example, [6].) Now $\mathcal{Q}\langle x_1, x_2\rangle$ = the closure of the semigroup generated by $x_1$ and $x_2$ is just the set product $(\mathcal{Q}\langle x_1\rangle)(\mathcal{Q}\langle x_2\rangle)$ which has again a countable of $\mathcal{H}$-classes. By induction then, $A$ has but a countable number of $\mathcal{H}$-classes.

Now if $\alpha$ is any continuous homomorphism defined on $A$ then $\alpha$ cut down to any $\mathcal{H}$-class $H$ is topologically equivalent to a homomorphism defined on the Schützenberger group of $H$, (See [3]). In particular, $\alpha(H)$ is again zero dimensional. Then, by the classical sum theorem of dimension theory, $\alpha(H)$ must be zero dimensional.

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# On saturated ideals and $P_\varkappa\lambda$

by

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**Abstract.** We present some results concerning saturated ideals on $P_\varkappa\lambda$. In particular, we prove that if $\lambda^{<\varkappa} = \lambda$ and $\varkappa$ is ethereal or $\lambda$-Shelah then $NS_{\varkappa\lambda}$, the ideal of non-stationary subsets of $P_\varkappa\lambda$ fails to be $\lambda^+$-saturated. Indeed, in the former case $\diamondsuit_{\varkappa\lambda}$ holds.

In this paper we present some results concerning saturated ideals on $P_\varkappa\lambda$. In § 1 we generalise some well-known properties of saturated ideals on $\varkappa$ to the $P_\varkappa\lambda$ context. For instance, we show that if $\varkappa = \mu^+$ then $P_\varkappa\lambda$ carries no $\lambda$-saturated ideals, that certain restrictions of $NS_{\varkappa\lambda}$, the ideal of non-stationary subsets of $P_\varkappa\lambda$ cannot be $\lambda^+$-saturated and that saturation is related to the GCH and a closure property of the generic ultrapower.

Our main results concerning $NS_{\varkappa\lambda}$ appear in §§ 2 and 3. In [1], Baumgartner, Taylor and Wagon introduced the notion of an $M$-ideal, and used in to prove (for instance) that if $\varkappa$ is weakly compact then the ideal of non-stationary subsets of $\varkappa$ is not $\varkappa^+$-saturated. § 2 contains analogous results for ideals on $P_\varkappa\lambda$: If $\varkappa$ is ethereal or $\lambda$-Shelah (and $\lambda^{<\varkappa} = \lambda$) then $NS_{\varkappa\lambda}$ is not $\lambda^+$-saturated.

In [10], Ketonen proved that if $2^{<\varkappa} = \varkappa$ and $\varkappa$ is ethereal then $\diamondsuit_\varkappa$ holds. In § 3 we adapt his argument to show that if $\lambda^{<\varkappa} = \lambda$ and $\varkappa$ is ethereal then $\diamondsuit_{\varkappa\lambda}$ holds.

Our set-theoretical notation and terminology is standard. Throughout $\varkappa$ will denote a regular uncountable cardinal and $\lambda$ a cardinal $\geqslant \varkappa$. $P_\varkappa\lambda = \{x \subset \lambda \mid |x| < \varkappa\}$ and $\lambda^{<\varkappa}$ is the cardinality of this set. For $x \in P_\varkappa\lambda$, $\hat{x} = \{y \in P_\varkappa\lambda \mid x \subset y\}$, $\varkappa_x = \varkappa \cap x$ and $\bar{x}$ denotes the order type of $x$. For $A \subseteq P_\varkappa\lambda$,

$$[A]^2_< = \{(x, y) \in A^2 \mid x \subset y \text{ and } |x| < |\varkappa \cap y|\}.$$

$A$ is said to be *unbounded* iff $(\forall x \in P_\varkappa\lambda)$ $(A \cap \hat{x} \neq \emptyset)$ and $I_{\varkappa\lambda}$ denotes the ideal of not unbounded subsets of $P_\varkappa\lambda$.

Throughout, $I$ will denote a proper, $\varkappa$-complete ideal on $P_\varkappa\lambda$ extending $I_{\varkappa\lambda}$ and $I^*$ the filter dual to $I$. If $A \in I^+$ $(= \{X \subseteq P_\varkappa\lambda \mid X \notin I\})$ then $I|A$ is the ideal on $P_\varkappa\lambda$ given by $I|A = \{X \subseteq P_\varkappa\lambda \mid X \cap A \in I\}$.

Clearly, all these concepts could be similarly defined for $P_\varkappa X$ where $X$ is any set of ordinals of cardinality $\geqslant \varkappa$.

§ **1.** In this section we briefly mention some simple properties of saturated ideals on $P_\varkappa\lambda$.

In [4], Jech proved that if $\varkappa$ is a successor cardinal and $\lambda$ is regular, then $\mathrm{NS}_{\varkappa\lambda}$ is nowhere $\lambda$-saturated. In fact, we may easily show that if $\varkappa$ is a successor cardinal then no ideal on $P_\varkappa\lambda$ is $\lambda$-saturated. Firstly, we need the following

LEMMA 1.1. *Suppose $\eta \geqslant \varkappa$, $\varkappa = \mu^+$ and there is a family of functions $\{f_\sigma \mid \sigma < \eta\}$ such that $f_\sigma\colon P_\varkappa\lambda \to \mu$ and $\{x \in P_\varkappa\lambda \mid f_\sigma(x) = f_\varrho(x)\} \in I$ whenever $\sigma < \varrho < \eta$. Then $I$ is not $\eta$-saturated.*

Proof. Let the family $\{f_\sigma \mid \sigma < \eta\}$ be as given; then, by $\varkappa$-completeness, for each $\sigma < \eta$ we may find a $\delta_\sigma < \mu$ such that $A_\sigma = f_\sigma^{-1}(\{\delta_\sigma\}) \in I^+$. By a well-known result of Tarski (see [5, Lemma 17.6]) there is a regular cardinal $\nu$ such that $\varkappa \leqslant \nu \leqslant \eta$ and $I$ is $\nu$-saturated. Choose $\delta < \mu$ such that $X = \{\sigma < \nu \mid \delta_\sigma = \delta\}$ has cardinality $\nu$; then the family $\{A_\sigma \mid \sigma \in X\}$ is easily seen to contradict the $\nu$-saturation of $I$.

THEOREM 1.2. *If $\varkappa = \mu^+$ then $I$ is not $\lambda$-saturated.*

Proof. For each $x \in P_\varkappa\lambda$ let $h_x\colon x \to \mu - \{0\}$ be injective, and for each $\alpha < \lambda$ let $f_\alpha\colon P_\varkappa\lambda \to \mu$ be given by $f_\alpha(x) = h_x(\alpha)$ if $\alpha \in x$; $f_\alpha(x) = 0$ otherwise. It is clear that $\{x \in P_\varkappa\lambda \mid f_\alpha(x) = f_\beta(x)\} \in I$ whenever $\alpha < \beta < \lambda$, and the result now follows immediately from Lemma 1.1.

Using the method of almost disjoint functions we may also show that certain restrictions of $\mathrm{NS}_{\varkappa\lambda}$ cannot be $\lambda^+$-saturated. We first need the following

LEMMA 1.3. *There exists a family $\{g_\sigma \mid \sigma < \lambda^+\}$ such that $g_\sigma\colon \lambda \to \lambda$ and $\bigcup\{\alpha < \lambda \mid g_\sigma(\alpha) = g_\varrho(\alpha)\} < \lambda$ whenever $\sigma < \varrho < \lambda^+$. If $2^{<\lambda} = \lambda$ then we may find $2^\lambda$ such functions.*

The proof for $\lambda$ regular is given in [2, III Lemma 4.10]. The case when $\lambda$ is a singular cardinal is similar.

THEOREM 1.4. *Suppose $\varkappa = \mu^+$, $\mu^\eta = \mu$, $I \supseteq \mathrm{NS}_{\varkappa\lambda}$ and $\{x \in P_{\varkappa\lambda} \mid \operatorname{cf}\bar{x} = \eta\} \in I^*$. Then $I$ is not $\lambda^+$-saturated.*

Proof. Let $\{g_\sigma \mid \sigma < \lambda^+\}$ be as in Lemma 1.3. For each $x \in P_\varkappa\lambda$ such that $\operatorname{cf}\bar{x} = \eta$ let $h_x\colon x^\eta \to \mu$ be injective and $g_x\colon \eta \to x$ be such that $g_x''\eta$ is cofinal in $x$. For each $\sigma < \lambda^+$ let $C_\sigma = \{x \in P_\varkappa\lambda \mid \operatorname{cf}\bar{x} = \eta$ and $(\forall\alpha \in x)\ (g_\sigma(\alpha) \in x)\}$, then $C_\sigma \in I^*$ and for each $x \in C_\sigma$ let $f_\sigma(x) = h_x(\langle g_\sigma(g_x(\delta)) \mid \delta < \eta\rangle)$. Suppose $\sigma < \varrho < \lambda^+$ and $A = \{x \in C_\sigma \cap C_\varrho \mid f_\sigma(x) = f_\varrho(x)\} \in I^+$. Pick $\gamma < \lambda$ such that

$$\{\alpha < \lambda \mid g_\sigma(\alpha) = g_\varrho(\alpha)\} \subseteq \gamma,$$

then for each $x \in A$ and $\delta < \eta$, $g_x(\delta) < \gamma$; hence $\gamma \notin x$, contradicting $A \in I^+$. The result now follows from Lemma 1.1.

If in addition $2^{<\lambda} = \lambda$, then it is clear (using Lemma 1.3) that $I$ is not $2^\lambda$-saturated.

A $P_\varkappa\lambda$-generalization of Kurepa's Hypothesis yields a similar result. Let $\mathrm{KH}_{\varkappa\lambda}$ denote the assertion "there exists a family $F \subseteq P(\lambda)$ such that $|F| = \lambda^+$ and for every infinite $x \in P_\varkappa\lambda$, $|\{a \cap x \mid a \in F\}| \leqslant |x|$".



THEOREM 1.5. *If $P_\varkappa\lambda$ carries a normal $\lambda^+$-saturated ideal then $\mathrm{KH}_{\varkappa\lambda}$ fails. Moreover, if $\varkappa = \mu^+$, then the assumption of normality is unnecessary.*

We leave the details to the reader. It is well-known (and indeed follows easily from Lemma 1.3) that the ideal on $\varkappa$, $I_\varkappa = \{X \subseteq \varkappa \mid |X| < \varkappa\}$ is not $\varkappa^+$-saturated. Analogously we have the following

THEOREM 1.6. *$I_{\varkappa\lambda}$ is not $\lambda^+$-saturated.*

Proof. We have two cases.

*Case* 1. $\lambda$ is regular. Let $\{Y_\sigma \mid \sigma < \lambda^+\} \subseteq I_\lambda^+$ witness that $I_\lambda$ is not $\lambda^+$-saturated. For each $\sigma < \lambda^+$ let $X_\sigma = \{x \in P_\varkappa\lambda \mid \bigcup x \in Y_\sigma\}$; then $X_\sigma \in I_{\varkappa\lambda}^+$ and the family $\{X_\sigma \mid \sigma < \lambda^+\}$ is easily seen to witness that $I_{\varkappa\lambda}$ is not $\lambda^+$-saturated.

*Case* 2. $\lambda$ is a singular cardinal, say $\operatorname{cf}\lambda = \eta$. Let $\langle \mu_\delta \mid \delta < \eta\rangle$ be a strictly increasing sequence of regular cardinals cofinal in $\lambda$ such that $\varkappa \leqslant \mu_0$ and for each $\delta < \eta$, $\mu_\delta > \bigcup\{\mu_\gamma \mid \gamma < \delta\}$. For each $\delta < \eta$ let $Y_\delta = \mu_\delta - \bigcup\{\mu_\gamma \mid \gamma < \delta\}$ and (by case 1) let $\{X_\delta^\varrho \mid \varrho < \mu_\delta^+\} \subseteq I_{\varkappa Y_\delta}^+$ witness that $I_{\varkappa Y_\delta}$ is not $\mu_\delta^+$-saturated. For each $f \in \prod\{\mu_\delta^+ \mid \delta < \eta\}$ let

$$X(f) = \{x \in P_\varkappa\lambda \mid (\forall\delta < \eta)(x \cap Y_\delta \in X_\delta^{f(\delta)} \cup \{\emptyset\})\}.$$

It is straightforward to check that $X(f) \in I_{\varkappa\lambda}^+$ and $X(f) \cap X(g) \in I_{\varkappa\lambda}$ whenever $f, g \in \prod\{\mu_\delta^+ \mid \delta < \eta\}$, $f \neq g$. Since $|\prod\{\mu_\delta^+ \mid \delta < \eta\}| > \lambda$, this completes the proof.

We close this section by mentioning three results concerning saturation whose proofs are similar to that of the corresponding result for ideals on $\varkappa$.

Firstly, recall ([9]) that $I$ is said to be (1) *seminormal* iff whenever $A \in I^+$, $\eta < \lambda$ and $f\colon A \to \eta$ is regressive, there is a $B \in P(A) \cap I^+$ such that $f \restriction B$ is constant; (2) weakly lean iff for each $A \in I^+$ there is a $B \in P(A) \cap I^+$ such that $|B| = \lambda$. If $D$ is $P(P_\varkappa\lambda)/I$-generic over $V$ (the ground model), let $M$ denote the associated generic ultrapower (see [6, § 2]).

THEOREM 1.7 (cf. [8, § 5]). *Suppose $\eta \leqslant \lambda$ and $I$ is seminormal and weakly lean. Then $P(P_\varkappa\lambda)/I \Vdash$ "$M^\eta \subseteq M$" iff $P(P_\varkappa\lambda)/I$ is $(\eta, <\lambda^+, \infty)$-distributive.*

THEOREM 1.8 (cf. [6, THEOREM 3.1.2]). *Assume that $2^\alpha = \alpha^+$ for each $\alpha < \varkappa$. If $\sigma > \lambda$ and $P_\varkappa\lambda$ carries a normal $\sigma$-saturated ideal, then $2^\lambda \leqslant \sigma$.*

The following theorem will be needed in §§ 2 and 3.

THEOREM 1.9 (cf. [1, Theorem 3.11]). *Suppose $I$ is normal, then $I$ is $\lambda^+$-saturated iff the ideals $I|A$ (for $A \in I^+$) are the only normal ideals on $P_\varkappa\lambda$ extending $I$.*

## § 2. $M$-ideals and $\mathrm{NS}_{\varkappa\lambda}$.

DEFINITION 2.1. $I$ is said to be an *$M$-ideal* iff $I$ is normal and $M(A) = \{x \in A \mid \varkappa_x$ is a weakly inaccessible cardinal and $A \cap P_{\varkappa_x}x \in \mathrm{NS}_{\varkappa_x x}^+\} \in I^*$ whenever $A \in I^*$.

The important fact concerning $M$-ideals is contained in the following

THEOREM 2.2. *If $A \in \mathrm{NS}_{\varkappa\lambda}^+$ then $\mathrm{NS}_{\varkappa\lambda}|A$ is not an $M$-ideal.*

Proof. Suppose not; then there exists a closed unbounded set $B \subseteq P_\varkappa\lambda$ such that $B \cap A \subseteq M(A)$. By a result of Menas ([11, Corollary 1.6]) we may find a function $g\colon \lambda^2 \to P_\varkappa\lambda$ such that $\{x \in P_\varkappa\lambda |\ (\forall \alpha, \beta \in x)\ (g(\alpha, \beta) \subseteq x)\} \subseteq B$. Clearly, $\varkappa$ is weakly inaccessible, and hence $C = \{x \in P_\varkappa\lambda |\ (\forall \alpha, \beta \in x)\ (g(\alpha, \beta) \subseteq x \text{ and } |g(\alpha, \beta)| < |\varkappa_x|)\} \in \mathrm{NS}^*_{\varkappa\lambda}$. Pick $x \in C \cap A$ such that $C \cap A \cap P_{\varkappa_x} x = \emptyset$, then $x \in M(A)$ and hence $A \cap P_{\varkappa_x} x \in \mathrm{NS}^+_{\varkappa_x x}$. Also $\varkappa_x$ is weakly inaccessible and so $C \cap P_{\varkappa_x} x \in \mathrm{NS}^*_{\varkappa_x x}$, a contradiction.

In [3], Carr proved that if $\lambda^{<\varkappa} = \lambda$ and $\varkappa$ is $\lambda$-Shelah, then for each $S \in \mathrm{NS}^+_{\varkappa\lambda}$, $\{x \in P_\varkappa\lambda |\ \varkappa_x$ is an inaccessible cardinal and $S \cap P_{\varkappa_x} x \in \mathrm{NS}^+_{\varkappa_x x}\} \in \mathrm{NSh}^*_{\varkappa\lambda}$. (Here $\mathrm{NSh}_{\varkappa\lambda}$ is the normal ideal on $P_\varkappa\lambda$ induced by the $\lambda$-Shelah property.) This together with Theorems 1.9 and 2.2 immediately yields the following

Corollary 2.3. ($\lambda^{<\varkappa} = \lambda$). *If $\varkappa$ is $\lambda$-Shelah then for each $A \in \mathrm{NSh}^+_{\varkappa\lambda}$, $\mathrm{NS}_{\varkappa\lambda}|A$ is not $\lambda^+$-saturated.*

Recall ([10]) that $\varkappa$ is said to be *ethereal* iff whenever $C \subseteq \varkappa$ is closed unbounded in $\varkappa$ and $\langle t_\sigma |\ \sigma < \varkappa\rangle$ is a sequence such that $(\forall \sigma < \varkappa)$ $(t_\sigma \subseteq \sigma$ and $|t_\sigma| = |\sigma|)$, there exists $\varrho, \sigma \in C$ such that $\varrho < \sigma$ and $|t_\varrho \cap t_\sigma| = |\varrho|$.

The following characterisation of ethereal cardinals follows immediately from results of Ketonen ([10, § 2]).

Theorem 2.4. *$\varkappa$ is ethereal iff whenever $C$ is closed unbounded in $\varkappa$ and $\langle t_\sigma, C_\sigma |\ \sigma < \varkappa\rangle$ is a sequence such that $(\forall \sigma < \varkappa)$ $(t_\sigma \subseteq \sigma$, $|t_\sigma| = |\sigma|$ and $C_\sigma$ is closed unbounded in $\sigma)$, there exists weakly inaccessible cardinals $\varrho, \sigma \in C$ such that $\varrho < \sigma$, $|t_\varrho \cap t_\sigma| = \varrho$ and $\varrho \in C_\sigma$.*

Let $M_{\varkappa\lambda} = \{x \in P_\varkappa\lambda |\ \varkappa_x$ is a weakly inaccessible cardinal and $|x| = \varkappa_x\}$. Let $E_{\varkappa\lambda}$ denote the set of all $X \subseteq P_\varkappa\lambda$ which do not have the property "whenever $C \in \mathrm{NS}^*_{\varkappa\lambda}$, $q\colon X \to \lambda$ and $\langle S_x, C_x |\ x \in X\rangle$ is a sequence such that $(\forall x \in X \cap M_{\varkappa\lambda})$ $(S_x \subseteq x, |S_x| = |x|$ and $C_x \in \mathrm{NS}^*_{\varkappa_x x})$, there exists a pair $(y, x) \in [X \cap M_{\varkappa\lambda} \cap C]^2_<$ such that $|S_y \cap S_x| = |y|$, $y \in C_x$ and $q(y) \in x$".

The reasons for introducing the sequence $\langle C_x |\ x \in X\rangle$ and the function $q\colon X \to \lambda$ are twofold; firstly, to make $E_{\varkappa\lambda}$ an $M$-ideal and, secondly, to ensure that Lemma 3.4 is true.

Following Menas ([11, Lemma 1.12]), we have

Theorem 2.5. *The following are equivalent*:

(a) *$\varkappa$ is ethereal*;

(b) $(\forall \lambda \geqslant \varkappa)$ *($E_{\varkappa\lambda}$ is a normal ideal on $P_\varkappa\lambda$)*;

(c) $(\exists \lambda \geqslant \varkappa)$ *($E_{\varkappa\lambda}$ is a normal ideal on $P_\varkappa\lambda$)*;

(d) $(\exists \lambda \geqslant \varkappa)$ $(P_\varkappa\lambda \notin E_{\varkappa\lambda})$.

Proof (a) $\to$ (b). We show that $P_\varkappa\lambda \notin E_{\varkappa\lambda}$. Suppose $C \in \mathrm{NS}^*_{\varkappa\lambda}$, $q\colon P_\varkappa\lambda \to \lambda$ and $\langle S_x, C_x |\ x \in P_\varkappa\lambda\rangle$ witness that $P_\varkappa\lambda \in E_{\varkappa\lambda}$. Since $\varkappa$ is ethereal, it is weakly inaccessible, and hence we may pick a sequence of elements of $C$, $\langle x_\sigma |\ \sigma < \varkappa\rangle$ such that for each $\sigma < \varkappa$, $x_\sigma \cup \{q(x_\sigma)\} \subseteq x_{\sigma+1}$, $|x_\sigma| < |\varkappa \cap x_{\sigma+1}|$, $\varkappa \cap x_\sigma$ is an ordinal and if $\lim(\sigma)$ then $x_\sigma = \bigcup \{x_\varrho |\ \varrho < \sigma\}$.

Let $k\colon x = \bigcup \{x_\sigma |\ \sigma < \varkappa\} \to \varkappa$ be a bijection and let $h\colon \varkappa \to \varkappa$, $g\colon \varkappa \to \varkappa$ be functions such that for each $\sigma < \varkappa$, $k''x_\sigma$ is an ordinal, $h(\sigma) = k''x_\sigma$ and $g(\sigma) = \varkappa_{x_\sigma}$.



The functions $h$ and $g$ are monotone increasing and continuous; hence we may find a closed unbounded subset of $\varkappa$, $H$ such that for each $\sigma \in H$, $\lim(\sigma)$ and $h(\sigma) = g(\sigma) = \sigma$.

If $\sigma \in H$ is regular then clearly $\{x_\varrho |\ \varrho < \sigma\}$ is closed unbounded in $P_\sigma x_\sigma$, and hence $C_\sigma = \{\varrho < \sigma |\ x_\varrho \in C_{x_\sigma}\}$ contains a closed unbounded set in $\sigma$. For each $\sigma \in H$ let $t_\sigma = k''S_{x_\sigma}$; then by Theorem 2.4 we may find weakly inaccessible cardinals $\varrho, \sigma \in H$ such that $\varrho < \sigma$, $\varrho \in C_\sigma$ and $|t_\varrho \cap t_\sigma| = \varrho$. But then $|S_{x_\varrho} \cap S_{x_\sigma}| = |x_\varrho|$, $x_\varrho \in C_{x_\sigma}$ and $q(x_\varrho) \in x_\sigma$, a contradiction.

The proof that $E_{\varkappa\lambda}$ is $\varkappa$-complete and normal is similar to [10, Proposition 2.3(3)].

(b) $\to$ (c) and (c) $\to$ (d) are trivial.

(d) $\to$ (a). Suppose $C$ is closed unbounded in $\varkappa$ and $\langle t_\sigma |\ \sigma < \varkappa\rangle$ is a sequence such that $(\forall \sigma < \varkappa)$ $(t_\sigma \subseteq \sigma$ and $|t_\sigma| = |\sigma|)$. For each $x \in P_\varkappa\lambda$ let $S_x = t_{\varkappa_x}$ if $x \in M_{\varkappa\lambda}$: $S_x = x$ otherwise. Clearly, $C' = \{x \in P_\varkappa\lambda |\ \varkappa_x \in C\}$ is closed unbounded in $P_\varkappa\lambda$, and hence, since $P_\varkappa\lambda \notin E_{\varkappa\lambda}$, we may find a pair $(y, x) \in [M_{\varkappa\lambda} \cap C']^2_<$ such that $|S_y \cap S_x| = |y|$. But then $\varkappa_y, \varkappa_x \in C$, $\varkappa_y < \varkappa_x$ and $|t_{\varkappa_y} \cap t_{\varkappa_x}| = |S_y \cap S_x| = |y| = \varkappa_y$.

It is easy to see from the definition of $E_{\varkappa\lambda}$ that for each $A \in E^+_{\varkappa\lambda}$, $E_{\varkappa\lambda}|A$ is an $M$-ideal, and hence from Theorems 1.9 and 2.2 we have

Corollary 2.6. *If $\varkappa$ is ethereal and $A \in E^+_{\varkappa\lambda}$ then $\mathrm{NS}_{\varkappa\lambda}|A$ is not $\lambda^+$-saturated.*

**§ 3. Etheral $\varkappa$ and $\diamondsuit_{\varkappa\lambda}$.** For $A \subseteq P_\varkappa\lambda$ let $\diamondsuit_{\varkappa\lambda}(A)$ denote the assertion "there exists a sequence $\langle S_x |\ x \in A\rangle$ such that for each $a \subseteq \lambda$, $\{x \in A |\ S_x = a \cap x\} \in \mathrm{NS}^+_{\varkappa\lambda}$". The following lemma gives a useful characterisation of $\diamondsuit_{\varkappa\lambda}(A)$.

Lemma 3.1 ([10]). *For $A \subseteq P_\varkappa\lambda$, $\diamondsuit_{\varkappa\lambda}(A)$ holds iff there exists a family $\{N_a |\ a \subseteq \lambda\} \subseteq P(A) \cap \mathrm{NS}^+_{\varkappa\lambda}$ such that for each $x \in A$ and $a, b \subseteq \lambda$, if $x \in N_a \cap N_b$ then $a \cap x = b \cap x$.*

Proof. If $\langle S_x |\ x \in A\rangle$ witnesses that $\diamondsuit_{\varkappa\lambda}(A)$ holds then for each $a \subseteq \lambda$, $N_a = \{x \in A |\ S_x = a \cap x\}$ is the required stationary set.

Conversely, if $\{N_a |\ a \subseteq \lambda\} \subseteq P(A) \cap \mathrm{NS}^+_{\varkappa\lambda}$ satisfies the given property, then for each $x \in A$ let $S_x = a \cap x$ whenever $x \in N_a$: $S_x = \emptyset$ if no such $a$ exists. The sequence $\langle S_x |\ x \in A\rangle$ is then easily seen to witness that $\diamondsuit_{\varkappa\lambda}(A)$ holds.

Lemma 3.1 immediately yields the following (well-known) corollary.

Corollary 3.2. *If $\diamondsuit_{\varkappa\lambda}(A)$ holds then $\mathrm{NS}_{\varkappa\lambda}|A$ is not $2^\lambda$-saturated.*

Ketonen ([10, Theorem 2.8]) proved that if $2^{<\varkappa} = \varkappa$ and $\varkappa$ is ethereal then $\diamondsuit_\varkappa$ holds. We now adapt his argument to prove the following

Theorem 3.3 ($\lambda^{<\varkappa} = \lambda$). *If $\varkappa$ is ethereal and $A \in E^+_{\varkappa\lambda}$ then $\diamondsuit_{\varkappa\lambda}(A)$ holds.*

Proof. For each $x \in P_\varkappa\lambda$, let $<_x$ be a well ordering of $x$ in order type $|x|$ and let $f_x\colon P(x) \to \lambda$ be injective such that $f_x''P(x) \cap f_y''P(y) = \emptyset$ whenever $x, y \in P_\varkappa\lambda$, $x \neq y$. Given $\alpha \in x$ and $a \subseteq \lambda$, let $h(x, \alpha, a) = f_y(a \cap y)$ where $y = \{\beta \in x |\ \beta <_x \alpha\}$. Let

$$Q_a = \{f_x(a \cap x) |\ x \in P_\varkappa\lambda\} \quad \text{and} \quad C_a = \{x \notin P_\varkappa\lambda |\ |\{\alpha \in x |\ h(x, \alpha, a) \in x\}| = |x|\};$$

then

LEMMA 3.4. *For each* $a \subseteq \lambda$, $C_a \in E^*_{\varkappa\lambda}$.

Proof of Lemma 3.4. Suppose not; then there exists an $a \subseteq \lambda$ such that $M_{\varkappa\lambda} - C_a \in E^+_{\varkappa\lambda}$. For each $x \in M_{\varkappa\lambda} - C_a$ there exists an $\alpha_x \in x$ such that if $\alpha \in x$ and $\alpha_x <_x \alpha$ then $h(x, \alpha, a) \notin x$; hence

$$C_x = \{\{\beta \in x \mid \beta <_x \alpha\} \mid \alpha \in x, \alpha_x <_x \alpha\} \in \mathrm{NS}^*_{\varkappa_x x}.$$

But then since $M_{\varkappa\lambda} - C_a \in E^+_{\varkappa\lambda}$, we may find a pair $(y, x) \in [M_{\varkappa\lambda} - C_a]^2_<$ such that $y \in C_x$ and $f_y(a \cap y) \in x$, a contradiction.

Now suppose that $A \in E^+_{\varkappa\lambda}$. For each $x \in P_\varkappa\lambda$, a set $r \subseteq x$ is said to be an $f$-$x$-set iff $|r| = |x|$ and $r \subseteq \{h(x, \alpha, a) \mid \alpha \in x\}$ for some $a \subseteq \lambda$.

We define a sequence $\langle r_x \mid x \in P_\varkappa\lambda\rangle$ by induction on $|x|$ as follows: If $x \notin M_{\varkappa\lambda} \cap A$ let $r_x = \varnothing$. Suppose $x \in M_{\varkappa\lambda} \cap A$ and $r_y$ has been defined for each $y \in P_{\varkappa_x} x$. If there exists an $f$-$x$-set $r \subseteq x$ such that $\{y \in P_{\varkappa_x} x \mid |r_y \cap r| < |y|\} \in \mathrm{NS}^*_{\varkappa_x x}$, then let $r_x$ be such an $r$. If no such $f$-$x$-set exists let $r_x = \varnothing$. Suppose $a \subseteq \lambda$; then we claim that

$$N_a = \{x \in P_\varkappa\lambda - \{\varnothing\} \mid |r_x \cap Q_a| = |x|\} \in \mathrm{NS}^+_{\varkappa\lambda}.$$

Suppose not then as in Theorem 2.2 we may find a function $g\colon \lambda^2 \to P_\varkappa\lambda$ such that $\{x \in P_\varkappa\lambda \mid (\forall \alpha, \beta \in x)\ (g(\alpha, \beta) \subseteq x)\} \subseteq P_\varkappa\lambda - N_a$ and since $\varkappa$ is weakly inaccessible, $B = \{x \in P_\varkappa\lambda \mid (\forall \alpha, \beta \in x)\ (g(\alpha, \beta) \subseteq x$ and $|g(\alpha, \beta)| < |\varkappa \cap x|)\} \in \mathrm{NS}^*_{\varkappa\lambda}$. Clearly, $\{x \in M_{\varkappa\lambda} \mid r_x \neq \varnothing\} \in E_{\varkappa\lambda}$ and hence by Lemma 3.4 we may find an $x \in M_{\varkappa\lambda} \cap A \cap B$ such that $r = \{h(x, \alpha, a) \mid \alpha \in x\} \cap x$ has cardinality $|x|$ and $r_x = \varnothing$. Since $x \in M_{\varkappa\lambda} \cap B$, $P_{\varkappa_x} x - N_a \in \mathrm{NS}^*_{\varkappa_x x}$, and, for each (non-empty) $y \in P_{\varkappa_x} x - N_a$, $|r_y \cap r| \leqslant |r_y \cap Q_a| < |y|$ (since $r \subseteq Q_a$), thus contradicting the fact that $r_x = \varnothing$.

Suppose now that $b, c \subseteq \lambda$ with $x \in N_b \cap N_c$. $r_x$ is of the form

$$r_x \subseteq \{h(x, \alpha, a) \mid \alpha \in x\}$$

for some $a \subseteq \lambda$ and hence, since $x \in N_b \cap N_c$, $b \cap x = a \cap x = c \cap x$. Finally, for each $a \subseteq \lambda$, $N_a \subseteq A$, and hence, by Lemma 3.1, $\diamondsuit_{\varkappa\lambda}(A)$ holds.

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