On inhomogeneity of products of compact F-spaces

by

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Abstract. In this note we prove the following Theorem. *Let X_i be an infinite compact F-space, for any $i \in I \neq \emptyset$, and let Y_j be a compact first countable space, for any $i \in J$. Then the product $\prod_{i\in I} X_i \times \prod_{j \in J} Y_j$ is not homogeneous.*

A space X is said to be an *F-space* if every bounded continuous real-valued function defined on any subset of X of the form $f^{-1}((0,1])$ for some continuous $f\colon X \to [0,1]$ can be extended to a continuous function defined on X. This means that $f^{-1}((0,1])$ is C^*-embedded. For basic facts concerning F-spaces see [1]. The best known compact F-space is the space $\omega^* = \beta\omega \setminus \omega$, where ω denotes the set of natural numbers with the discrete topology and $\beta\omega$ denotes the Čech–Stone compactification of ω. K. Kunen [2] proved that there is a point j in ω^* such that $j \notin \operatorname{cl} A$ for any countable set A. In fact, he proved that there are 2^c such points. This result will play a central role in proof of our Theorem.

Theorem was proved a few years ago, by K. Kunen and independently by R. Frankiewicz and P. Zbierski. The proofs were essentially the same. Since we have not yet obtained a more general result (say stating that the product of compact infinite F-spaces by a compact space is not homogeneous) we have decided to publish this result.

1. Definitions and lemmas. Let p be an ultrafilter over ω; that is, $p \in \omega^*$, and let $\{x_n|\ n \in \omega\}$ be an arbitrary sequence in a topological space X. A point x is said to be a *p-limit of* $\{x_n|\ n\in\omega\}$ ($x = p\text{-}\lim\{x_n|\ n \in \omega\}$) iff:

$$\forall_u \ (\text{if } u \text{ is an open neighborhood of } x \text{ then } \{n|\ x_n \in u\} \in p).$$

Let $p, q \in \omega^*$; then $p \underset{\mathrm{RK}}{\leqslant} q$ iff there is a function $f\colon \omega \to \omega$ such that $p = \beta f(q)$ where βf is the Stone–Čech extension of f.

The following lemma is an immediate consequence of Kunen's Theorem mentioned in introduction.

1.1. LEMMA. *There are $p, q \in \omega^*$ such that $p \not\leq_{RK} q$ and p is not a q-limit of any countable sequence $\{x_n \mid n \in \omega\} \subseteq \omega^*$, s.t. $p \notin \{x_n \mid n \in \omega\}$.*

Proof. By Kunen's Theorem there is a subset $A \subseteq \omega^*$ of size 2^c such that $\forall p \in A$ $(p \notin \mathrm{cl}\{x_n \mid n \in \omega\}$ iff $\forall_n p \neq x_n)$. Now let q be any point in A. Then, if $B = \{t \in \omega^* \mid t \leq_{RK} q\}$ then $|B| = 2^\omega$. Let $p \in A \setminus B$.

1.2. LEMMA. *Let $p, q \in \omega^*$ and let X be a compact F-space, let $\{x_n \mid n \in \omega\}$ be a discrete set in X and $\{y_n \mid n \in \omega\}$ be any subset of X. Assume that*

$$x = p\text{-}\lim\{x_n \mid n \in \omega\} = q\text{-}\lim\{y_m \mid m \in \omega\}.$$

Then one of following holds:

(i) *$\{m\colon q_m = x\} \in q$;*

(ii) *$p \leq_{RK} q$;*

(iii) *p is equal to q-$\lim\{z_n \mid n \in \omega\}$ for some $\{z_n \mid n \in \omega\} \subseteq \omega^*$ such that $z_n \neq z_m$ for $m \neq n$.*

Proof. Assume that (i) does not hold, i.e. $A = \{m \mid y_m \neq x\} \in q$, and one of the following holds:

(1) $\{m \in A \mid y_m \in \mathrm{cl}\{x_n \mid n \in \omega\}\} \in q$ (then, of course (iii) holds);
(2) $B = \{m \in A \mid y_m \in \mathrm{cl}\{x_n \mid n \in \omega\}\} \notin q$; that is,

$$B = \{m \in A \mid y_m \notin \mathrm{cl}\{x_n \mid n \in \omega\}\} \in q.$$

Then we have disjoint cozero sets U_n, which are open neighborhoods of x_n, and a cozero set V disjoint from the U_n's such that

$$\forall_{m \in B}\, y_m \in \bigcup\{U_n \mid n \in \omega\} \text{ or } y_m \in V.$$

Since X is an F-space and $x \in \mathrm{cl} \bigcup \{U_n \mid n \in \omega\}$,

$$x \notin \mathrm{cl}\, V$$

and so $C = \{m \in B \mid y_m \in \bigcup \{U_n \mid n \in \omega\}\} \in q$. Let $\varphi\colon C \to \omega$ be defined by

$$\varphi(n) = m \quad \text{iff} \quad y_n \in U_m.$$

Then φ makes p and q comparable in $\leq_{RK}$, so (ii) holds.

2. Lemma on products. Let X be a compact space and $A = \{x_n \mid n \in \omega\}$ be any discrete subset of X.

The set $A = \{x_n \mid n \in \omega\}$ has property (β) if there is a family of open disjoint sets $\{U_n \mid n \in \omega\}$ such that $x_n \in U_n$ and for arbitrary set $A \subseteq \omega$ we have

$$\mathrm{cl}[\bigcup_{n \in A} U_n] \cap \mathrm{cl}[\bigcup_{n \notin A} U_n] = \emptyset.$$

Of course, if X is an F-space then property (β) is equivalent to discreteness.

For $K \supseteq L$, let $\pi_L\colon \prod_{i \in K} X_i \to \prod_{i \in L} X_i$ be the projection.

2.1. LEMMA. *For any compact X_i and any set $\{x_n \mid n \in \omega\} \subseteq \prod_{i \in K} X_i$ the following are equivalent.*

(a) *$\{x_n \mid n \in \omega\}$ has property (β) in $\prod_{i \in K} X_i$.*

(b) *For some countable $L \subseteq K$, $\{\pi_L(x_n) \mid n \in \omega\}$ has property (β) in $\prod_{i \in L} X_i$.*

Proof. If (a) holds, we may assume that U_n's are open basic (in product topology) sets. Thus $\mathrm{cl}[\bigcup_{n \in A} U_n]$ depends only on countably many coordinates and hence we get (b). The converse implication is obvious.

Remark. In particular, if there is an i such that $\{\pi_i(x_n) \mid n \in \omega\}$ has property (β) then (a) holds.

3. The proof of the Theorem. Let $\{x_n \mid n \in \omega\} \subseteq \prod_{i \in I} X_i \times \prod_{j \in J} Y_i$ be such that

(1) $\forall_{i \in I}\{\pi_{\{i\}}(x_n) \mid n \in \omega\}$ is discrete;
(2) $\forall_{j \in J} \forall_{n, m \in \omega} \pi_{\{j\}}(x_n) = \pi_{\{j\}}(x_m)$.

By the remark after 2.1 $\{x_n \mid n \in \omega\}$ has property (β). Let p and q be as in 1.1 and let

$$x = p\text{-}\lim\{x_n \mid n \in \omega\} \quad \text{and} \quad y = q\text{-}\lim\{x_n \mid n \in \omega\}.$$

We claim that there is no homeomorphism $h\colon \prod_{i \in I} X_i \times \prod_{j \in J} Y_j$ for which $h(y) = x$.

Let $y_m = h(x_m)$. Then $\{y_m \mid m \in \omega\}$ has property (β) and $x = q$-$\lim\{y_m \mid m \in \omega\}$.

By 2.1, let $P \subseteq I \cup J$ be countable such that $\{\pi_p(y_m) \mid m \in \omega\}$ has property (β). Let $I_1 = P \cap I$ and $J_1 = P \cap J$. For $i \in P$ write

$$\pi_{\{i\}}(x) = p\text{-}\lim\{\pi_{\{i\}}(x_n) \mid n \in \omega\} = q\text{-}\lim\{\pi_{\{i\}}(y_n) \mid n \in \omega\}.$$

Now for $i \in I_1$ let $A_i = \{m\colon \pi_{\{i\}}(y_m) = \pi_{\{i\}}(x)\}$. By the choice of p and q and by 1.2, $A_i \in p$. Since P is countable, there is an $F \subseteq \omega$, F infinite and

(a) $\forall_{i \in I_1}(|F \setminus A_i| < \omega)$;
(b) $\forall_{i \in J_1} \forall_U (U$ open neighborhood of $\pi_{\{i\}}(x)$

$$|\{m \in F \mid \pi_{\{i\}}(y_m) \notin U\}| < \omega.$$

(It is possible to fix F, since $\forall_{i \in J_1}$ Y_i is first countable.) Thus $\{\pi_P(x_m) \mid m \in F\}$ converges in the normal sense, i.e. modulo the Fréchet ideal, to $\pi_P(x)$; this is a contradiction to (β).

References

[1] W. Comfort and S. Negrepontis, *The theory of ultrafilters*, Berlin 1974.
[2] K. Kunen, *Another point in βN*, in *Colloquium in Topology*, ed. Czásár, Amsterdam 1979.

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The structure of orbits in dynamical systems

by

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Abstract. Spaces which are both locally homeomorphic to $\boldsymbol{Q}\times\boldsymbol{R}$, the topological product of the rationals $\boldsymbol{Q}$ and the reals $\boldsymbol{R}$, and arcwise connected are studied. It is shown that such spaces are the image of $\boldsymbol{R}$ under a one-to-one and continuous map having the arc lifting property.

A necessary and sufficient condition for a separable and metrizable space X to be the orbit in some flow is presented. The following structure theorem is obtained.

A space X is the orbit of a Poisson-stable and aperiodic motion if and only if X is homeomorphic to the suspension of a universally transitive homeomorphism of $\boldsymbol{Q}$.

Unless explicitly stated otherwise all spaces under consideration are separable and metrizable.

1. Introduction.

1.1. The following problem will be discussed. What are necessary and sufficient conditions for a separable and metrizable space X to be the orbit in some continuous dynamical system (or flow)? In this paper a topological characterization of orbits of flows is presented. A structure theorem for orbits of flows is discussed as well. The classification problem is only lightly touched upon.

For locally compact orbits the situation is rather simple. There are only three homeomorphism types of locally compact orbits. In a flow each locally compact orbit is either a singleton, or a simple closed curve or a topological copy of the real line. And obviously each of these spaces can be endowed with the dynamical structure of an orbit. The reader is referred to Subsections 1.3 and 1.4 for more details about these remarks.

The topological structure of orbits which are not locally compact is much more complicated and has not yet been studied in great detail thus far. The orbits which are not locally compact are precisely the orbits of the motions which are (positively or negatively) Poisson-stable, but not periodic. The structure theorem is presented in Section 5. It will be shown that an aperiodic and Poisson-stable motion can be viewed as the suspension of a discrete dynamical system on the space of the rationals $\boldsymbol{Q}$. Such a discrete system is generated by a so-called universally transitive homeo-