THEOREM 3 (CH). If the weight of \( X \) is not greater than \( \omega_1 \), \( X \in \mathcal{L} \) and every compact subset of \( X \) is a \( G_\sigma \)-set then \( X \) is \( \sigma \)-compact.

Proof. By Theorem 1, \( X \) satisfies (\( * \)). Put \( f(K) = K \) for every \( K \in \mathcal{H}(X) \). Then it follows from (\( * \)) that \( X \) is \( \sigma \)-compact.

THEOREM 4 (CH). If the weight of \( X \) is not greater than \( \omega_1 \), \( X \in \mathcal{L} \) and \( X \) does not contain uncountable compact subsets then \( X \) with the topology induced by \( G_\sigma \)-subsets, with respect to the original topology, is a Lindelöf space (see [N], for a related result).

Proof. It is enough to observe that if \( F = \{ x_n : n < \omega \} \) is a compact subset of \( X \) and \( x_\alpha \in G_\sigma \) is a \( G_\sigma \)-subset of \( X \) for \( n < \omega \) then there is a \( G_\sigma \)-subset \( H \) of \( X \) such that \( F \subseteq H = \bigcup \{ G_\alpha : n \in \omega \} \).

Remark 1. Theorem 3 may be improved a little bit, namely the following statement is true if the weight of \( X \) is not greater than \( \omega_1 \), \( X \in \mathcal{L} \) and every compact subset of \( X \) is of the \( G_\sigma \)-type then \( X \) is \( \sigma \)-compact if and only if every metric element of \( \mathcal{L} \) is \( \sigma \)-compact. Hint: Put \( C_n = \{ c \in I : \rho(c, x_n) < \epsilon \} \); with every \( x \in \mathcal{L} \), \( n \in \omega \), define big-sets as non-\( \sigma \)-compact sets and \( I = \{ x_n : n < \omega \} \) \( \subseteq \mathcal{L} \) such that for every \( x \therefore \) there is an \( \alpha \in A \) satisfying \( x_{\alpha n} = x_{\alpha n} \).

Remark 2. It follows from Theorem 4 that \( X \) from [A] does not belong to \( \mathcal{L} \) as an uncountable space without uncountable compact subsets in which every point is of the \( G_\sigma \)-type.

Let me finish this note with the following

QUESTION. Assume that (CH) holds and \( X \) is such that every closed subset of \( X \) of weight not greater than \( \omega_1 \) satisfies (\( * \)). Does \( X \) necessarily satisfy (\( * \))?

Remark 3. Positive answer to this question would yield a positive answer to Michael’s conjecture.

References


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Correction to: Adding a random or a Cohen real: topological consequences and the effect on Martin’s axiom

by

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This paper appeared in Fundamenta Mathematicae 103 (1979), 47-60 pp. and Shelah has recently written to me that there is a serious problem with Theorem 5.3, p. 57. This states that if \( MA_{\aleph_1} \) holds in a model \( M \) then it still holds in \( M[x] \) where \( x \) is a Cohen or random real over \( M \); and if \( MA_{\aleph_1} \) holds in a model \( M \) then it still holds in \( M[x] \) where \( x \) is a Cohen real over \( M \). The statement about \( MA_{\aleph_1} \) is false: Todorčević noticed that when \( x \) is Cohen the statement conflicts with a result of Shelah's that appears in his paper on taking the inaccessible away from Solovay's proof that all sets are Lebesgue measurable (Israel Journal of Mathematics 48 (1984) 1-47 pp.). Shelah then noticed that his result can be modified to show that the statement about \( MA_{\aleph_1} \) is false when \( x \) is random. The problems with the proof of this false theorem are, in the Cohen case, that the auxiliary partial order \( Q \) relies on maximal finite antichains being able to decide nearly everything, when, in fact, they seldom do; in the random case \( Q \) was not carefully defined and, in fact, fails to be transitive.

On the other hand, the second part of Theorem 5.3 — if \( MA_{\aleph_1} \) holds in \( M \) then it holds in \( M[x] \) where \( x \) is Cohen over \( M \) — is true. Perhaps the easiest proof was noticed several years ago by Baumgartner and Tall, and is sketched here.

Recall that \( MA_{\aleph_1} \) is equivalent to the statement \( P(C) \): for every centered family \( \mathcal{D} \) on \( \omega \) of size less than \( C \) there is some infinite \( A \in \mathcal{D} \) with \( A \in B \) mod finite for all \( B \in \mathcal{D} \).

So assume \( \mathcal{D} = \{ B_I : I \in I \} \) is a Cohen forcing name for a centered family on \( \omega \) of size less than \( C \) where \( \mathcal{A} \) is some infinite set and \( A \in \mathcal{A} \) mod finite for all \( B \in \mathcal{A} \). We may assume that \( \mathcal{D} \) is forced to be closed under finite intersections. Let \( Q \) be the set of all triples \( \langle s, t, B \rangle \) where \( s \) is a finite Cohen condition, \( t \) is a finite subset of \( \omega \), and \( s \in B \). The order on \( Q \) is \( \langle s, t, B \rangle \leq \langle s', t', B' \rangle \) if and only if \( s \subseteq s' \) and \( t \subseteq t' \) and \( s \in B' \).

So assume \( \mathcal{D} = \{ B_I : I \in I \} \) is a Cohen forcing name for a centered family on \( \omega \) of size less than \( C \) where \( \mathcal{A} \) is some infinite set and \( A \in \mathcal{A} \) mod finite for all \( B \in \mathcal{A} \). We may assume that \( \mathcal{D} \) is forced to be closed under finite intersections. Let \( Q \) be the set of all triples \( \langle s, t, B \rangle \) where \( s \) is a finite Cohen condition, \( t \) is a finite subset of \( \omega \), and \( s \in B \). The order on \( Q \) is \( \langle s, t, B \rangle \leq \langle s', t', B' \rangle \) if \( s \subseteq s' \) and \( t \subseteq t' \) and \( s \in B' \). So assume \( \mathcal{D} = \{ B_I : I \in I \} \) is a Cohen forcing name for a centered family on \( \omega \) of size less than \( C \) where \( \mathcal{A} \) is some infinite set and \( A \in \mathcal{A} \) mod finite for all \( B \in \mathcal{A} \). We may assume that \( \mathcal{D} \) is forced to be closed under finite intersections. Let \( Q \) be the set of all triples \( \langle s, t, B \rangle \) where \( s \) is a finite Cohen condition, \( t \) is a finite subset of \( \omega \), and \( s \in B \). The order on \( Q \) is \( \langle s, t, B \rangle \leq \langle s', t', B' \rangle \) if \( s \subseteq s' \) and \( t \subseteq t' \) and \( s \in B' \). So assume \( \mathcal{D} = \{ B_I : I \in I \} \) is a Cohen forcing name for a centered family on \( \omega \) of size less than \( C \) where \( \mathcal{A} \) is some infinite set and \( A \in \mathcal{A} \) mod finite for all \( B \in \mathcal{A} \). We may assume that \( \mathcal{D} \) is forced to be closed under finite intersections. Let \( Q \) be the set of all triples \( \langle s, t, B \rangle \) where \( s \) is a finite Cohen condition, \( t \) is a finite subset of \( \omega \), and \( s \in B \). The order on \( Q \) is \( \langle s, t, B \rangle \leq \langle s', t', B' \rangle \) if \( s \subseteq s' \) and \( t \subseteq t' \) and \( s \in B' \). So assume \( \mathcal{D} = \{ B_I : I \in I \} \) is a Cohen forcing name for a centered family on \( \omega \) of size less than \( C \) where \( \mathcal{A} \) is some infinite set and \( A \in \mathcal{A} \) mod finite for all \( B \in \mathcal{A} \). We may assume that \( \mathcal{D} \) is forced to be closed under finite intersections. Let \( Q \) be the set of all triples \( \langle s, t, B \rangle \) where \( s \) is a finite Cohen condition, \( t \) is a finite subset of \( \omega \), and \( s \in B \). The order on \( Q \) is \( \langle s, t, B \rangle \leq \langle s', t', B' \rangle \) if \( s \subseteq s' \) and \( t \subseteq t' \) and \( s \in B' \).