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Received 20 January 1986

The structure of orbits in dynamical systems

by

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Abstract. Spaces which are both locally homeomorphic to $\boldsymbol{Q}\times\boldsymbol{R}$, the topological product of the rationals $\boldsymbol{Q}$ and the reals $\boldsymbol{R}$, and arcwise connected are studied. It is shown that such spaces are the image of $\boldsymbol{R}$ under a one-to-one and continuous map having the arc lifting property.

A necessary and sufficient condition for a separable and metrizable space X to be the orbit in some flow is presented. The following structure theorem is obtained.

A space X is the orbit of a Poisson-stable and aperiodic motion if and only if X is homeomorphic to the suspension of a universally transitive homeomorphism of $\boldsymbol{Q}$.

Unless explicitly stated otherwise all spaces under consideration are separable and metrizable.

1. Introduction.

1.1. The following problem will be discussed. What are necessary and sufficient conditions for a separable and metrizable space X to be the orbit in some continuous dynamical system (or flow)? In this paper a topological characterization of orbits of flows is presented. A structure theorem for orbits of flows is discussed as well. The classification problem is only lightly touched upon.

For locally compact orbits the situation is rather simple. There are only three homeomorphism types of locally compact orbits. In a flow each locally compact orbit is either a singleton, or a simple closed curve or a topological copy of the real line. And obviously each of these spaces can be endowed with the dynamical structure of an orbit. The reader is referred to Subsections 1.3 and 1.4 for more details about these remarks.

The topological structure of orbits which are not locally compact is much more complicated and has not yet been studied in great detail thus far. The orbits which are not locally compact are precisely the orbits of the motions which are (positively or negatively) Poisson-stable, but not periodic. The structure theorem is presented in Section 5. It will be shown that an aperiodic and Poisson-stable motion can be viewed as the suspension of a discrete dynamical system on the space of the rationals $\boldsymbol{Q}$. Such a discrete system is generated by a so-called universally transitive homeomorphism. See Section 4. Some information about discrete systems, including a very simple orbit classification, is presented in 1.3.

The most striking geometric feature of orbits of motions which are aperiodic as well as Poisson-stable is that these orbits are locally homeomorpnic to $Q\times R$, the topological product of the rationals and the reals. Some basic lemmas about such spaces are presented in Section 2. Because the spaces under discussion are not locally compact, we could not fall back on the results and methods of [9] and [16]. The methods employed in this paper are totally different. In Section 3 we discuss the pasting together of topological copies of $Q\times R$. This is the cornerstone for the topological characterization of orbits of flows. A space which is both locally homeomorphic to $Q\times R$ and arcwise connected is called a *P-manifold* in this paper. The main result is as follows. A separable and metrizable space X is an orientable P-manifold if and only if X can be endowed with the structure of an orbit in some flow which is both aperiodic and Poisson-stable. This result is proved in Section 4 by endowing such a P-manifold with the structure of the suspension of a suitable homeomorphism of Q onto itself.

In view of the preceding results the classification of orbits of flows can be reduced to another classification problem, namely that of homeomorphisms of Q onto itself. Here, however, the picture is far from complete. See Section 5 for more details.

1.2. The background material for this paper can be found in [3], [12], [22] or [23]. To fix the notations and for the convenience of the reader a listing is given of the most frequently occurring notions.

Let G be the topological group of either the reals R or the integers Z. A *dynamical system* on a space X is a continuous mapping $\pi\colon X\times G\to X$ such that for all $x\in X$ and for all $s,t\in G$

(i) $\pi(x,0)=x$, and
(ii) $\pi(\pi(x,s),t)=\pi(x,s+t)$.

For each $x\in X$ the mapping $\pi_x\colon G\to X$, defined by $\pi_x(t)=\pi(x,t)$, is called the *motion through* x. The motion π_x is a continuous mapping of G into X and its image $\{\pi_x(t)|\ t\in G\}$ is called the *orbit of* x in the system π. For each $t\in G$ the mapping $\pi^t\colon X\to X$, defined by $\pi^t(x)=\pi(x,t)$, is called a *transition*. The transition π^t is a homeomorphism with inverse π^{-t}.

If $G=R$, then usually the dynamical system is said to be *continuous* or it is called a *flow*. If $G=Z$, then the dynamical system is called *discrete*. In this case the system is completely determined by the transition π^1. Thus a discrete dynamical system can be viewed as a pair (X,f) of a topological spaces X together with a homeomorphism $f\colon X\to X$.

Let $\pi\colon X\times G\to X$ be a dynamical system and let $x\in X$. The orbit of the point x is denoted by $\Gamma(x)$. As for each $t\in G$ the transition π^t maps $\Gamma(x)$ onto itself, the set $\Gamma(x)$ is invariant and the restriction of π to $\Gamma(x)\times G$ defines a dynamical system of $\Gamma(x)$. The *positive limit set* $\Omega(x)$ of x is defined as follows: $y\in\Omega(x)$ if and only if, for some sequence (t_n) from G, $t_n\to\infty$ and $\pi(x,t_n)\to y$. If $x\in\Omega(x)$, then the point x and also the motion π_x are called *positively Poisson-stable*. In a similar way the *negative limit set* $A(x)$ of x and *negative Poisson stability* are defined. *Poisson-stable* means positively or negatively Poisson-stable. A motion which is both positively and negatively Poisson-stable is said to be *two-sided Poisson-stable*. Equivalence of dynamical systems is defined as follows.

The discrete dynamical systems (X,f) and (Y,g) are called *equivalent* or *conjugated* whenever there exists a homeomorphism $h\colon X\to Y$ such that $h\circ f=g\circ h$. The flows $\pi\colon X\times R\to X$ and $\varrho\colon Y\times R\to Y$ are said to be *topologically equivalent* if there is a homeomorphism $h\colon X\to Y$ which maps each orbit of π onto an orbit in the system ϱ and preserves the orientation of orbits. In the special case that X and Y consist of a single orbit, the topological equivalence of π and ϱ amounts to the existence of a homeomorphism of X onto Y. This follows from the results of Ura [25]. See also [12], sections (1.26) and (2.50) in particular.

In the proof of the characterization theorem, to be presented in Section 4, the notion of the suspension of a discrete system is used. Let (X,f) be a discrete dynamical system. On $X\times R$ an equivalence relation $\sim$ is defined by $(x,r)\sim(y,s)$ if and only if, for some $m\in Z$, $r=s+m$ and $y=f^m(x)$. For each $(x,t)\in X\times R$ the equivalence class of (x,t) is denoted by $[x,t]$. Let Y denote the quotient space $X\times R/\sim$. The flow $\varrho\colon Y\times R\to Y$, defined by $\varrho([x,t],s)=[x,t+s]$, is called the *suspension of* (X,f). The subspace $\{[x,0]|\ x\in X\}$ is identified with X. It is to be observed that the restriction of ϱ^1 to X coincides with f. Also, if two discrete systems are conjugated, then their suspensions are topologically equivalent.

1.3. In this subsection some of the statements of 1.1. are elaborated. First discrete systems are discussed. The following theorem must be part of the folklore. As we have not found it in the literature, we have included a proof.

THEOREM. *Let (X,f) be a discrete system. Then each orbit of the system is homeomorphic to one of the following*: (i) *a finite set*, (ii) *the integers Z, or* (iii) *the rationals Q.*

Proof. Let $\Gamma(x)=\{f^n(x)|\ n\in Z\}$ be an orbit. As any point of $\Gamma(x)$ is mapped into any other point of $\Gamma(x)$ by a transition, the orbit $\Gamma(x)$ is homogeneous. Also, clearly $\Gamma(x)$ is countable. Now x is either isolated in $\Gamma(x)$ or not. It follows that if (i) and (ii) do not hold, $\Gamma(x)$ is a countably infinite and dense in itself space. Then according to a theorem of Sierpiński ([6] 1.3H, [24]) it is homeomorphic to Q.

The following observations are to be made. According to the convention made at the beginning of the paper in the formulation of the theorem it was tacitly assumed that the space X is metrizable. The following example shows that this condition cannot be omitted.

EXAMPLE 1. In [11] 4.22, 1d, an example is presented exhibiting the following phenomenon. The additive group of the integers Z can be endowed with a topology under which it is a nonmetrizable topological group. Addition of 1 defines a dynamical system in which each orbit coincides with Z (endowed with a nonmetrizable topology).

Although there are only countably many homeomorphism types of orbits, there are uncountable many distinct (i.e. nonconjugated) discrete systems. This is illustrated by the following example which is also relevant to the discussion in Section 5.

EXAMPLE 2. Let α and β be unimodular complex numbers with the property that the argument is an irrational multiple of 2π between 0 and 2π. The set $A = \{\alpha^n |\ n \in \boldsymbol{Z}\}$ is dense in the unit-circle S^1, whence homeomorphic to $\boldsymbol{Q}$. The shift σ_A is defined by $\sigma_A(\alpha^n) = \alpha^{n+1}$; the system (A, σ_A) may be seen as a dynamical system on $\boldsymbol{Q}$. In a similar way on the set $B = \{\beta^n |\ n \in \boldsymbol{Z}\}$ the shift σ_B is defined. Now, if $\alpha \neq \beta$, then the systems (A, σ_A) and (B, σ_B) cannot be conjugated. If this is false, then there is an equivalence $h\colon A \to B$. By replacing h by the composition of h and a suitable transition of (B, σ_B) if necessary, we may assume that $h(1) = 1$. Then $h(\alpha) = \beta$, as h is an equivalence. It follows that h is an isomorphism of topological groups, which is continuous at the identity. Hence, h is uniformly continuous and extendable to S^1. This results is an equivalence of rotations with different rotation numbers (e.g. [12]), a contradiction.

1.4. The main purpose of this paper is to prove the following theorem for flows.

THEOREM. *Let $\pi\colon X \times \boldsymbol{R} \to X$ be a flow. Then each orbit of the system π is homeomorphic to one of the following*: (i) *a singleton*, (ii) *the unit circle* S^1, (iii) *the reals* $\boldsymbol{R}$, *or* (iv) *an orientable P-manifold.*

The definition of an orientable P-manifold and a part of the proof are postponed till Sections 3 and 4.

Proof. Let $x \in X$. The orbit $\Gamma(x)$ is either compact or not. If $\Gamma(x)$ is compact, then (i) or (ii) holds ([12] 2.35). If $\Gamma(x)$ is not compact, then the motion π_x is aperiodic. We distinguish between the following, mutually exclusive cases: (a) π_x is not Poisson-stable, and (b) π_x is Poisson-stable.

Case (a) occurs if and only if (iii) holds ([8], Theorem 1).

The discussion of case (b) is postponed.

Remark. Although the results in this paper are formulated for flows, with some minor modifications all results also hold for partial flows or local dynamical system.

This can be quite easily verified and also follows from more general observations in [5], in particular Theorem 2.

2. Embedding matchboxes.

2.1. The orbits of motions which are both aperiodic and Poisson-stable are locally homeomorphic to $\boldsymbol{Q} \times \boldsymbol{R}$. That is the contents of the theorem below.

DEFINITION. A space X is said to be *locally homeomorphic to* $\boldsymbol{Q} \times \boldsymbol{R}$ if each point x of X has an open neighborhood which is homeomorphic to $\boldsymbol{Q} \times \boldsymbol{R}$.

THEOREM. *Let $\pi\colon X \times \boldsymbol{R} \to X$ be any flow. Let $x \in X$. Suppose that the motion π_x is aperiodic and Poisson-stable. Then the orbit $\Gamma(x)$ of x is locally homeomorphic to $\boldsymbol{Q} \times \boldsymbol{R}$.*

Proof. Without loss of generality we may assume that $X = \Gamma(x)$. Let $y \in \Gamma(x)$.

By [10], Chapter IV, Theorem 2.17, and Chapter VI, Theorem 2.12, there exist a closed subset S of X and a real number $\eta > 0$ such that

$y \in S$,
$g\colon S \times [-\eta, \eta] \to \pi(S \times [-\eta, \eta])$ is a homeomorphism, and
$\pi(S \times [-\eta, \eta])$ is a neighborhood of y.

Such a set S is called an *η-section*. In passing we note that the existence of sections has first been established by Whitney [26].

We shall show y to have an open neighborhood which is homeomorphic to $\boldsymbol{Q} \times \boldsymbol{R}$. Let W be an open neighborhood of y which is contained in $\pi(S \times [-\eta, \eta])$. Let N be an open neighborhood of y in S such that $\pi(N \times (-\varepsilon, \varepsilon)) \subset W$ for some ε such that $0 < \varepsilon < \eta$. Then $\pi\colon N \times (-\varepsilon, \varepsilon) \to \pi(N \times (-\varepsilon, \varepsilon))$ is a homeomorphism. Because $N \times (-\varepsilon, \varepsilon)$ is open in $S \times [-\eta, \eta]$, the set $\pi(N \times (-\varepsilon, \varepsilon))$ is open in W. It remains to be shown N homeomorphic to $\boldsymbol{Q}$. First it is to be observed that for each $z \in N$ there is exactly one $t(z) \in \boldsymbol{R}$ such that $z = \pi(x, t(z))$, because π_x is aperiodic. We have

$$\pi(\{x\} \times (t(z) - \varepsilon, t(z) + \varepsilon)) = \pi(\{z\} \times (-\varepsilon, \varepsilon)).$$

Because π restricted to $N \times (-\varepsilon, \varepsilon)$ is a topological embedding, the collection of intervals $\{(t(z) - \varepsilon, t(z) + \varepsilon) |\ z \in N\}$ is pairwise disjoint, whence countable. It follows that N is countable. Without loss of generality we may assume that x is positively Poisson-stable. Let $z \in N$. Then z is positively Poisson-stable as well and there exists a sequence (t_k) in $\boldsymbol{R}$ such that $t_k \to \infty$ and $\pi(z, t_k) \to z$. It follows that $\pi(z, t_k) \in \pi(N \times (-\varepsilon, \varepsilon))$ for k sufficiently large. Because $N \times (-\varepsilon, \varepsilon)$ and $\pi(N \times (-\varepsilon, \varepsilon))$ are homeomorphic and z is not a periodic point, z is not isolated in N. Thus N is countable and dense in itself. By a theorem of Sierpiński ([6] 1.3H, [24]) N is homeomorphic to $\boldsymbol{Q}$.

2.2. NOTATION. Throughout the paper the following notation will be used.

$$F = \{(x, y) \in \boldsymbol{R}^2 |\ x \in \boldsymbol{Q},\ -1 \leqslant y \leqslant 1\}.$$
$$E = \{(x, y) \in \boldsymbol{R}^2 |\ x \in \boldsymbol{Q},\ -1 < y < 1\}.$$

The set F is called the *standard matchbox*. For each $x \in \boldsymbol{Q}$, then set $\{x\} \times [-1, 1]$ is called a *match in F*.

The natural projections of F onto $\boldsymbol{Q}$ and $[-1, 1]$ are denoted by pr_1 and pr_2 respectively. It is to be observed that both pr_1 and pr_2 are open mappings. Because $[-1, 1]$ is compact, the mapping pr_1 is closed as well.

Let X be a space. Suppose that $h\colon F \to X$ is a topological embedding such that $h(F)$ is closed and $h(E)$ is open in X. Then the set $V = h(F)$ is called a *matchbox in X*. In this situation the induced map $h\colon F \to V$ is called a *parametrization of V*. The sets $h(\{x\} \times [-1, 1])$, $x \in \boldsymbol{Q}$, are called *matches of V*. In the terminology of manifold theory, the map h^{-1} is a chart of V. As also the embedding of E is involved, a formulation using charts is somewhat cumbersome.

2.3. We now discuss the existence of matchboxes in spaces.

PROPOSITION AND DEFINITION. Let X be a space which is locally homeomorphic to $Q\times R$. Suppose that $x\in X$ and W is a neighborhood of x. Then there is a matchbox V in X and a parametrization $h\colon F\to V$ such that $h(0,0)=x$, V is a neighborhood of x and $V\subset W$. The set V is called a *matchbox neighborhood of* x.

Proof. Let $h\colon Q\times R\to W_0$ be a homeomorphism onto an open neighborhood W_0 of x. As $Q\times R$ is homogeneous we may assume $h(0,0)=x$. Choose an open neighborhood U_0 of x such that

$$x\in U_0\subset \mathrm{cl}_X U_0\subset W_0\cap W.$$

Because h is a homeomorphism, for some irrational number α and for some $\varepsilon>0$ the set $E'=\{(x,y)|\ x\in Q, x\in(-\alpha,\alpha), y\in(-\varepsilon,\varepsilon)\}$ is mapped by h into U_0. As E' is open in $Q\times R$, the set $h(E')$ is open in W_0, whence open in X. Let

$$F'=\{(x,y)|\ x\in Q, x\in(-\alpha,\alpha),\ y\in[-\varepsilon,\varepsilon]\}$$

Then $F'=\mathrm{cl}_{Q\times R}E'$ and consequently $h(F')\subset\mathrm{cl}_X h(E')\subset\mathrm{cl}_X U_0$. Also $h(F')$ is a closed subset of W_0. It follows that $h(F')$ is a closed subset of $\mathrm{cl}_X U_0$. As $\mathrm{cl}_X U_0$ is closed in X, $h(F')$ is closed in X as well. Now the proposition easily follows.

The corollary below is now obvious.

DEFINITION. A space X is called *atriodic* if X does not contain three arcs each having a point p as a common endpoint and not intersecting otherwise.

COROLLARY. *If a space X is locally homeomorphic to $Q\times R$, then it is atriodic.*

2.4. The following is a key lemma. Roughly speaking one may say that the intersection of a matchbox and an arc consists of finitely many arcs only.

LEMMA. *Let X be a space which is locally homeomorphic to $Q\times R$. Let $h\colon F\to V$ be a parametrization of a matchbox V in X. Let the topological embedding $g\colon[0,1]\to X$ be a parametrization of the arc J.*

Then there is a partition $0\leqslant a_1\leqslant b_1<a_2<b_2<\ldots<a_n\leqslant b_n\leqslant 1$ of $[0,1]$ such that

(i) $V\cap J=\{g(t)|\ t\in[a_i,b_i],\ i=1,\ldots,n\}$;

(ii) *$g([a_i,b_i])$ is a match of V, $2\leqslant i\leqslant n-1$;*

(iii) *if $a_1>0$, then $g([a_1,b_1])$ is a match of V; if $a_1=0$, then $g([a_1,b_1])$ is a possibly degenerated arc, which is contained in a match of V, such that $g(b_1)\in V\setminus h(E)$;*

(iv) *if $b_n<1$, then $g([a_n,b_n])$ is a match of V; if $b_n=1$, then $g([a_n,b_n])$ is a possibly degenerated arc, which is contained in a match of V, such that $g(a_n)\in V\setminus h(E)$.*

Proof. Suppose that $g(t)=h(x,s)$ for some $t\in(0,1)$, $x\in Q$ and $s\in(-1,1)$. Then $g(t-\varepsilon,t+\varepsilon)\subset h(E)$ for some $\varepsilon>0$, as $h(E)$ is an open neighborhood of (x,s). Because $g(t-\varepsilon,t+\varepsilon)$ is connected and $h(E)$ is homeomorphic to $Q\times R$, we have $g(t-\varepsilon,t+\varepsilon)\subset h(\{x\}\times(-1,1))$. Now let

$$C=\{x\in Q|\ J\cap h(\{x\}\times(-1,1))\neq\emptyset\}.$$

It follows that for each $x\in C$ the set $g^{-1}(h\{x\}\times(-1,1)))$ is an open interval in $[0,1]$. This interval is denoted by (a_x,b_x). Observe that $g([a_x,b_x])=h(\{x\}\times[-1,1])$ with the possible exception of the cases $a_x=0$ and $b_x=1$. Also for all $x,y\in C$, if $x\neq y$, then $[a_x,b_x]\cap[a_y,b_y]=\emptyset$. Now assume that C is infinite. Because of the compactness of $[0,1]$ we may assume that some sequence $\frac{1}{2}(a_{x_n}+b_{x_n})$ converges to some $z\in[0,1]$. As the length of (a_{x_n},b_{x_n}) must tend to zero, the sequences (a_{x_n}) and (b_{x_n}) converge to z also. It follows that $g(z)$ belongs to the disjoint closed sets $h(Q\times\{-1\})$ and $h(Q\times\{1\})$, a contradiction. Finally it should be observed that it cannot occur that for some $t\in(0,1)$, $\varepsilon>0$ and $x\in Q$ the intersection

$$g((t-\varepsilon,t+\varepsilon))\cap h(\{x\}\times[-1,1])$$

is equal to $\{h(x,-1)\}$ or $\{h(x,1)\}$. Indeed X is atriodic by Corollary 2.3.

2.5. Now we investigate the following situation. A matchbox V is contained in another matchbox, which without loss of generality may be assumed to be the standard one.

DEFINITION. Let F be the standard matchbox. Let K be a closed and open subset of Q. Suppose that $t,b\colon K\to[-1,1]$ are continuous functions such that $b<t$. The set $W=\{(x,y)|\ x\in K, b(x)\leqslant y\leqslant t(x)\}$ is called a *simple matchbox in F with base K*.

As Q is homeomorphic to K, W is a matchbox in F.

LEMMA. *Suppose that V is a matchbox in F and $h\colon F\to V$ is a parametrization of V. Suppose $x\in Q$. Let $y=h(x,0)$ and $z=\mathrm{pr}_1(y)$. Suppose that*

$$h(\{x\}\times[-1,1])=V\cap\mathrm{pr}_1^{-1}(z).$$

Then there are a clopen neighborhood K of z in Q and a simple matchbox W with base K such that $W=V\cap\mathrm{pr}_1^{-1}(K)$.

Proof. The subset $h(\{x\}\times[-1,1])$ of F is denoted by $\{z\}\times[s,t]$. Without loss of generality we may assume that $(z,s)=h(x,-1)$ and $(z,t)=h(x,1)$. Choose ε such that $0<\varepsilon<\dfrac{t-s}{2}$. It is to be observed that $h|\ Q\times\{1\}$ and $h|\ Q\times\{-1\}$ are continuous. There exists a clopen neighborhood B of x in Q such that for each $w\in B$ we have $s-\varepsilon<\mathrm{pr}_2h(w,-1)<s+\varepsilon$ and $t-\varepsilon<\mathrm{pr}_2h(w,1)<t+\varepsilon$. Now ε has been chosen in such a way that, for each $w\in B$, $\mathrm{pr}_2(h(w,\theta_w))=\dfrac{s+t}{2}$ for some θ_w. It follows that $\mathrm{pr}_1(h(\{w_1\}\times[-1,1]))\cap\mathrm{pr}_1(h(\{w_2\}\times[-1,1]))=\emptyset$ for distinct $w_1,w_2\in B$. Let us write $C=B\times[-1,1]$ and $D=F\setminus C$. Both $h(C)$ and $h(D)$ are matchboxes in F and, $h(C\cap E)$ as well as $h(D\cap E)$ are open in F. It follows that both $\mathrm{pr}_1(h(C))$ and $\mathrm{pr}_1(h(D))$ are clopen in Q. Define $K=\mathrm{pr}_1(h(C))\setminus\mathrm{pr}_1(h(D))$. Clearly $z\in K$ and $V\cap\mathrm{pr}_1^{-1}(K)=h(C)\cap\mathrm{pr}_1^{-1}(K)$. Now $h(B\times\{1\})$ is closed and so is $h(B\times\{1\})\cap\mathrm{pr}_1^{-1}(K)$. From the observation above about the choice of ε it follows that for each $v\in K$ the set $\mathrm{pr}_1^{-1}(v)\cap h(B\times\{1\})$ consists of exactly one point.

The set $h(B\times\{1\})\cap \mathrm{pr}_1^{-1}(K)$ can be considered the graph of a continuous function $t\colon K\to[-1,1]$. See [27]. 7.1, Problem 108. In a similar way, by considering $h(B\times\{-1\})$, a continuous function $b\colon K\to[-1,1]$ is defined. Finally let $W=\{(x,y)|\ x\in K,\ b(x)\leqslant y\leqslant t(x)\}$.

3. *P*-manifolds.

3.1. Definition. A space X is called a *P-manifold* if it is arcwise connected and locally homeomorphic to $\boldsymbol{Q}\times\boldsymbol{R}$.

The orbits of aperiodic and Poisson-stable motions clearly are P-manifolds. Actually the letter P in "P-manifolds" refers to Poisson. Now we shall show that a P-manifold is a one-to-one continuous image of the real line. This provides a parametrization of the P-manifold.

Theorem. *Suppose that X is a P-manifold. Then there exists a bijective and continuous mapping $f\colon \boldsymbol{R}\to X$.*

Proof. First it is to be observed that X is uniquely arcwise connected. That is, for any two points p and q of X there is a unique arc, denoted by $\widehat{pq}$, which starts at p and ends up in q. This follows from the fact that X cannot contain a topological copy of S^1, since X is atriodic and arcwise connected. Now we are going to define an order on X (cf. [20]). We pick a point $p\in X$. Let V be a matchbox neighborhood of p (Proposition 2.3). Let I be the match of V such that $p\in I$. Write $I\setminus\{p\}=I^+\cup I^-$ such that I^+ and I^- are connected. For each point $x\in X\setminus\{p\}$ there is a unique arc $\widehat{xp}$ and either $\widehat{xp}\cap I^+\neq\emptyset$ or $\widehat{xp}\cap I^-\neq\emptyset$. For each $x\in X$ we shall write

$$x\in R_p \text{ if } x=p \text{ or } \widehat{xp}\cap I^+\neq\emptyset, \quad \text{and}$$
$$x\in L_p \text{ if } x=p \text{ or } \widehat{xp}\cap I^-\neq\emptyset.$$

Both R_p and L_p are arcwise connected and $R_p\cap L_p=\{p\}$. For all x and y with $x\neq y$ we define x to be less than y, $x<y$, in the following cases:

(i) $x\in L_p$ and $y\in R_p$;
(ii) $x,y\in L_p$ and $y\in\widehat{px}$;
(iii) $x,y\in R_p$ and $x\in\widehat{py}$.

It is easily checked that $<$ is a linear order on X. We shall show that $(X,<)$ has the order type λ of the real numbers with the usual order (e.g. [14], [18]).

First a countable family of parametrizations $h_n\colon F\to V_n$, $n=0,1,2,\ldots$ is selected in such a way that $\{h_n(E)|\ n=0,1,\ldots\}$ is an open base of X. As each $h_n(E)$ is homeomorphic to $\boldsymbol{Q}\times\boldsymbol{R}$, there is a countable subset D of X such that for each n and each match J of V_n the set $J\cap D$ is dense in J. It is to be observed that every match J of every V_n endowed with the order induced by $<$ is order-isomorphic to a closed interval. From Proposition 2.3 and Lemma 2.4 it follows that the set D is dense in the order $<$. From these observations it is clear that $(X,<)$ has the order type λ of $\boldsymbol{R}$. Now let $f\colon\boldsymbol{R}\to X$ be any order isomorphism. Every match J of every V_n, being order-isomorphic to a closed interval, is by f^{-1} topologically mapped onto a closed interval. It follows that for each $n=0,1,2,\ldots$, the set $f^{-1}(h_n(E))$ is a countable union of open intervals and that f is continuous.

3.2. In the preceding subsection it was shown that for any P-manifold X there exists bijective and continuous $f\colon\boldsymbol{R}\to X$. Spaces of this type have been the object of several studies e.g. [2], [19], [20] and [21]. We now discuss a property of P-manifolds which is not shared by all real curves.

Definition. Let $f\colon\boldsymbol{R}\to X$ be a mapping. We shall say that *f has the arc lifting property* if for each arc $j\colon[0,1]\to X$ there is a unique arc $\tilde{j}\colon[0,1]\to\boldsymbol{R}$ such that $f\circ\tilde{j}=j$.

Remark. We indiscriminately use the term "arc" for the embedding of the unit interval as well as for the image of the embedding, as has been done in the preceding subsection.

Theorem. *Suppose that the space X is atriodic. Then each bijective and continuous mapping $f\colon\boldsymbol{R}\to X$ has the arc lifting property.*

Proof. The proof is very similar to the proof of Theorem 1.25 in [1] stating a somewhat stronger result for aperiodic motions. Let $j\colon[0,1]\to X$ be an arc. Write $C=j([0,1])$ and $I=\{t\in\boldsymbol{R}|\ f(t)\in C\}$.

To prove the theorem it is sufficient to show that I is compact, because then clearly the restriction of f to I is a topological embedding.

First we shall show that there exists a strictly increasing sequence $t_n\to\infty$ such that $t_n\notin I$. It this is false, then $[t,\infty)\subset I$ for some $t\in\boldsymbol{R}$. As $[t,\infty)=\bigcup\{[t,n]|\ n=1,2,\ldots\}$ and the restriction of f to $[t,n]$ is a topological embedding for each n, $f([t,\infty))$ is a half-open interval in C. Let p be the unique limit point of $f([t,\infty))$ in C which is not contained in $f([t,\infty))$. Let $p=f(q)$. Then $q\notin[t,\infty)$ and $[q-\varepsilon,q+\varepsilon]\cap[t,\infty)=\emptyset$ for some $\varepsilon>0$. As the restriction of f to $[q-\varepsilon,q+\varepsilon]$ is a topological embedding, it follows that the three arcs $f([q-\varepsilon,q])$, $f([q,q+\varepsilon])$ and $\mathrm{cl}\,f([t,\infty))$ have only the point p is common. This contradicts the fact that X is atriodic. Similarly there is a strictly decreasing sequence $s_n\to-\infty$ such that $s_n\notin I$. Write $A_0=\{f(u)|\ s_0\leqslant u\leqslant t_0\}\cap C$ and for $n\geqslant 1$,

$$A_n=\{f(u)|\ s_{n+1}\leqslant u\leqslant s_n \text{ or } t_n\leqslant u\leqslant t_{n+1}\}\cap C.$$

Then $\{A_n|\ n=0,1,\ldots\}$ is a pairwise disjoint family of closed subsets of C. It follows by Sierpiński's theorem ([1], [17]) that C equals A_n for some n. It has been proved now that I is bounded. As I is closed as well, I is compact.

Examples. The dumbbell and the figure eight are curves ([2]) which lack the arc lifting property.

3.3. Let X be a P-manifold. The bijective and continuous mappings $\boldsymbol{R}\to X$ in a natural way fall into two classes, the directions of X. This enables us to define orientation. It is shown that orbits are orientable. The following lemma is relevant to the definitions below.

LEMMA. *Suppose that X is a P-manifold. For $i = 1, 2$ let f_i: $\boldsymbol{R} \to X$ be bijective and continuous mappings. Then $f_2^{-1} \circ f_1$ is a homeomorphism.*

Proof. The map $f_2^{-1} \circ f_1$ is a bijection which, restricted to any closed interval, is a topological embedding, since f_2 has the arc lifting property.

DEFINITION. Suppose that X is a P-manifold. Let f_i: $\boldsymbol{R} \to X$ be bijective and continuous mappings, $i = 1, 2$. We say that f_1 is *equivalent to* f_2 if $f_2^{-1} \circ f_1$ is increasing. An equivalence class of continuous and bijective mappings $\boldsymbol{R} \to X$ is called a *direction for* X.

PROPOSITION. *Suppose that X is a P-manifold. Then there are precisely two directions for X.*

Proof. The easy proof is left to the reader.

DEFINITION. Suppose that X is a P-manifold with direction $\{f\}$, the equivalence class of f: $\boldsymbol{R} \to X$. Let V be a matchbox in X and h: $F \to V$ a parametrization. We shall say that V is *coherently directed by* h if for each $x \in \boldsymbol{Q}$ the map $f^{-1} \circ h_x$: $[-1, 1] \to \boldsymbol{R}$, defined by $f^{-1} \circ h_x(t) = f^{-1}(h(x, t))$, is increasing. The P-manifold X is called *orientable* if there is a matchbox V in X and a parametrization h: $F \to V$ such that V is coherently directed by h.

Orbits of aperiodic and Poisson-stable motions are orientable P-manifolds. That is the contents of the following theorem.

THEOREM. *Let π: $X \times \boldsymbol{R} \to X$ be a flow. Let $x \in X$. Suppose that the motion π_x is aperiodic and Poisson-stable. Then the orbit $\Gamma(x)$ of x is an orientable P-manifold with direction $\{\pi_x\}$.*

Proof. The notation of the proof of Theorem 2.1 is used. As in that proof $y \in \Gamma(x) = X$ and π: $N \times (-\varepsilon, \varepsilon) \to \pi(N \times (-\varepsilon, \varepsilon))$ is a homeomorphism. Here N is homeomorphic to $\boldsymbol{Q}$, thus showing y to have an open neighborhood which is homeomorphic to $\boldsymbol{Q} \times \boldsymbol{R}$.

Now we shall show that the matchbox $\pi(N \times [-\varepsilon, \varepsilon])$ is coherently directed by π_x. For each $z \in N$ and $s \in [-\varepsilon, \varepsilon]$, $\pi(z, s) = \pi(\pi(x, t(z)), s) = \pi(x, t(z)+s)$. It follows that $\pi_x^{-1} \circ \pi_z(s) = t(z)+s$ and $\pi_x^{-1} \circ \pi_z$: $[-\varepsilon, \varepsilon] \to \boldsymbol{R}$ is increasing.

Remark. From the proof above it follows that each point of an orbit has a matchbox neighborhood which can be coherently directed. From the results in Section 4 it will be clear that the same holds in any orientable P-manifold.

EXAMPLE. A well-known example by Knaster of an indecomposable continuum in the plane provides an example of a nonorientable P-manifold X ([17]). Let E denote the set of endpoints of the Cantor set C, i.e., E is the set of points in the unit interval $[0, 1]$ the tryadic expansion of which has no 1's and eventually either 0's or 2's. The space X^* consists of all semicircles in the upper half of the plane with centre $(\frac{1}{2}, 0)$ through the points of E and of all semicirles in lower half of the plane with centre $\left(\frac{5}{2 \cdot 3^n}, 0\right)$ through the points x of E such that $\frac{2}{3^n} \leqslant x \leqslant \frac{1}{3^{n-1}}$, $n \geqslant 1$. Now let $X = X^* \setminus \{(0, 0)\}$.

It is easily seen that X is a P-manifold. That X is not orientable can be proved directly by inspection of any bijective and continuous mapping $\boldsymbol{R} \to X$. The easiest way to reach the conclusion that X is not orientable is as follows. If X is orientable, then, by the results of Section 4, X is the orbit of an aperiodic and Poisson-stable motion. It can be shown however that planar Poisson-stable motions must be periodic (cf. [1], [23]).

3.4. Now we are going to examine the pasting together of matchboxes.

DEFINITION. Let X be a P-manifold. Suppose that J is an arc in X with parametrization g: $[0, 1] \to X$. Suppose that V is a matchbox in X and h: $F \to V$ is a parametrization of V. We shall say that V is a *matchbox along* J if for some $x \in \boldsymbol{Q}$

(i) $J \cap V = h(\{x\} \times [-1, 1])$, and

(ii) the map $t \to g^{-1}(h(x, t))$ is increasing.

PROPOSITION. *Let X be a P-manifold. Suppose that J is an arc in X and that $x \in J$ and x is not an endpoint of J. Let W be a neighborhood of x. Then there is a matchbox neighborhood V of x such that V is a box along J and $V \subset W$.*

Proof. Let g: $[0, 1] \to X$ be a parametrization of J. Let p and q be the endpoints of J. Let h: $F \to V'$ be a parametrization of V' such that $x = h(0, 0)$ and $V' \subset (X \setminus \{p, q\}) \cap W$. Let $0 \leqslant a_1 \leqslant b_1 < \ldots < a_n \leqslant b_n \leqslant 1$ be the partition of $[0, 1]$ as is mentioned in Lemma 2.4. As the cases $a_1 = 0$ and $b_n = 1$ do not occur, for some $i \in \{0, \ldots, n\}$ and for a unique $z \in \boldsymbol{Q}$ we have $x \in g([a_i, b_i]) = h(\{z\} \times [-1, 1])$. Because the intersection of J and V' consists of finitely many matches only, there is a clopen neighborhood K of z in $\boldsymbol{Q}$ such that $h(K \times [-1, 1]) \cap J = h(\{z\} \times [-1, 1])$. We write $V = h(K \times [-1, 1])$. Let k be any homeomorphism of F onto $K \times [-1, 1]$. As a parametrization for V we take either $h \circ k$ or the composition of the reflection $(x, t) \to (x, -t)$ of F and $h \circ k$. This depends on condition (ii).

THE PASTING THEOREM. *Let X be a P-manifold and L an arc in X. For $i = 1, 2$, let V_i be a matchbox along L and let h_i be a parametrization of V_i. Let q_i be the point in $\boldsymbol{Q}$ such that $L \cap V_i = h_i(\{q_i\} \times [-1, 1])$, $i = 1, 2$. Suppose that for some $s_1, s_2 \in (-1, 1)$, $L \cap V_1 \cap V_2 = h_1(\{q_1\} \times [s_1, 1]) = h_2(\{q_2\} \times [-1, s_2])$. Then there are clopen neighborhoods A_i of q_i in $\boldsymbol{Q}$, $i = 1, 2$, and a matchbox V along L and a parametrization h: $F \to V$ such that*

$$V = h_1(A_1 \times [-1, 1]) \cup h_2(A_2 \times [-1, 1]), \text{ and}$$

$$h_1^{-1} h(\boldsymbol{Q} \times \{-1\}) = A_1 \times \{-1\}, h_2^{-1} h(\boldsymbol{Q} \times \{1\}) = A_2 \times \{1\}.$$

Moreover, if V_1 (or V_2) is coherently directed by h_1 (or h_2), then V is coherently directed by h.

Proof. For $i = 1, 2$ write $h_i(E) = U_i$. It is to be noticed that $h_1(q_1, -1) \notin V_2$ and $h_2(q_2, 1) \notin V_1$. Now b_1 and t_1 are chosen such that $\{q_1\} \times [b_1, t_1] \subset h_1^{-1}(U_1 \cap U_2)$. Using the compactness of $[b_1, t_1]$ and the continuity of h_1 a clopen neighborhood A_1'

of q_1 is selected such that, for all $y \in A'_1$, $h_1(y, -1) \notin V_2$ and

$$\{y\} \times [b_1, t_1] \subset h_1^{-1}(U_1 \cap U_2).$$

The simple matchbox $A'_1 \times [b_1, t_1]$ is denoted by B. The set $h_2^{-1}h_1(B)$ is a matchbox in F. By Lemma 2.5 there is a clopen neighborhood A'_2 of q_2 and a simple matchbox $C = \{(x, y)|\ x \in A'_2,\ b_2(x) \leqslant y \leqslant t_2(x)\}$ such that $C = h_2^{-1}h_1(B) \cap \mathrm{pr}_1^{-1}(A'_2)$. Now it is to be observed that $h_2(q_2, -1) = h_1(q_1, s_1) \in U_1$ and

$$\mathrm{pr}_2(h_1^{-1}(h_2(q_2, -1))) = \mathrm{pr}_2(q_1, s_1) = s_1 < b_1.$$

Hence a clopen neighborhood A_2 of q_2 can be selected such that $A_2 \subset A'_2$ and, for all $y \in A_2$, $h_2(y, 1) \notin V_1$, $h_2(y, -1) \in U_1$ and $\mathrm{pr}_2(h_1^{-1}(h_2(y, -1))) < b_1$.

Define $D_2 = C \cap \mathrm{pr}_1^{-1}(A_2)$ and $A_1 = \mathrm{pr}_1 h_1^{-1}h_2(D_2)$. Let $D_1 = B \cap \mathrm{pr}_1^{-1}(A_1)$. Then D_1 and D_2 are simple matchboxes and $h_2^{-1}h_1\colon D_1 \to D_2$ is a homeomorphism. For any $y \in A_2$, the set $\{y\} \times [b_2(y), t_2(y)]$ is a match of D_2, whence also of C and $h_2^{-1}h_1(B)$. It follows that for some $x \in A_1$, $\{y\} \times [b_2(y), t_2(y)] = h_2^{-1}h_1(\{x\} \times [b_1, t_1])$. And, because $\mathrm{pr}_2 h_1^{-1}h_2(y, -1) < b_1$, we have $h_2^{-1}h_1(x, b_1) = (y, b_2(y))$ and $h_2^{-1}h_1(x, t_1) = (y, t_2(y))$. Because A_i is homeomorphic to the graph $\{(s, t_i(s)|\ s \in A_i\}$, $i = 1, 2$, it follows that the map $g_{12}\colon A_1 \to A_2$, defined by $g_{12}(x) = \mathrm{pr}_1 h_2^{-1}h_1(x, t_1(x))$ is a homeomorphism. Moreover, for all $x \in A_1$,

$$(*) \qquad h_2^{-1}h_1(x, t_1(x)) = (g_{12}(x), t_2(g_{12}(x))).$$

The box V is now obtained by pasting D_1 onto D_2 via $h_2^{-1} \circ h_1$. The precise definition of the parametrization h of V is as follows. Let $g_1\colon Q \to A_1$ be any homeomorphism and let $g_2 = g_{12} \circ g_1$. For $x \in Q$ define

$$\begin{aligned} h(x, t) &= h_1(g_1(x), \tfrac{3}{2}[(t+\tfrac{1}{3})+b_1(t+1)]) && \text{for } -1 \leqslant t \leqslant -\tfrac{1}{3};\\ &= h_1(g_1(x), \tfrac{3}{2}[-b_1(t-\tfrac{1}{3})+t_1(t+\tfrac{1}{3})]) && \text{for } -\tfrac{1}{3} \leqslant t \leqslant \tfrac{1}{3};\\ &= h_2(g_2(x), \tfrac{3}{2}[-t_2(g_2(x))(t-1)+(t-\tfrac{1}{3})]) && \text{for } \tfrac{1}{3} \leqslant t \leqslant 1. \end{aligned}$$

The map h is a topological embedding on each of the closed sets $Q \times [-1, -\frac{1}{3}]$, $Q \times [-\frac{1}{3}, \frac{1}{3}]$ and $Q \times [\frac{1}{3}, 1]$. The definitions of h agree on the sets $Q \times \{-\frac{1}{3}\}$ and $Q \times \{\frac{1}{3}\}$, as can be verified by application of (*). As

$$h(E) = h_1(A_1 \times (-1, 1)) \cup h_2(A_2 \times (-1, 1)),$$

the union of two open sets, it is clear now, that h and $V = h(F)$ satisfy the required conditions. The statement about the coherent directedness follows from the following observation. Let $f\colon \boldsymbol{R} \to X$ be bijective and continuous. Then, for any $x \in Q$, the map $t \to f^{-1}(h(x, t))$ of $[-1, 1]$ into $\boldsymbol{R}$ is increasing if and only if the map

$$t \to f^{-1}(h_1(g_1(x), \tfrac{3}{2}[(t+\tfrac{1}{3})+b_1(t+1)]))$$

of $[-1, -\frac{1}{3}]$ into $\boldsymbol{R}$ is increasing.

COROLLARY. *Let X be a P-manifold and L an arc in X. For $i = 1, \ldots, n$ ($n \geqslant 3$), let V_i be a matchbox along L and $h_i\colon F \to V_i$ a parametrization. Let q_i be the point in Q such that $L \cap V_i = h_i(\{q_i\} \times [-1, 1])$ $i = 1, \ldots, n$.*

Suppose that $L \cap V_{i-1} \cap V_{i+1} = \emptyset$ for $i = 2, \ldots, n-1$.

Suppose that there exist s_i, t_i with $-1 < s_i < t_i < 1$, $i = 1, \ldots, n$, such that

$$L \cap V_i \cap V_{i+1} = h_i(\{q_i\} \times [t_i, 1]) = h_{i+1}(\{q_{i+1}\} \times [-1, s_{i+1}]) \quad \textit{for } i = 1, \ldots, n-1.$$

Then there are clopen neighborhoods A_i of q_i in Q, $i = 1, \ldots, n$, and a matchbox V along L with parametrization h such that

$$V = \bigcup \{h_i(A_i \times [-1, 1])|\ i = 1, \ldots, n\}, \quad \text{and}$$

$$h_1^{-1}h(Q \times \{-1\}) = A_1 \times \{-1\},\ h_n^{-1}h(Q \times \{1\}) = A_n \times \{1\}.$$

Moreover, if one of the V_i is coherently directed by h_i, then V is coherently directed by h.

4. Characterization theorem.

4.1. In this section we shall fill in the gap in the proof of Theorem 1.4 by presenting a proof of the following theorem.

THEOREM. *A space X is homeomorphic to the orbit of an aperiodic and Poisson-stable motion in some dynamical system if and only if X is an orientable P-manifold.*

The "only if" part of the proof can be found in Subsection 3.3. A proof of the "if" part is given in the following subsections. Here an outline of the proof is presented. The notation, which we are going to introduce now, will be used throughout the section.

Suppose that X is an orientable P-manifold. We shall define a dynamical system which contains one orbit only. This orbit is homeomorphic to X. Let $g\colon \boldsymbol{R} \to X$ be a bijective and continuous mapping by which the direction $\{g\}$ for X is determined. Let V be a coherently directed matchbox and $h\colon F \to V$ a parametrization. The *zero-section* of V is the set $Z = \{h(x, 0)|\ x \in Q\}$. It is a topological copy of Q. In 4.2 the so-called *Poincaré map* $p\colon Z \to Z$ will be defined. It is to be observed that in the dynamical system which will be constructed the set Z is locally a section (i.e., each point of Z has neighborhood the intersection of which with Z is a section) and the mapping p is the Poincaré map associated with Z. It is indicated that without loss of generality there are only two cases to be considered:

(i) the Poincaré map $p\colon Z \to Z$ is a homeomorphism,

(ii) there is a point $q \in Z$ such that $p\colon Z \to Z \setminus \{q\}$ is a homeomorphism.

In both cases the mapping p is *universally transitive*, i.e., there is a point $y \in Z$ such that Z is the set of all $p^n(y)$, which are defined, $n \in \boldsymbol{Z}$. Case (i) will be discussed in 4.3 and case (ii) in 4.4.

4.2. The notation of 4.1 is used. From Theorem 3.2 it follows that $g^{-1}(V)$ is a countable infinite collection of pairwise disjoint arcs. In view of Lemma 2.4 the collection is also discrete. Let Z be the zero-section of V. It follows that $g^{-1}(Z)$ is a countably infinite and discrete set. By this discussion the following definition is justified. For each $y \in Z$ we let $D(y) = \{t \in \boldsymbol{R} \mid t > g^{-1}(y) \text{ and } g(t) \in Z\} = g^{-1}(Z) \cap (g^{-1}(y), \infty)$. Now whenever $D(y) \neq \emptyset$, we define

$$p(y) = g(\min D(y)).$$

The mapping $y \to p(y)$ is called the *Poincaré map.*

LEMMA. *For all $y \in Z$ with the exception of at most one $p(y)$ is well defined. Furthermore $|Z \setminus p(Z)| \leqslant 1$.*

Proof. It was observed above that $g^{-1}(Z)$ is countably infinite and discrete. So $\inf(g^{-1}(Z)) = -\infty$ or $\sup(g^{-1}(Z)) = \infty$ and one of the following cases occurs:

(i) $\inf(g^{-1}(Z)) = -\infty$ and $\sup(g^{-1}(Z)) = \infty$,
(ii) $\min(g^{-1}(Z)) = q \in g^{-1}(Z)$ and $\sup(g^{-1}(Z)) = \infty$,
(iii) $\inf(g^{-1}(Z)) = -\infty$ and $\max(g^{-1}(Z)) = q \in g^{-1}(Z)$.

Clearly, in cases (i) and (ii) the mapping $p: Z \to Z$ is well defined. In case (iii) p is well defined on $Z \setminus \{q\}$. In case (i) $p: Z \to Z$ is bijective, in case (ii) $p: Z \to Z \setminus \{q\}$ is bijective and in case (iii) $p: Z \setminus \{q\} \to Z$ is bijective. It is not difficult to see that apart from the directional properties the cases (ii) and (iii) may be considered the same (cf. 3.3, proposition).

Remark. From the observations in the proof of the preceding l mma it is clear that only two cases are to be considered, namely

(i) the Poincaré map $p: Z \to Z$ is bijective, and
(ii) there exists $q \in Z$ such that $p: Z \to Z \setminus \{q\}$ is bijective.

In both cases the mapping p clearly is universally transitive and therefore fixed-point-free.

THEOREM. *The Poincaré map $p: Z \to Z$ is a topological embedding.*

Proof. We only discuss the case that p is bijective, as the other case is very similar. Let $x \in Z$. We shall show that p is continuous at x. Write $y = p(x)$ and observe that $y \neq x$. Let W_1 and W_2 be disjoint clopen neighborhoods of x and y respectively in Z.

The points $h^{-1}(x)$ and $h^{-1}(y)$ are denoted by $(x', 0)$ and $(y', 0)$ respectively. The points $g^{-1}(x)$ and $g^{-1}(y)$ are denoted by x'' and y'' respectively. Now in X the arc $J = g([x'', y''])$ is considered which begins in x and ends up in y. Because V is coherently directed, it can be seen that the arc J consists of the following three consecutive parts. The first part connects x with $h(x', 1)$. It is the arc $\{h(x', t) \mid 0 \leqslant t \leqslant 1\}$. The second arc begins in $h(x', 1)$ and ends up in $h(y', -1)$. It is denoted by J'. The third part connects $h(y', -1)$ and y. It is the arc $\{h(y', t) \mid -1 \leqslant t \leqslant 0\}$. Now we are going to define a sequence of matchboxes along the arc J. The first box is the image under h of the box $F_1 = h^{-1}(W_1) \times [0, 1]$ and the last box in the sequence is the image under h of the box $F_2 = h^{-1}(W_2) \times [-1, 0]$. Observe that these two boxes cover the first and third part of the arc J. For each point z of the arc J' a matchbox neighborhood V_z along J is selected such that $V_z \subset X \setminus Z$ (2.3). The collection $\{\operatorname{int} V_z \mid z \in J'\}$ is an open collection whose union contains J'. By compactness of J' there is a finite subcollection $\{\operatorname{int} V_{z_1}, \ldots, \operatorname{int} V_{z_n}\}$, the union of which contains J'. We may assume that the collection is minimal, i.e., no subcollection of it covers J'. Because of the minimality, after rearrangement of the z_i if necessary, the sequence $h(F_1), V_{z_1}, \ldots, V_{z_n}, h(F_2)$ satisfies the hypotheses of Corollary 3.4. In this way one gets a matchbox W along J with parametrization $k: F \to W$ and clopen sets W_1' and W_2' in Z such that

$$W \subset h(F_1) \cup V_{z_1} \cup \ldots \cup V_{z_n} \cup h(F_2), \quad x \in W_1' \subset W_1, \, y \in W_2' \subset W_2 \quad \text{and}$$
$$k(Q \times \{-1\}) = W_1' \quad \text{and} \quad k(Q \times \{1\}) = W_2'.$$

That is, the "bottom" of the matchbox $W = k(F)$ is the subset W_1' of W_1 and the "top" is the subset W_2' of W_2. As W is coherently directed as well, it is clear that the Poincaré map sends the "bottom" to the "top". It follows that the sets W_1' and W_2' are homeomorphic. The theorem follows.

From the proof of the theorem we also get the following corollary.

COROLLARY. *In each point x of Z there exist a closed and open neighborhood W and a parametrization $k: F \to k(F) \subset X$ such that $k(F)$ is a matchbox in X, $k(Q \times \{-1\}) = W$, $k(Q \times \{1\}) = p(W)$ and $k(F) \cap Z = W \cup p(W)$.*

4.3. We now discuss the case (i), namely $p: Z \to Z$ is bijective. Throughout the notation of (4.1) is used. Using the last corollary and the fact that Z is separable and metrizable, we get a countable cover $\{W_1, W_2, \ldots\}$ of Z and a sequence of parametrizations $\{k_1, k_2, \ldots\}$ such that for each $i = 1, 2, \ldots$

(i) W_i is a closed subset of X and an open subset of Z;
(ii) $k_i: F \to k_i(F)$ is a parametrization of the matchbox $k_i(F)$; and
(iii) $k_i(Q \times \{-1\}) = W_i$, $k_i(Q \times \{1\}) = p(W_i)$ and $k_i(F) \cap Z = W_i \cup p(W_i)$.

After having replaced W_i by $W_i \setminus (W_1 \cup \ldots \cup W_{i-1})$ we may assume that the collection $\{W_1, W_2, \ldots\}$ is disjoint. As each W_i is clopen in Z, the collection $\{W_1, W_2, \ldots\}$ is locally finite.

Now it is to be observed that the collection $\{k_i(F) \mid i = 1, 2, \ldots\}$ is a locally finite closed cover of the space X. This can be seen as follows. That each $k_i(F)$ is closed, is obvious, $i = 1, 2, \ldots$ Because $\{W_i\}$ is disjoint, the collection $\{k_i(F) \setminus Z\}$ is disjoint as well in view of (iii) and Lemma 2.4. As $k_i(F) \setminus Z = k_i(E)$, $i = 1, 2, \ldots$, it follows that $\{k_i(F) \setminus Z \mid i = 1, 2, \ldots\}$ is a disjoint open cover of $X \setminus Z$. Both $\{W_i\}$ and $\{p(W_i)\}$ are locally finite and closed collections in Z. As V is a neighborhood of Z in X which is homeomorphic to F, and so has a product structure, it can be concluded that $\{k_i(F)\}$ is a locally finite collection in X.

Now we let $Z \times [0, 1] = Y$. The collection $\{W_i \times [0, 1] \mid i = 1, 2, \ldots\}$ is a disjoint and clopen cover of Y. The map $\pi: Y \to X$ is defined as follows. Let $i = 1, 2, \ldots$ The map $k_i: F \to X$ is a topological embedding such that $k_i(Q \times \{-1\}) = W_i$ and

$k_i(Q\times\{1\}) = p(W_i)$. The embedding k_i is used to glue $W_i\times[0,1]$ into X in such a way that $\pi(x,0) = x$ and $\pi(x,1) = p(x)$, $x\in W_i$. In this way a topological embedding π of $W_i\times[0,1]$ into X is defined. As π is defined on each member of $\{W_i\times[0,1]\}$, it is defined on Y. Because $\{W_i\times[0,1]\}$ is locally finite and closed, the mapping π is continuous. The mapping π is closed, because $\{k_i(F)\}$ is a locally finite and closed collection in X. It follows that π is a quotient map and that X is homeomorphic to $Y/\sim$, where $y\sim y'$ iff $\pi(y) = \pi(y')$. As is easily seen this amounts to $(z,1)\sim(p(z),0)$ for all $z\in Z$. It follows that X is homeomorphic to the phase space of the suspension of (Z,p).

4.4. In this subsection we discuss the case (ii) of Remark 4.2, namely there exists $q\in Z$ such that $p\colon Z\to Z\setminus\{q\}$ is bijective. Let $g\colon R\to X$ be the map by which X is directed. We may assume that $g(0) = q$. Observe that 0 is the smallest real number t such that $g(t)\in Z$.

We shall show first that there is no sequence $t_k\to-\infty$ such that the sequence $g(t_k)$ converges to some point $x\in X$, thus showing that the negative limit set of the mapping g must be empty (cf. [2], [21]).

Suppose this is false. Then for some $x\in X$ and some sequence $t_k\to-\infty$ we have $g(t_k)\to x$. As g is bijective, $x = g(s)$ for a unique $s\in R$. We now consider two cases, namely $s\geqslant 0$ and $s<0$. In the case $s\geqslant 0$ let $k\colon F\to W$ be the parametrization of a matchbox W such that $g([0,s])\subset k(E)$ and each match of W hits Z as many times as $g([0,s])$ does.

Such a matchbox W is easily obtained by applying the techniques of (4.2). We consider the arc $g([-1,s+1]) = J$. First for each $z\in g([0,s])$ we take a matchbox neighborhood V_z along J. Care is taken that for $z\notin Z$ the intersection $V_z\cap Z = \emptyset$. Then a minimal subcover of $\{\operatorname{int} V_z|\ z\in g([0,s])\}$ is selected and finally the pasting theorem is applied.

We may assume that W is coherently directed.

Now let W_0 be a matchbox neighborhood of x such that $W_0\subset W$.

As $g(t_k)\to x$, $g(t_k)\in W_0$ for some $t_k<0$. By "traveling" along the matches of $g(t_k)$ we see that $g(w)\in Z$ for some $w<0$. This contradicts the fact that 0 is the smallest real number t such that $g(t)\in Z$.

The cases $s<0$ is treated in a similar fashion. Let $k\colon F\to W$ be the parametrization of a matchbox W such that $g([s,0])\subset k(E)$ and each match of W hits Z exactly once. Because $g((-\infty,s])$ is connected, it cannot occur that $g((-\infty,s])\subset W$. Because $g(t_k)\to x$ and $t_k\to-\infty$, it follows that there exist $u_1,u_2\in R$ such that $u_1<u_2<s$ and $g(u_2)\notin W$, whilst $g(u_1)\in W$.

We may assume that W is coherently directed.

Now by "travelling" along the fiber of $g(u_1)$ we get $g(w)\in Z$ for some w with $w<u_2$. But 0 is the smallest number t such that $g(t)\in Z$. A contradiction.

Having established that the negative limit set of g is empty, we see that the set $\{g(n)|\ n = 0,-1,-2,\ldots\}$ is discrete. Then we can find a discrete collection $\{V_n|\ n = 0,-1,-2,\ldots\}$ such that, for each n, V_n is a matchbox neighborhood of $g(n)$ which is coherently directed. Let $V^* = \bigcup\{V_n|\ n = 0,-1,-2,\ldots\}$. Then V^* is a matchbox and the Poincaré map of its zero-section is bijective. So the result of 4.3 applies.

The proof of the following is now obvious.

COROLLARY. *Suppose that X is an orientable P-manifold. Then there exists a coherently directed matchbox V in X such that the Poincaré map of the zero-section of V is a homeomorphism.*

5. Structure theorem.

5.1. Putting together the results of the preceding sections we obtain the following

THEOREM. *Let $\pi\colon X\times R\to X$ be a flow. Let $x\in X$. The motion π_x is aperiodic and Poisson-stable if and only if π_x is topologically equivalent to the suspension of a discrete system (Q,h), where h is a universally transitive homeomorphism.*

Proof. Let $h\colon Q\to Q$ be a homeomorphism. If h is universally transitive, then h is aperiodic and Poisson-stable. It follows that the suspension of (Q,h) is an aperiodic and Poisson-stable motion. Thus the "if" part is proved. To obtain a proof of the "only if" part we may assume that $X = \Gamma(x)$. The orbit $\Gamma(x)$ is a P-manifold which is directed by the motion π_x (3.3).

Using Corollary 4.4 we find a coherently directed matchbox V such that the Poincaré map $p\colon Z\to Z$ of the zero-section of V is a homeomorphism. By the proof presented in 4.3 we see that $\Gamma(x)$ is homeomorphic to the phase space of the suspension of (Z,p). From the general observations about equivalences (1.2), it follows that π_x is topological equivalent to the suspension of (Z,p).

5.2. In the light of Theorem 5.1 it is of interest to know when two discrete systems on Q have equivalent suspensions.

DEFINITION 1. Let $h\colon Q\to Q$ be a homeomorphism which is two-sided Poisson-stable. Let C be a closed and open subset of Q. For each $x\in C$ the number $n(h,C,x)$ is the least integer $m\geqslant 1$ such that $h^m(x)\in C$; for this least integer m, the point $h^m(x)$ is denoted by $r(h,C)(x)$. The map $r(h,C)$ is called the *first return homeomorphism.*

Observe that $r(h,C)$ is well-defined because h is positively Poisson-stable. The map $r(h,C)$ is onto C because h is negatively Poisson-stable.

PROPOSITION. *Let $h\colon Q\to Q$ be a homeomorphism which is two-sided Poisson-stable. Let C be a closed and open subset of Q. Then the map $r(h,C)$ is a homeomorphism $C\to C$ which is two-sided Poisson-stable.*

Proof. Write $C = \bigcup\{C_n|\ n = 1,2,\ldots\}$ where

$$C_n = \{x|\ h^i(x)\notin C,\ i = 1,2,\ldots,n-1, \text{ and } h^n(x)\in C\}.$$

Each C_n is closed and open, and $\{C_n\}$ is a partition of C. On C_n the map $r(h,C)$ equals h^n, $n = 1,2,\ldots$

DEFINITION 2. Leg $g,h\colon Q\to Q$ be homeomorphisms which are two-sided Poisson-stable. The homeomorphisms g and h are called *first return equivalent* if there are closed and open subsets C and D of Q such that $r(g, C)$ are $r(h, D)$ are conjugated.

Remarks. Obviously, equivalent homeomorphisms are first return equivalent. The converse of this statement is false. See Example 1 of the next subsection.

The notions of first return homeomorphism and first return equivalence have their counterparts in ergodic theory, namely induced system and Kakutani equivalence ([4], [13]).

THEOREM. *Suppose that g and h are homeomorphisms of Q onto itself which are two-sided Poisson-stable and universally transitive. Then the suspensions of (Q, g) and (Q, h) are topologically equivalent if and only if g and h are first return equivalent.*

Proof. Suppose that C is a clopen subset of Q. The suspension of (Q, g) is an aperiodic and Poisson-stable motion (5.1). Its phase space is an orientable P-manifold. Now C and $r(g, C)$ can be considered to be the zero-level of a coherently oriented matchbox and the Poincaré map. It follows that (Q, g) and $(C, r(g, C))$ have topologically equivalent suspensions. The "if" part easily follows. To prove the "only if" part suppose that the suspensions X and Y of (Q, g) and (Q, h) are topologically equivalent. Let $f\colon X \to Y$ be any homeomorphism, sending orbits to orbits and preserving orientation. As each of the systems consists of one orbit only, we may assume that $f([0, 0]) = [0, 0]$. We also may assume that the P-manifolds X and Y are directed by the motions. In Y we consider the matchbox

$$V = \{[y, t]|\ -\sqrt{2} \leqslant y \leqslant \sqrt{2},\ -\tfrac{1}{4} \leqslant t \leqslant \tfrac{1}{4}\}.$$

Then, because f is a homeomorphism, for some irrational number β and for some real number r the subset $W = \{[x, s]|\ -\beta \leqslant x \leqslant \beta,\ -r \leqslant s \leqslant r\}$ of X is topologically embedded in V. Both V and W may be regarded as standard matchboxes. Thus $f(W)$ is a matchbox in V with parametrization f. By Lemma 2.4 the intersection of $f(W)$ and $\{(0, t)|\ -\frac{1}{4} \leqslant t \leqslant \frac{1}{4}\} = J$ consists of finitely many arcs. By using the argument which has been employed in the proof of Proposition 3.4, it can be seen that β can be chosen in such a way that $f(W) \cap J$ consists of one arc only. Then, in view of Lemma 2.5, β can also be chosen in such a way that $f(W)$ is a simple matchbox with base K. Let $V' = \{[y, t]|\ y \in K,\ -\frac{1}{4} \leqslant t \leqslant \frac{1}{4}\}$ and let the mapping $[y, t] \to [y, 0]$ be denoted by pr. Now, the zero-section $\{[x, 0]|\ |x| \leqslant \beta\}$ of W is inspected. The Poincaré map of this section is precisely the first return map $r(g, (-\beta, \beta))$. Similarly, the Poincaré map of $\{[y, 0]|\ y \in K\}$ is the first return homeomorphism $r(h, K)$. Now $\mathrm{pr} \circ f$ induces a homeomorphism of the zero-section of W onto the zero-section of V'. This homeomorphism is a conjugation between the Poincaré maps of the sections.

5.3. Now an example is presented exhibiting two homeomorphisms of Q, which are first return equivalent, but yet not equivalent. The author would like to acknowledge his indebtedness to M. Keane for suggesting the example.

EXAMPLE 1 (cf. [15]). As usual, $\{a\}$ denotes the fractional part of a, $a \in \boldsymbol{R}$. We shall define two systems (Q, σ_A) and (Q, σ_B) which are first return equivalent, but yet not equivalent.

Let $\alpha, \beta \in [0, 1)$ be irrational algebraic numbers such that $\frac{1}{2} < \alpha$ and $\alpha = \dfrac{1}{2-\beta}$. Note that $\alpha \neq \beta$. Let $\gamma \in [0, 1)$ be an irrational transcendental number.

$A = \{\{\gamma + n\alpha\}|\ n \in \boldsymbol{Z}\}$ and σ_A is the shift $\sigma_A(\{\gamma+n\alpha\}) = \{\gamma+(n+1)\alpha\}$.

$B = \left\{\left\{\dfrac{\gamma}{\alpha} + n\beta\right\}\Big|\ n \in \boldsymbol{Z}\right\}$ and σ_B is the shift $\sigma_B(x) = \{x+\beta\}$, $x \in B$.

As a matter of fact A and B are equivalent to the systems A and B discussed in 1.3 Example 2. Thus A and B are homeomorphic to Q. It has been indicated that the systems A and B are not equivalent. Now write $U = (0, \alpha) \cap A$. Then U is closed and open in A and we shall show that $r(\sigma_A, U)$ is conjugated to σ_B thus showing that (A, σ_A) and (B, σ_B) are first return equivalent. We write r instead of $r(\sigma_A, U)$. We have

$$\begin{aligned} r(x) &= x+2\alpha-1 = x+\alpha\beta, && \text{for } 0 < x < 1-\alpha \\ &= x+\alpha-1 = x+\alpha(\beta-1), && \text{for } 1-\alpha < x < \alpha. \end{aligned}$$

The map $h\colon U \to (0, 1)$ is defined by $h(x) = \dfrac{x}{\alpha}$, $x \in U$. Now, easy computation shows $h \circ r = \sigma_B \circ h$.

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Residuality of the set of embeddings into Nagata's n-dimensional universal spaces

by

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Abstract. We prove that the set of homeomorphic embeddings of an n-dimensional metrizable space X of weight $\tau \geqslant \aleph_0$ into the universal n-dimensional Nagata's space $K_n(\tau) \subset S(\tau)^{\aleph_0}$, $S(\tau)$ being the standard τ-star-space, is residual in the function space of all continuous mappings of X into $S(\tau)^{\aleph_0}$. This answers in a strong form a question posed by K. Kuratowski (see [N2], p. 260). The proof is based on a classical Baire-category method.

1. Introduction. The aim of this paper is to extend some classical embedding results for n-dimensional separable metrizable spaces to nonseparable spaces. More specifically, we show that, given an n-dimensional metrizable space X of weight $\tau \geqslant \aleph_0$, the embeddings of X into Nagata's universal space $K_n(\tau)$ (a generalization of the classical Nöbeling's universal space; see [E], Theorem 1.11.5) form a residual set in the space of all mappings of X into the universal metrizable space $S(\tau)^{\aleph_0}$, where $S(\tau)$ is the star-space of weight τ.

This result answers (in a strong form) a question in [N2], which J. Nagata attributes to K. Kuratowski; an answer to the original question follows also from [P1], where some refinements of Nagata's embedding theorems for n-dimensional and countable-dimensional metrizable spaces ([N3], Theorems VI. 5 and [N1], Theorem 9) are given.

In this paper embedding theorems are obtained by the classical Baire-category method, while the embedding problems dealt with in the paper [P1] do not admit such an approach. In particular, the set of all embeddings of a countable-dimensional metric space of weight $\tau \geqslant \aleph_0$ into $K_\infty(\tau) = \bigcup_{n=1}^{\infty} K_n(\tau)$, which is dense in $C(X, S(\tau)^{\aleph_0})$ by [P], Corollary 2.2, may not be residual in $C(X, S(\tau)^{\aleph_0})$ (see Remark 3.7).

2. Notation and definitions. Our terminology follows [E] and [N3]. By dimension we understand the covering dimension dim. The term function and a symbol $f: X \to Y$ always denotes a continuous function. By I we denote the unit interval $[0, 1]$, by Q — the set of rationals in I, by N — the set of integers and by I^ω — the Hilbert cube. A family $\mathscr{A}$ of subsets of a metric space (X, ϱ) is δ-discrete, if $\varrho(A, B) \geqslant \delta$ for every distinct $A, B \in \mathscr{A}$, where

$$\varrho(A, B) = \min\{\varrho(a, b)\colon a \in A, b \in B\}.$$