$$\mathscr{H}'_3 = \{f \in C(X, I^\omega)\colon \overline{f(X)} \subset N_n^\omega\},$$

is suffices to prove that

(11) the set $\mathscr{H}'_3$ is residual in $C(X, I^\omega)$.

But this was in fact proved in Proposition 3.3, since we can identify I with $S(A)$, where $|A| = 2$ (the only modification of the proof consists in replacing $\mathscr{K}$ by the family $\mathscr{K}' = \{K = \{q_1, \ldots, q_{n+1}\}\colon q_i \in Q \text{ for } i = 1, \ldots, n+1\}$).

4.4. Proof of Proposition 3.6. Recall that if A is a member of a family $\mathscr{A}$ of subsets of X, then $S(A, \mathscr{A}) = S^1(A, \mathscr{A}) = \bigcup \{B \in A\colon B \cap \mathscr{A} \neq \emptyset\}$,

$$S^n(A, \mathscr{A}) = S(S^{n-1}(A, \mathscr{A}), \mathscr{A})$$

and $S^\infty(A, A) = \bigcup\limits_{n=1}^{\infty} S^n(A, \mathscr{A})$.

Let $\mathscr{N}_1, \mathscr{N}_2, \ldots$ be a sequence of star-finite coverings of X such that the family $\{S(x, \mathscr{N}_i)\colon i = 1, 2, \ldots\}$ is a neighborhood basis at each point $x \in X$. For $i \in N$ let

$$\mathscr{S}_i = \{S^\infty(N, \mathscr{N}_i)\colon N \in \mathscr{N}_i\}.$$

Since $\mathscr{N}_i$ is star-finite, we have $\mathscr{S}_i = \{S_\alpha\colon \alpha \in A_i\}$, where $S_\alpha \cap S_\beta = \emptyset$ for $\alpha \neq \beta$ and $S_\alpha = \bigcup \{N_{\alpha j}\colon j \in N\}$, where $N_{\alpha j} \in \mathscr{N}_i$. We can assume that for each $i \in N$, $A_i = A$, where $|A| = \tau$. For each $i \in N$ take an open covering $\mathscr{P}_i$ of X such that $\mathscr{P}_i = \{P_{\alpha j}\colon \alpha \in A_i, j \in N\}$, where $\overline{P_{\alpha j}} \subset N_{\alpha j}$ and define $U_{ij} = \bigcup \{N_{\alpha j}\colon \alpha \in A_i\}$ and $F_{ij} = \bigcup \{\overline{P_{\alpha j}}\colon \alpha \in A_i\}$. For $i, j \in N$ define

$$\mathscr{F}_{ij} = \{\varphi \in C(X, B(\tau) \times I^\omega)\colon \varrho(\varphi(F_{ij}), \varphi(X \setminus U_{ij})) > 0\}$$

and

$$\mathscr{G}_i = \{\varphi \in C(X, B(\tau) \times I^\omega)\colon \text{the family } \{\varphi(S_\alpha)\}_{\alpha \in A_i} \text{ is } \delta\text{-discrete for some } \delta > 0\}.$$

It is easy to see that

(12) the sets $\mathscr{F}_{ij}$ and $\mathscr{G}_i$ are open in $C(X, B(\tau) \times I^\omega)$ for every $i, j \in N$.

We will show that

(13) the sets $\mathscr{F}_{ij}$ and $\mathscr{G}_i$ are dense in $C(X, B(\tau) \times I^\omega)$ for every $i, j \in N$.

Take an arbitrary $\varepsilon > 0$ and $\varphi = (c, f): X \to B(\tau) \times I^\omega$, where

$$c = \{c_m\}^\infty_{1}: X \to B(\tau) = \mathop{\mathrm{P}}_{m=1}^{\infty} D_m$$

and D_m is the set A with the discrete topology for every $m \in N$ and

$$f = \{f_m\}_{m=1}^\infty: X \to I^\omega = \mathop{\mathrm{P}}_{m=1}^{\infty} I_m,$$

where $I_m = I$ for $m = 1, 2, \ldots$ Take $k \in N$ such that $2^{-k} < \varepsilon/2$.

Define $\psi = (d, g): X \to B(\tau) \times I^\omega$ as follows: $d = \{d_m\}_{m=1}^\infty$ and $g = \{g_m\}_{m=1}^\infty$, where $d_m = c_m$ and $g_m = f_m$ for $m \neq k$, $d_k(x) = \alpha$, if $x \in S_\alpha$ for some $\alpha \in A_i$, and $g_k: X \to I$ is any function such that $g_k(F_{ij}) = 1$ and $g_k(X \setminus U_{ij}) = 0$. It is easy to verify that $\psi \in \mathscr{F}_{ij} \cap \mathscr{G}_i$ and $d(\psi, \varphi) < \varepsilon$.

Now we check that

(14) $\bigcap_{i,j=1}^{\infty} (\mathscr{F}_{ij} \cap \mathscr{G}_i) \subset \mathscr{H}_4$.

Let $\varphi \in \bigcap_{i,j=1}^{\infty} (\mathscr{F}_{ij} \cap \mathscr{G}_i)$. We will show that

(15) for every $x \in X$ and a closed set $F \subset X$ such that $x \notin F$ we have $\varphi(x) \notin \overline{\varphi(F)}$.

Indeed, there exist $i, j \in N$ and $\alpha \in A_i$ such that $x \in P_{\alpha j} \subset N_{\alpha j} \subset X \setminus F$. Let $F' = F \cap (X \setminus U_{ij})$ and $F'' = F \cap U_{ij} \subset F \cap \bigcup \{N_{\beta j}: \beta \in A_i, \beta \neq \alpha\}$. Since $\varphi \in \mathscr{F}_{ij}$, we have $\varrho((\varphi(x), \varphi(F')) > 0$.

Since $\varphi \in \mathscr{G}_i$ and $N_{\alpha j} \subset S_\alpha$, the family $\{\varphi(N_{\alpha j})\}_{\alpha \in A_i}$ is δ-discrete for some $\delta > 0$, and thus $\varrho(\varphi(x), \varphi(F'')) \geqslant \delta > 0$. It follows that $\varrho(\varphi(x), \varphi(F)) > 0$, i.e $\varphi(x) \notin \overline{\varphi(F)}$. The condition (15) implies that φ is a homeomorphic embedding, i.e. $\varphi \in \mathscr{H}_4$. By (12), (13) and (14) the set $\mathscr{H}_4$ is residual in $C(X, B(\tau) \times I^\omega)$.

Added in proof. As was proved by H. Toruńczyk [T], for every completely metrizable space X of weight $\leqslant \tau$, the set of all closed embeddings of X into $S(\tau)^{\aleph_0}$ is residual in $C(X, S(\tau)^{\aleph_0})$ equipped with the limitation topology. Let us note that the methods of our paper can be modified to show that Proposition 3.3. and Theorem 3.1 also hold if $C(X, S(\tau)^{\aleph_0})$ is considered with the limitation topology. In particular, we obtain the following result:

For every n-dimensional completely metrizable space X of weight $\leqslant \tau$ the set of all embeddings of X onto a closed subset of $S(\tau)^{\aleph_0}$ contained in $K_n(\tau)$ is residual in the space $C(X, S(\tau)^{\aleph_0})$ equipped with the limitation topology.

This strengthens some results of A. Waśko, Bull. London Math. Soc. 18 (1986), 293–198 and Y. Hattori, A note on universal spaces for finite dimensional complete metric spaces, preprint.

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UNIVERSITY OF WARSAW
Pałac Kultury i Nauki, IX p.
00-901 Warszawa, Poland

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