Results on automorphisms of recursively saturated models of PA

by

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Abstract. We characterize those automorphisms of a countable recursively saturated model of arithmetic which can be extended to automorphisms of a given elementary end extension of the model. We calculate the cardinality of the set of nonextendable automorphisms and (in two special cases) the cardinality of the set of extendable automorphisms.

1. Introduction. Throughout the paper the characters M and N denote countable recursively saturated models of PA. If M is such a model then

$$E(M) = \{(M, I): I \prec M \text{ and } I \text{ is recursively saturated}\}.$$

A great deal is known about the family of elementary cuts in recursively saturated models of PA (cf. [Kot], [Smo] and references in these papers). But many questions still remain without answer. In this paper we present some results connected with the problem of classification of isomorphism types of elements of $E(M)$.

If F is a function with a domain $\mathfrak{A}$ and if $X \subseteq \mathfrak{A}$, then by $F * X$ we denote the image of X under F. By $\mathrm{Aut}(\mathfrak{A})$ we denote the set of all automorphisms of $\mathfrak{A}$ and for any $X \subseteq \mathfrak{A}$ we write

$$A(X) = \{F * X: F \in \mathrm{Aut}(\mathfrak{A})\}.$$

Most of our results concern the cardinality of $A(X)$, for some special X and $\mathfrak{A}$. A tool for calculating $|A(X)|$ is given by the following lemma.

1.1. Lemma (Kueker, Reyes [Ku]). *Let X be a subset of a countable structure $\mathfrak{A}$. Suppose that for every finite sequence $\bar{a}$ of elements of $\mathfrak{A}$ there are $b_1, b_2 \in \mathfrak{A}$, $b_1 \in X$, $b_2 \notin X$ such that $(\mathfrak{A}, \bar{a}, b_1) \cong (\mathfrak{A}, \bar{a}, b_2)$ then $|A(X)| = 2^{\aleph_0}$.*

As a consequence we have the following theorem.

1.2. Theorem (Schlipf [Sch]). *If $X \subseteq \mathfrak{A}$, $(\mathfrak{A}, X)$ is a countable, recursively saturated structure and X is not definable in $\mathfrak{A}$, then $|A(X)| = 2^{\aleph_0}$.*

In the above theorem and in the rest of the paper definability means definability with parameters.

In Section 4 we give a generalization of Schlipf's result to the case where X is a subset of a recursively saturated model M and X can be coded in a recursively saturated elementary end extension of M.

We use standard notation. In particular, if a is an element of a model M, then D_a denotes the set of elements coded by a in M and $(a)_i$ denotes the ith term of the sequence coded by a. If I is an initial segment of a model M, in symbols $I \subseteq_e M$, then $X \subseteq I$ is coded in M if for some $a \in M$, $X = I \cap D_a$.

The language of PA is denoted by L, Form is the set of formulas of L and Term is the set of all terms (= L-definable functions) of L.

2. (M, N)-isomorphisms. Let $I \subseteq_e M$, $J \subseteq N$ and let F be an isomorphism of I onto J. We say that F is an (M, N)-*isomorphism* if for every $A \subseteq I$ coded in M, $F * A$ is coded in N and for every $B \subseteq J$ coded in N, $F^{-1} * B$ is coded in M (this definition was introduced in [Kos 1]).

We say that $I \subseteq_e M$ is coded by ω from above in M if there is an $\alpha \in M$ coding an descending sequence of skies of a nonstandard length such that

$$I = \inf\{(\alpha)_n \colon n \in \omega\}.$$

2.1. THEOREM. *Let M and N be countable, recursively saturated models of* PA. *If $(M, I) \in E(M)$, $(N, J) \in E(N)$, neither of I and J is coded by ω from above in M and N respectively and $F\colon I \cong J$ is an (M, N)-isomorphism, then there exists $G\colon M \cong N$ such that $F \subseteq G$.*

The proof of Theorem 2.1 can be given by the usual back and forth construction based on the following lemma.

2.2. LEMMA. *Let M, N, I, J and F be as in Theorem 2.1. If $\bar a$ and $\bar b$ are finite sequences of elements of M and N, respectively, such that for every $x \in I$, $(M, \bar a, x)$ $(N, \bar b, F(x))$, then for every $a \in M$ there exists $b \in N$ such for every $x \in I$, $(M, \bar a, a, x) \equiv (N, \bar b, bF(x))$.*

Proof. Take $a \in M$. Let $\alpha \in M$ be such that

$$\forall x \in I \;\; M \models \varphi(\bar a, a, x) \leftrightarrow \langle \ulcorner \varphi \urcorner, x\rangle \in D_\alpha,$$

where $\ulcorner \varphi \urcorner$ denotes the Gödel number of φ.

By the assumption on F there exists $\beta \in N$ such that $J \cap D_\beta = F * D_\alpha$.

Since F is an isomorphism, for every $c \in J$ and every $\varphi \in$ Form we have:

$$(*) \qquad N \models \exists v\; \forall x < c\; (\varphi(\bar b, v, x) \leftrightarrow \langle \ulcorner \varphi \urcorner, x\rangle \in D_\beta)$$

(to check this, for every $c \in J$ replace β by a suitable $\beta_c \in J$).

Let $\{\varphi_n\}_{n \in \omega}$ be a recursive enumeration of Form. Consider the type $s(w)$:

$$\{(w)_n = \max\{z\colon \exists v\; \forall x < z \bigwedge_{i<n} (\varphi_i(\bar b, v, x) \leftrightarrow \langle \ulcorner \varphi_i \urcorner, x\rangle \in D_\beta)\}\colon n \in \omega\}.$$

(We can assume that the enumeration $\{\varphi_n\}$ is such that this maximum always exists.)

In view of $(*)$, $s(w)$ is consistent and if $e \in N$ is its realization then for every $n \in \omega$, $J < (e)_n$. Further since J is not coded by from above in N, $J < (e)_c$ for some nonstandard c. Hence the type $t(v)$:

$$\{\forall x < (e)_c \bigwedge (\varphi_i(\bar b, v, x) \leftrightarrow \langle \ulcorner \varphi_i \urcorner, x\rangle \in D_\beta)\colon n \in \omega\}$$

is consistent. Now if b realizes $t(v)$ in N then for all $x \in I$ we have $(M, \bar a, a, x) \equiv (N, \bar b, b, F(x))$; this finishes the proof. ■

2.3. COROLLARY. *If M and N are countable, recursively saturated models then for every $(M, I) \in E(M)$, $(N, J) \in E(N)$ such that neither of I and J is coded by ω from above in M and N respectively, (M, I) is isomorphic to (N, J) iff there exists an (M, N)-isomorphism of I onto J.*

2.4. COROLLARY. *Let M and N be countable, recursively saturated models and let $(M, I) \in E(M)$, $(N, J) \in E(N)$ be such that neither of I and J is coded by ω from above in M and N respectively. If $F\colon I \to J$ is an elementary embedding such that for every $A \subseteq I$ coded in M there exists $\beta \in N$ such that $F * A = D_\beta \cap F * I$, then there exist $K \prec_{\mathrm{cof}} N$ and $G\colon M \cong K$ such that $F \subseteq G$.*

Proof (sketch). To construct G it is enough to know that the conclusion of Lemma 2.2 is true also under the assumptions of our corollary. Observe that the statement $(*)$ from the proof of Lemma 2.2 is true for all $c \in F * I$. Since we may assume that $F * I$ is cofinal in J, this is just enough for continuation of the proof and the result follows. ■

3. Extendable automorphisms. Now we consider the following question. Suppose that $(M, I) \in E(M)$, $(N, J) \in E(N)$ and $(M, I) \cong (N, J)$.

What is the cardinality of the set of all (M, N)-isomorphisms of I onto J? This can be reduced to the question: suppose that $(M, I) \in E(M)$; how many automorphisms of I can be extended to automorphisms of M? A partial answer is given below.

If $X \subseteq M$, then by Form(X) we denote the set of formulas of L with an additional predicate for X. If $\bar a$ is a finite sequence of elements of M, then

$$\mathrm{Tp}(\bar a, x) = \{\Phi(\bar v)\colon \Phi(\bar v) \in \mathrm{Form}(X) \text{ and } (M, X) \models \Phi(\bar a)\}.$$

Let $\{\Phi_n\}_{n \in \omega}$ be a recursive enumeration of Form(X); then for $n \in \omega$ we have

$$\mathrm{Tp}^n(\bar a, X) = \mathrm{Tp}(\bar a, X) \cap \{\Phi_1, \ldots, \Phi_n\}.$$

3.1. LEMMA. *If X is a subset of a countable model M such that (M, X) is recursively saturated and X is infinite, then for every finite sequence $\bar a$ of elements of M there exists $b \in M$ such that*

$$|\{x \in X\colon \mathrm{Tp}(\bar a, b, X) = \mathrm{Tp}(\bar a, x, X)\}| = \aleph_0.$$

Proof. Suppose, on the contrary, that for some $\bar a$

$$\forall b \in X\; \exists r \in \omega\;\; |\{x \in X\colon \mathrm{Tp}(\bar a, b, X) = \mathrm{Tp}(\bar a, x, X)\}| = r.$$

Consider the type $t(v)$:

$$\{|\{x \in X\colon \mathrm{Tp}^n(\bar a, v, X) = \mathrm{Tp}^n(\bar a, x, X)\}| > r\colon r, n \in \omega\} \cup \{v \in X\}.$$

Since (M, X) is recursively saturated, $t(v)$ cannot be consistent. Hence we can find $r_0, n_0 \in \omega$ such that

$$\forall b \in X\ |\{x \in X:\ \mathrm{Tp}^{n_0}(\bar{a}, b, X) = \mathrm{Tp}^{n_0}(\bar{a}, x, X)\}| < r_0 .$$

But this implies that X is finite, a contradiction. ■

For $I \subseteq_e M$ let $\mathrm{Aut}_M(I)$ be the set of (M, M)-automorphisms of I.

We say that $I \subseteq_e M$ *almost rigid in* M if for some $a \in M$

$$I = \sup\{t(a):\ t(a) \in I \text{ and } t \in \mathrm{Term}\} \quad \text{or}$$

$$I = \inf\{t(a):\ t(a) \notin I \text{ and } t \in \mathrm{Term}\} .$$

3.2. THEOREM. *Let M be a countable, recursively saturated model and let $I \subseteq_e M$ be such that either (M, I) is recursively saturated or I, $I \neq \omega$, is almost rigid in M, then $|\mathrm{Aut}_M(I)| = 2^{\aleph_0}$.*

Proof. The proof can be carried out by the standard back and forth procedure with Lemma 3.1 used to split automorphisms on elements of I. We leave the details to the reader. When (M, I) is recursively saturated, the lemma applies to $X = I$. When I is almost rigid in M, we first take an $a \in M$ witnessing the almost rigidness of I and then we construct a family of automorphism satisfying the condition $F(a) = a$ (hence $F * I = I$). In this case the lemma applies to

$$X = \{x \in M:\ x < c\} ,$$

for any nonstandard $c \in I$. ■

Remarks. 1. Of course, if I is almost rigid then (M, I) is not recursively saturated. We have many examples of cuts which are of neither of the forms mentioned above. For instance, it can be shown easily that if $I \subseteq_e M$, $I \neq \omega$ is strong in M, then I is not almost rigid in M and there are many nonrecursively saturated structures (M, I) with I strong in M.

2. One of the corollaries to the Arithmetized Completeness Theorem says that for every nonstandard model M of PA there is a recursively saturated model N such that $M \subseteq_e N$. If $|\mathrm{Aut}(M)| < 2^{\aleph_0}$, then, of course, (N, M) cannot be recursively saturated. Theorem 3.2 implies that in this case also M cannot be almost rigid in N.

3. We conjecture that Theorem 3.2 is true for all $(M, I) \in E(M)$, for countable, recursively saturated M.

4. Nonextendable automorphisms.

4.1. THEOREM. *For every countable, recursively saturated model M and every $(M, I) \in E(M)$, $|\mathrm{Aut}(I) - \mathrm{Aut}_M(I)| = 2^{\aleph_0}$.*

Theorem 4.1 will follow from a more general Theorem 4.4 below.

Recall that for $X \subseteq M$, $A(X) = \{F * X:\ F \in \mathrm{Aut}(M)\}$.

The proof of Lemma 3.4 from [Kos2] gives also the following result.

4.2. LEMMA. *If X is a cofinal subset of a countable, recursively saturated model M such that, for every $a \in M$, $\{x \in X:\ x < a\}$ is finite, then $|A(X)| = 2^{\aleph_0}$.*

4.3. COROLLARY. *If M and X are as in Lemma 4.2 and X can be defined in (M, Y) by a formula of a countable language with a finite number of parameters, then $|A(Y)| = 2^{\aleph_0}$.*

4.4. THEOREM. *Let M be a countable, recursively saturated model and let $(M, I) \in E(M)$. Then for every $X \subseteq I$ coded in M, if X is not definable in I, then $|A(X)| = 2^{\aleph_0}$.*

Proof. Let $\alpha \in M$ be such that $X = I \cap D_\alpha$ and let $\bar{a}$ be a finite sequence of elements of I.

Let $\{\varphi_n\}_{n \in \omega}$ be a recursive enumeration of Form and let

$$\mathrm{tp}^n(\bar{a}) = \mathrm{tp}(\bar{a}) \cap \{\varphi_0, \ldots, \varphi_n\} .$$

Consider the type $t(v, w, u, \bar{a})$:

$$\{v, w < u\} \cup \{\mathrm{tp}^n(\bar{a}, v) = \mathrm{tp}^n(\bar{a}, w):\ n \in \omega\} \cup \{v \in D_\alpha \,\&\, w \notin D_\alpha\} .$$

If for every $\bar{a}$ in I there is a $c \in I$ such that $t(v, w, c, \bar{a})$ is consistent, then, using the recursive saturation of M, we see that the assumption of Kueker–Reyes lemma is satisfied and hence $|A(X)| = 2^{\aleph_0}$.

Now suppose that for some $\bar{a}$ in I, for every $c \in I$ $t(v, w, c, \bar{a})$ is not consistent. Let

$$\Phi(r, u) = \forall v, w < u\ [\mathrm{tp}^r(\bar{a}, v) = \mathrm{tp}^r(\bar{a}, w) \to (v \in X \leftrightarrow w \in X)] .$$

By our assumption we have

$$\forall c \in I\ \exists r \in \omega\ \Phi(r, c) .$$

We will consider two cases.

Case 1. $\exists r_0 \in \omega\ \forall c \in I\ \Phi(r_0, c)$.

For every $b \in M$ the r_0-type of b is the set

$$\{\varphi(\bar{a}, v):\ M \models \varphi(\bar{a}, b) \text{ and } \varphi \in \{\varphi_0, \ldots, \varphi_{r_0}\}\} .$$

Let $t_0(\bar{a}, v), \ldots, t_k(\bar{a}, v)$ be the collection of all distinct r_0-types realized by elements of X.

Now, it can be easily verified that

$$\forall x \in I\ x \in X \leftrightarrow I \models \bigvee_{i=0}^{k} \bigwedge t_i(\bar{a}, x) .$$

This contradicts the fact that X is not definable in M.

Case 2. $\forall r \in \omega\ \exists c \in I\ \neg\Phi(r, c)$.

Notice that if $r_1 < r_2$ then

$$I \models \forall u\ (\Phi(r_1, u) \to \Phi(r_2, u)) .$$

Hence the function $f\colon \omega \to I$, $f(r) = \max\{u\colon \Phi(r, u)\}$, is well defined and $f * \omega$ is a cofinal subset of M. Also, for every $a \in I$, $\{x \in f * \omega\colon x < a\}$ is finite; hence by Corollary 4.3 we have $|A(X)| = 2^{\aleph_0}$. ■

Theorem 4.1 follows directly from Theorem 4.4 and the fact that if M is countable and recursively saturated and if $(M, I) \in E(M)$, then the family of subsets of I coded in M is countable and contains nondefinable sets (M is not a conservative extension of I).

Theorem 4.4 gives also a new proof of a well-known lemma.

4.5. Lemma (Kaufmann [Ka]). *If X is a nondefinable subset of a countable, recursively saturated model M, then there exists a recursively saturated model N such that $M \prec_e N$ and X is not coded in N.*

Let us also mention the following easy consequence of Theorem 4.4.

4.6. Corollary. *Let M be a countable, recursively saturated model. If $(M, I) \in E(M)$ and F is an isomorphism of I onto J, then there exists N such that $(M, I) \cong (N, J)$ but F cannot be extended to any isomorphism of M onto N.*

Combining Corollary 4.6 with the proofs of Theorems 2.4 and 3.5 from [Kos2] we get the following result.

4.7. Theorem ($\diamondsuit$). *For every countable, recursively saturated model M there is a family of $2^{\aleph_1}$ pairwise nonisomorphic elementary end extensions of M, such that each extension in this family is recursively saturated, ω_1-like and rigid.*

5. Sets coded in recursively saturated elementary end extensions. Theorem 4.4 gives rise to a natural question: which subsets of countable recursively saturated models can be coded in recursively saturated elementary end extensions? Let us state a few remarks in this direction.

5.1. Observation. *If S is a partial inductive satisfaction class on a model M and $X \subseteq M$ is such that the structure (M, S, X) satisfies the full induction schema, then there exists a recursively saturated elementary end extension N of M in which X is coded.*

5.2. Corollary. *If S is a partial inductive satisfaction class on a countable model M, then $|A(S)| = 2^{\aleph_0}$.*

Proof. This follows from Theorem 4.4 and Observation 5.1. ■

5.3. Corollary. *If X is a subset of a countable model M such that the structure (M, X) is recursively saturated and (M, X) satisfies the induction schema, then there exists a recursively saturated elementary end extension N of M in which X is coded.*

Proof. This follows from Observation 5.1 by an easy resplendency argument. ■

A characterization of subsets of countable models of PA which can be coded in elementary end extensions was given in [KP]. Using this characterization, the assumptions of Observation 5.1 and Corollaries 5.2, 5.3 can be essentially weakened. A precise formulation lies slightly outside the scope of this paper. Let us only mention the following corollary.

5.4. Corollary. *If M is countable, $X \subseteq M$ is such that (M, X) is recursively saturated and X can be coded in an elementary end extension of M, then X can be coded in a recursively saturated elementary end extension of M.*

We have examples showing that for every countable recursively saturated model M there are $X \subseteq M$ such that (M, X) satisfies the induction schema and X cannot be coded in any recursively saturated elementary end extension of M, but still $|A(X)| = 2^{\aleph_0}$.

Problems. 1. Assume that (M, X) satisfies the induction schema. Is the converse of Observation 5.1 true?

2. A subset X of a model M is called a *class* if for every $a \in M$,

$$X \cap \{x \in M\colon x < a\}$$

is coded in M.

Assume that X is a nondefinable class of a countable recursively saturated model M. Is it true that $|A(X)| = 2^{\aleph_0}$?

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