Destroying precaliber $\aleph_1$: an application of a Δ-system lemma for closed sets

by

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Abstract. We construct a precaliber $\aleph_1$ partial order P which has an uncountable antichain in a forcing extension which preserves ω_1. Unless there are many measurables in an inner model, we construct a precaliber $\aleph_1$ partial order Q which has an uncountable antichain in a forcing extension which preserves stationary subsets of ω_1. We use a Δ-system lemma for closed sets disjoint from a fixed everywhere stationary set.

§ 1. Introduction. The countable chain condition in partial orders has been studied extensively, in part, because of its usefulness in forcing arguments. It is well known, for example, that countable chain condition partial orders preserve cardinals and that the property is preserved in iterations. Products of countable chain condition partial orders, however, may fail to have the countable chain condition and a countable chain condition partial order may lose this property in an extension of the set-theoretic universe (a Suslin tree provides an example in both cases). These pathologies disappear, though, if the countable chain condition is strengthened to σ-centred (a subset of a partial order is centred if every finite subset of it has a lower bound; a partial order is σ-centred if it is the union of countably many centred subsets).

In this paper we will examine what happens to partial orders which have an intermediate property known as precalibre $\aleph_1$. A partial order has *precaliber* $\aleph_1$ if any uncountable set has an uncountable centred subset.

Since σ-centred partial orders have precalibre $\aleph_1$ and remain σ-centred in any extension of the set-theoretic universe, the question arises as to whether or not the countable chain condition can be destroyed in partial orders with precalibre $\aleph_1$. The first theorem answers the question.

THEOREM 1. *There is a precaliber $\aleph_1$ partial order which has an uncountable antichain in a forcing extension of the universe which preserves ω_1.*

This forcing extension does not, however, preserve stationary subsets of ω_1 and also does not preserve cardinals (unless CH is true) so we are led to the next two theorems:

THEOREM 2. *There is a precaliber $\aleph_1$ partial order which has an uncountable antichain in a forcing extension of the universe which preserves stationary subsets of ω_1 unless there are many measurables in an inner model.*

Furthermore, the provisional nature of this result is necessary because if it is consistent that there is a supercompact cardinal then it is consistent that there is no such partial order.

THEOREM 3 ($MA_{\aleph_1}$ + covering lemma). *Any precaliber $\aleph_1$ partial order remains of precaliber $\aleph_1$ in forcing extensions of the universe which preserve cardinals.*

The set-theoretic hypothesis cannot be removed by Theorem 1. It is even consistent that proper partial orders destroy precalibre $\aleph_1$.

THEOREM 4. *It is consistent that there is a precalibre $\aleph_1$ partial order which has an uncountable antichain in a proper extension of the universe.*

This forcing extension collapses ω_2 so we have:

QUESTION 1. *Is it consistent that there is a precaliber $\aleph_1$ partial order which has the countable chain condition in a forcing extension which preserves cardinals but is not precaliber $\aleph_1$?*

In this paper, we also investigate a Δ-system lemma for closed sets disjoint from a fixed stationary set. In [3], we showed that if S is a stationary subset of ω_1 and $\{A_\alpha: \alpha\in\omega_1\}$ are closed sets disjoint from S, then there is an uncountable $A\subset\omega_1$ and $\beta\in\omega_1$ such that $\{A_\alpha-\beta: \alpha\in A\}$ is a disjoint family. We need a version of this lemma for higher cardinals $\varkappa$. If S is a subset of $\varkappa$ which is stationary in each ordinal in $\varkappa$ of uncountable cofinality and $\{A_\alpha: \alpha\in\omega_1\}$ are closed sets disjoint from S, then there is an uncountable $A\subset\omega_1$ and $\beta\in\varkappa$ such that $\{A_\alpha-\beta: \alpha\in A\}$ are disjoint. We show that β cannot be replaced by a countable subset of $\varkappa$: there are $\{A_\alpha: \alpha\in\omega_1\}$ closed sets in $\omega_1\cdot\omega+1$ of order-type $\omega+1$ such that for each uncountable $A\subset\omega_1$ and countable $B\subset\omega_1\cdot\omega+1$, $\{A_\alpha-B: \alpha\in A\}$ is *not* a disjoint family.

§ 2. Partial orders from trees.

The partial orders of this paper are constructed from trees. This is an idea of Baumgartner [2]: if T is a tree, then let $\boldsymbol{P}(T)$ be the partial order of finite antichains ordered by inclusion. The equivalences between properties of T and the chain conditions of $\boldsymbol{P}(T)$ are:

LEMMA 1 [2]. *T has no ω_1-branches if and only if $\boldsymbol{P}(T)$ has the countable chain condition.*

LEMMA 2. *The following are equivalent.*

(i) *T has neither ω_1-branches nor Suslin subtrees.*

(ii) *Any uncountable subset of T contains an uncountable antichain.*

(iii) *$P(T)$ has precaliber $\aleph_1$.*

(iv) *$P(T)$ is productively countable chain condition (i.e. any product of $P(T)$ and a countable chain condition partial order has the countable chain condition).*

Proof of Lemma 2. (i) ⇒ (ii) Let X be an uncountable subset of T which contains no uncountable antichain. Either X contains an ω_1-branch, in which case, T does too or X is an uncountable tree with neither uncountable branches nor uncountable antichains: that is, a Suslin subtree of T.

(iv) ⇒ (i) If T has an ω_1-branch B, then $\{\{b\}: b\in B\}$ is an uncountable antichain in $\boldsymbol{P}(T)$. If T has a Suslin subtree X then $\{(x, \{x\}): x\in X\}$ is an uncountable antichain in $X\times\boldsymbol{P}(T)$ where X has the tree order.

(ii) ⇒ (iii) It suffices to show that if any uncountable subset of T contains an uncountable antichain, then $\boldsymbol{P}(T)$ has precaliber $\aleph_1$. Let A be an uncountable subset of $\boldsymbol{P}(T)$. Without loss of generality, we assume that for some $n\in\omega$ and every $a\in A$, $|a| = n$. We prove by induction on n that there is an uncountable $X\subset A$ such that X is centred.

($n = 1$) In this case $\{a\in T: \{a\}\in A\}$ is an uncountable subset of T. By hypothesis it contains an uncountable antichain B, and then $\{\{a\}: a\in B\}$ is an uncountable centred subset of A.

($n = 2$) We enumerate A as $\{\{a_\alpha^0, a_\alpha^1\}: \alpha\in\omega_1\}$ and assume that $\{\{a_\alpha^0\}: \alpha\in\omega_1\}$ and $\{\{a_\alpha^1\}: \alpha\in\omega_1\}$ are both centred families in $\boldsymbol{P}(T)$. Set

$$X(\alpha) = \{\beta\in\omega_1: a_\alpha^1 \leqslant a_\beta^0 \text{ or } a_\alpha^0 \leqslant a_\beta^1\}.$$

Then there are two cases:

(i): $|X(\alpha)| = \omega_1$ for some $\alpha\in\omega_1$.

Let $X = \{a\in A: a_\alpha^0 \leqslant a^1\}$ for this α; X is uncountable and we claim X is centred. Suppose $a \neq b\in X$, and suppose $a\cup b$ is not an antichain in T. Then without loss of generality either $a^0\leqslant b^1$ or $a^1\leqslant b^0$. In the second case, we have $a_\alpha^0\leqslant a^1\leqslant b^0$, contradicting our assumption that $\{\{a_\alpha^0\}: \alpha\in\omega_1\}$ is pairwise compatible in $P(T)$. In the first case, a_α^0 and a^0 are both below b^1, and hence are comparable in T; again, this is a contradiction.

(ii): For each $\alpha\in\omega_1$, $|X(\alpha)|\leqslant\omega$.

Define $\alpha R\beta \leftrightarrow \beta\in X(\alpha)$ or $\alpha\in X(\beta)$ and let $\sim$ be the transitive closure of R. Then the equivalence classes are countable; hence there are uncountably many of them. Let X be an uncountable set of pairwise inequivalent elements. If $\alpha, \beta\notin X(\beta)$ then $\alpha\notin X$ and $\beta\notin X(\alpha)$, so $a_\alpha\cup a_\beta$ is an antichain in T.

In either case, we have shown that X is linked. But $\boldsymbol{P}(T)$ has the property that linked families are centred; thus X is an uncountable centred family.

($n>2$) Let $A = \{\{a_\alpha^0, a_\alpha^1, \ldots, a_\alpha^{n-1}\}: \alpha<\omega_1\}$. By the induction hypothesis we may suppose that each of $\{\{a_\alpha^0, \ldots, a_\alpha^{n-2}\}: \alpha<\omega_1\}$, $\{\{a_\alpha^1, \ldots, a_\alpha^{n-1}\}: \alpha<\omega_1\}$ and $\{\{a_\alpha^0, a_\alpha^{n-1}\}: \alpha<\omega_1\}$ is centred. From these it follows that for any $\alpha, \beta<\omega_1$, $\{a_\alpha^0, \ldots, a_\alpha^{n-1}\}\cup\{a_\beta^0, \ldots, a_\beta^{n-1}\}$ is an antichain, (since to check this we only need to look at two elements at a time), so that $\{\{a_\alpha^0, \ldots, a_\alpha^{n-1}\}: \alpha<\omega_1\}$ is centred.

LEMMA 3. (i) *If $\boldsymbol{P}(T)$ is σ-centred then T is the union of countably many antichains.*

(ii) *If T is union of countably many antichains and $|T|\leqslant 2^\omega$ then $P(T)$ is σ-centred.*

(iii) *If T is the $(2^\omega)^+$-ary tree of height 2 then $\boldsymbol{P}(T)$ is not σ-centred.*

Proof of Lemma 3 (ii). Let $T = \bigcup\{A_n:\ n\in\omega\}$ where each A_n is an antichain. We construct an antichain $C(A, U)\subset T$ for each $A\subset T$ which is the union of finitely many A_n and for $U\in \mathrm{SEQ}$ where SEQ is the set of finite sequences of clopen sets in 2^ω. We need some definitions: If A is a subset of T, then let

$$\mathrm{UP}(A) = \{t\in T:\ (\exists a\in A)\, a<t\}$$

and let $\mathrm{DOWN}(A)\subset A$ be the antichain of minimal elements of A. If U is a clopen subset of 2^ω and A is a subset of T then let $U(A) = \{a\in A:\ \pi(a)\in U\}$ where π is a fixed one-to-one mapping from T into 2^ω.

Let C_0 be $\mathrm{DOWN}(A)$.

Let C_i be $\mathrm{DOWN}\big(\mathrm{UP}((2^\omega - U(i-1))(C_{i-1}))\big)$ for each $i\in\operatorname{dom}U$ when $i>0$.

Let $C(A, U) = \bigcup\{U(i)(C_i):\ i\in\operatorname{dom}U\}$, $P = \bigcup\{[C(A, U)]^{<\omega}:\ A$ is the union of finitely many A_n and $U\in\mathrm{SEQ}\}$. $[C(A, U)]^{<\omega}$ is centred because each $C(A, U)$ is an antichain: each element of C_j is above some element of C_i when $i<j$, in fact, some element of $(2^\omega - U(i))C_i$. Thus if $t\in C_j$ and $s\in U(i)(C_i)$, t is above some $s'\in(2^\omega - U(i))(C_i)$ since s' and s are incompatible so are t and s.

To see that any finite antichain F is in $\boldsymbol{P}$, find $A\supset F$ which is the union of n many A_i.

To define $U(0)$: whenever $f\in F$ and $f\in\mathrm{DOWN}(A)$, let $f\in U(0)(A)$ whenever $f\in F$ and $f\notin\mathrm{DOWN}(A)$, let $e\in\mathrm{DOWN}(A)$ be such that $e<f$ and let $e\notin U(0)\mathrm{DOWN}(A)$. These finitely-many requirements on $U(0)$ can be accomplished.

To define $U(i)$ when $i<n$, whenever $f\in F$ and $f\in C_i$, let $f\in U(i)(C_i)$ and whenever $f\in F$ and $F\notin C_i$, let $e\in C_i$ be such that $e<f$ and let $e\in U(i)(C_i)$. This construction ensures $F\subset C(A, U)$ since even when each $U(i) = \emptyset$, $C_n\neq\emptyset$ implies that A contains a chain of size $n+1$.

(iii): Suppose $\boldsymbol{P}(T) = \bigcup\{A_i:\ i\in\omega\}$ where each A_i is centred. Let L be the functions from 1 into $(2^\omega)^+$ (i.e. the middle level of T). For each $i\in\omega$, let $L_i = L\cap(\bigcup A_i)$. Find $\alpha, \beta\in(2^\omega)^+$ such that, for each $i\in\omega$,

$$|\{\{(0, \alpha)\}, \{(0, \beta)\}\}\cap L_i| \neq 1\,.$$

This is possible since each L_i is a subset of L and so the L_i's can only distinguish at most 2^ω subsets of L. The antichain $\{\{(0, \alpha)\}, \{(0, \beta), (1, 0)\}\}$ is not an element of any A_i.

§ 3. The Δ-system lemma. A Δ-system lemma for uncountable families of closed subsets of ω_1 disjoint from a fixed stationary set was proved in [3].

Theorem 5. *Let S be a stationary subset of ω_1. Let $\{A_\alpha:\ \alpha\in\omega_1\}$ be an uncountable family of closed sets disjoint from S. There is an uncountable $A\subset\omega_1$ and $\gamma\in\omega_1$ such that $\{A_\alpha - \gamma:\ \alpha\in A\}$ is a disjoint family.*

This theorem is a special case of a Δ-system lemma for closed subsets of $\varkappa$ disjoint from a fixed everywhere stationary set.

Theorem 6. *Let $\varkappa$ be a cardinal of uncountable cofinality and S be a subset of $\varkappa$ which is stationary in each ordinal of uncountable cofinality. Let $\{A_\alpha:\ \alpha\in\omega_1\}$ be an uncountable family of closed sets disjoint from S. There is an uncountable $A\subset\omega_1$ and $\gamma\in\varkappa$ such that either*

(i) $\max A_\alpha = \gamma$ $(\alpha\in A)$, *or*

(ii) $\{A_\alpha - \gamma:\ \alpha\in A\}$ *is a disjoint family of nonempty sets.*

Proof. Let $\delta\leqslant\varkappa$ be the least ordinal such that $\{\alpha\in\omega_1:\ A_\alpha\subset\delta\}$ is uncountable. If $\operatorname{cf}(\delta) = \omega$, then (i) holds. Thus we can assume $\operatorname{cf}(\delta) = \omega_1$. We construct $\{\mu_\alpha:\ \alpha\in\omega_1\}$ and $\{\mathscr{B}_\alpha:\ \alpha\in\omega_1\}$ by induction. If $\{\mu_\beta:\ \beta<\alpha\}$ and $\{\mathscr{B}_\beta:\ \beta<\alpha\}$ are defined, then let μ_α be minimal such that for $\beta<\alpha$ and $A\in\mathscr{B}_\beta$ we have $\mu_\alpha>\max A$ and let $\mathscr{B}_\alpha$ be such that $\{A-\mu_\alpha:\ A\in\mathscr{B}_\alpha\}$ is (in $\{A_\beta-\mu_\alpha:\ \beta\in\omega_1\}$) a maximal pairwise disjoint family of nonempty sets. If any $\mathscr{B}_\alpha$ is uncountable, then (ii) holds. By minimality of δ each μ_α is defined and $\{\mu_\alpha:\ \alpha\in\omega_1\}$ is a closed unbounded set in δ. Find a limit ordinal α such that $\mu_\alpha\in S$. Find $A\in\mathscr{B}_\alpha$. Since $A\cap S\neq 0$, $\mu_\alpha\notin A$ and A is closed, there is $\beta<\alpha$ such that $A\cap\mu_\beta = 0$. Each element of $\mathscr{B}_\beta$ is contained in μ_α and so A is disjoint from each element of $\mathscr{B}_\beta$ outside μ_β which contradicts the maximality of $\mathscr{B}_\beta$.

Theorem 6 cannot be improved to get disjointness outside a countable set as in Theorem 5.

Theorem 7. *There is an uncountable family $\{A_\alpha:\ \alpha\in\omega_1\}$ of subsets of $\omega_1\cdot\omega+1$ such that, for each $A\in[\omega_1]^{\omega_1}$ and $B\in[\omega_1\cdot\omega+1]^\omega$, $\{A_\alpha - B:\ \alpha\in A\}$ is not a disjoint family and such that each A_α consists of a cofinal ω-sequence of successor ordinals in $\omega_1\cdot\omega$ and the limit point $\omega_1\cdot\omega$.*

Proof. Let $\{C_\alpha:\ \alpha\in\omega_1\}$ be a family of countable sets of successor ordinals in ω_1 such that, for each $A\in[\omega_1]^{\omega_1}$ and $D\in[\omega_1]^\omega$, $\{C_\alpha - D:\ \alpha\in A\}$ is not a disjoint family. This is possible by letting $\{C_\alpha:\ \alpha\in\omega_1\}$ be an increasing family whose union is the set of successor ordinals in ω_1. The C_αs do not work: they are not closed, so let

$$A_\alpha = \{\omega_1\cdot n + C_\alpha^i:\ i\leqslant n,\ n\in\omega\}\cup\{\omega_1\cdot\omega\}$$

where $\{C_\alpha^i:\ i\in\omega\}$ lists C_α. Each A_α has order type $\omega+1$. Suppose $A\in[\omega_1]^{\omega_1}$ and $B\in[\omega_1\cdot\omega+1]^\omega$. Let $D = \{\alpha\in\omega_1:\ (\exists n\in\omega)\,\omega_1\cdot n+\alpha\in B\}$. D is countable so there are $\alpha, \beta\in A$ such that $C_\alpha - D$ and $C_\beta - D$ intersect, say at $\gamma\in\omega_1$. Find $n\in\omega$ such that $\gamma\in\{C_\alpha^i:\ i\leqslant n\}\cap\{C_\beta^i:\ i\leqslant n\}$. Then $\omega_1\cdot n+\gamma\in A_\alpha\cap A_\beta$ but $\omega_1\cdot n+\gamma\notin B$ since $\gamma\notin D$.

§ 4. The proofs.

Definition. If $\varkappa$ is a regular uncountable cardinal and E is a stationary set of ω-limits in $\varkappa$, then $Q(E)$ is the tree of closed (in $\varkappa$) subsets of E ordered by end-extension.

Lemma 4. (1) *$Q(E)$ is ω-distributive (and thus preserves ω_1).*

(2) (Shelah) *$\varkappa>\omega_1$ implies $Q(E)$ preserves stationary subsets of ω_1.*

Proof. Let $\{A_n:\ n\in\omega\}$ be maximal antichains in $Q(E)$. Let S be a stationary subset of ω_1 and let D be a $Q(E)$-name for a closed unbounded subset of ω_1. Let $\{N_\alpha\in\varkappa\}$ be an elementary chain of elementary submodels, of cardinality less than $\varkappa$, of the universe containing ω, S, D, $\{A_n:\ n\in\omega\}$ (and each countable ordinal when

$\varkappa > \omega_1$). Let C be a closed unbounded set in $\varkappa$ such that $\alpha \in C$ implies $\varkappa \cap \dot{N}_\alpha = \alpha$. Find $\alpha \in C \cap E$ (greater than ω_1 when $\varkappa > \omega_1$). Let $\{\alpha_n: n \in \omega\}$ be an increasing sequence cofinal in α. If $\varkappa > \omega_1$, let $\{N^\beta: \beta \in \omega_1\}$ be an increasing sequence of elementary submodels of N_α such that $N^0 \supset \{\alpha_n: n \in \omega\}$. Let T be a closed unbounded set in ω_1 such that $\beta \in T$ implies $\omega_1 \cap N^\beta = \beta$. Find $\beta \in T \cap S$. Construct an increasing, in $Q(E)$, sequence of conditions $\{a_n: n \in \omega\} \subset N_\alpha$ such that each a_n extends some element of A_n and $\max a_n > \alpha_n$, and if $\varkappa > \omega_1$, a sequence $\{\gamma_n: n \in \omega\}$ cofinal in β such that $a_n \Vdash \gamma_n \in D$. Now $\bigcup \{a_n: n \in \omega\} \cup \{\alpha\}$ is a closed subset of E which extends some element of each A_n and forces $\beta \in D \cap S$.

The basic lemma we need is:

LEMMA 5. *If $\varkappa$ is a cardinal of uncountable cofinality and E is a subset of $\varkappa$ which does not contain a closed unbounded subset of any ordinal of uncountable cofinality, then $Q(E)$ is a tree in which any uncountable set contains an uncountable chain.*

Proof. Apply Theorem 6.

We can prove Lemma 4 also by noting that if $Q(E)$ contains an ω_1-branch, then E contains a closed unbounded subset of an ordinal of uncountable cofinality and that if $Q(E)$ contains a Suslin tree, then there is a notion of forcing having the countable chain condition which adds an ω_1-branch and is a closed unbounded subset of an ordinal of uncountable cofinality. Any closed unbounded set in ω_1 in a forcing extension by a countable chain condition partial order contains a closed unbounded set in the ground model.

We can now prove the basic theorems:

Proof of Theorem 1. Let E be a stationary costationary subset of ω_1. Forcing with $Q(E)$ adds an ω_1-branch to $Q(E)$, and hence, by Lemma 1, an uncountable antichain to $P(Q(E))$. By Lemma 4, ω_1 is preserved in this extension, and it follows from Lemma 2 and 5 that $P(Q(E))$ has precaliber $\aleph_1$.

Proof of Theorem 2. Apply the proof of Theorem 1 except that E must be a stationary subset of $\varkappa$ which does not contain a closed unbounded set of any ordinal of uncountable cofinality. This is possible at the successor of a singular cardinal unless there are many measurables in an inner model ([3] pp. 123–124). The necessity of this provision is proved by noting that if a supercompact cardinal is consistent then so is Martin's Maximum [5] which states that any $\aleph_1$ dense sets in a partial order $\boldsymbol{P}$ which preserves stationary subsets of ω_1 can be intersected with a filter. Construct dense sets in $\boldsymbol{P}$ which decide each of the elements of the uncountable antichain in a forcing extension. The filter decides an uncountable antichain in the ground model.

Proof of Theorem 3. Let $\boldsymbol{P}$ be a precaliber $\aleph_1$ partial order and Q be a partial order which preserves $\aleph_2$ in a model V of $\mathrm{MA}_{\aleph_1}$ and the covering lemma over an inner model K. We show that $\boldsymbol{P}$ is precaliber $\aleph_1$ in V^Q. Let A be an uncountable subset of $\boldsymbol{P}$ in V^Q. The covering lemma implies that there is a set of ordinals of cardinality $\aleph_1$ (in V^Q) in K which enumerates some $B \subset \boldsymbol{P}$ (in V where the enumeration is) which contains A in V^Q. Of course the covering lemma applies to any forcing extension of the universe [4]. Since cardinals are preserved, this set of ordinals has cardinality $\aleph_1$ in V and so, by $\mathrm{MA}_{\aleph_1}$, B is σ-centred (see [6]) and remains so in V^Q. Thus, in V^Q, A is subset of a σ-centred set.

Proof of Theorem 4. Let G be an unfilled (ω_2, ω_2^*) gap in $\mathscr{P}(\omega)$/FIN. Let $\boldsymbol{P}$ be the partial order which fills G (see p. 931 of [1]). By the proof of Theorem 4.2 of [1], $\boldsymbol{P}$ has precaliber $\aleph_1$. Let Q be the partial order which collapses ω_2 to ω_1 with countable conditions. In V^Q, G is still an unfilled gap (Q does not add any new subsets of ω) and so, by Theorem 4.2 of [1], there is a partial order $\boldsymbol{R}$ with the countable chain condition which adds an uncountable antichain to $\boldsymbol{P}$. Thus in V, $\boldsymbol{P}$ has precaliber $\aleph_1$, but in $V^{Q*\boldsymbol{R}}$, $\boldsymbol{P}$ has an uncountable antichain. But $Q * \boldsymbol{R}$ is an iteration of countably a closed partial order and a partial order with the countable chain condition and is, therefore, proper.

We close with another question:

QUESTION 2. *If $\boldsymbol{P}$ is a partial order in which every set of cardinality 2^ω is σ-centred then is $\boldsymbol{P}$ absolutely* C.C.C.?

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