$\varepsilon > 0$ such that $N^*(q, 5\varepsilon) = \{x \in X: d^*(x, q) < 5\varepsilon\} \subset f(U)$. Choose $0 < \delta < \varepsilon$ such that, for $V = N^*(p, \delta)$, we have $V \subset U$ and $f(V) \subset N^*(q, \varepsilon)$. We claim that for all $x, y \in V$, $d^*(f(x), f(y)) \geqslant Ed^*(x, y)$. For, suppose $d^*(f(x), f(y)) < Ed^*(x, y)$ for some $x, y \in V$. Then there exists a continuum K containing $f(x)$ and $f(y)$ such that $\operatorname{diam} K < Ed^*(x, y)$. Since $d^*(x, y) < 2\varepsilon$, $\operatorname{diam} K < 4\varepsilon$, and since $d^*(f(x), q) < \varepsilon$, this implies that $K \subset N^*(q, 5\varepsilon) \subset f(U)$. Thus, for $L = f^{-1}(K) \cap U$, the restriction $f|L: L \to K$ is a homeomorphism. It follows that L is a continuum containing x and y, with $\operatorname{diam} L \leqslant E^{-1} \cdot \operatorname{diam} K < d^*(x, y)$, a contradiction.

COROLLARY 7. *No compact connected manifold with boundary admits an open local expansion.*

QUESTION. Does there exist a local expansion for any compact connected manifold with boundary?

A final observation: if the definition of local expansion is relaxed by requiring only that, for some open cover $\mathscr{U}$ of X, $d(f(x), f(y)) > d(x, y)$ for all $x, y \in U \in \mathscr{U}$, $x \neq y$, then Theorems 1 and 2, and their corollaries, remain valid (but we do not see how to prove Theorem 3 in this setting).

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INSTITUTE OF MATHEMATICS
WROCŁAW UNIVERSITY
Pl. Grunwaldzki 2
50-384 Wrocław

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On infinite words and dimension raising homomorphisms

by

R. P. Hunter (University Park)

Abstract. There exists a 2-generator compact zero dimensional semigroup which admits a continuous homomorphism onto a one dimensional semigroup. An abelian finitely generated compact zero dimensional semigroup admits no dimension raising homomorphisms.

It is well known that a compact topological group cannot admit dimension raising homomorphisms. Indeed, if G is such a group of dimension n, then any continuous homomorphism must decrease the dimension by that of the kernel.

It is also well known that a compact semigroup may admit dimension raising homomorphisms. See, for example, [1], [7], and [8]. The first example of a dimension raising homomorphism of a compact semigroup was observed by R. J. Koch.

From the nature of the various examples there is an understandable viewpoint that such homomorphisms are part of a theory that somehow is essentially abelian in nature. This is consistent, of course, with the fact that any pathology in the topological structure of a compact connected group is due to the abelian part. Similar considerations hold for compact connected monoids.

Playing a central role in such constructions are compact semigroups which are zero dimensional. Cone constructions will then easily yield appropriate higher dimensional examples. The earliest examples were of this sort.

Now among the compact zero dimensional semigroups, those which are (topologically) finitely generated would appear, as is the case for groups, to be more predictable. Indeed, it is the case that a compact finitely generated zero dimensional *abelian* semigroup admits no dimension raising homomorphisms.

It is, therefore, mildly surprising that a finitely generated compact semigroup may well admit such homomorphisms.

The semigroup in question is due to Boasson and Nivat and is of interest in the theory of languages [5]. It is studied there for entirely different reasons from those considered here.

The purpose of this note is to present the following two contrasting results:

THEOREM A. *There exists a two generator compact zero dimensional semigroup which admits a (continuous) dimension raising homomorphism. The semigroup may* *be taken as a zero dimensional compactification of the free semigroup on two generators.*

THEOREM B. *A compact finitely generated zero dimensional abelian semigroup admits no dimension raising homomorphisms.*

Proof of Theorem A. We recall the semigroup of Boasson and Nivat as in [5]: Let V be a finite alphabet. V^* the free monoid on V, V^ω the set of all infinite words on V and let $V^\infty = V^* \cup V^\omega$. Define the distance between two different elements of V^∞ as 2^{-k} where k is the length of the longest common left factor of the two elements. In this semigroup, V^* is as given, V^ω consists of left zero elements. If $v \in V^*$ and $w \in V^\omega$ the product is juxtaposition vw. Now V^∞ is known to be a compact semigroup. In V^∞ the semigroup V^* is open and dense. Moreover, the set V^ω can be seen to be homeomorphic to the cantor set.

For simplicity we limit ourselves to the case where V has two elements say 0 and 1. Note that the topology we are using on V^ω is equivalent to the usual product topology where an infinite word is viewed as an infinite tuple, i.e. a point of $X\{0, 1\} = P$.

Let C denote the usual cantor ternary set taken from the unit interval by the removal of middle thirds. Let φ denote the classical and canonical homeomorphism between C and the (countable) cartesian product $P = X\{0, 1\}$.

Letting C_n denote the nth stage in the construction of the cantor set we recall the definition of $\varphi\colon C \to P$

$$\varphi(x) = (a_1, a_2, a_3, \ldots)$$

where

$$a_m = \begin{cases} 0 & x \in \text{an odd interval in } C_m, \\ 1 & x \in \text{an even interval in } C_m. \end{cases}$$

Recall that C_m is composed of 2^m disjoint intervals, the first odd the second even and so forth, from left to right.

We have already noted that P may be canonically identified with V^ω. Composing this with φ we obtain the homeomorphism γ

$$\gamma(x) = a_1 a_2 a_3 \ldots$$

Now suppose that x and y are complementary endpoints of C. That is to say, they are endpoints of the same bounded complementary domain of C. We suppose then that $x < y$ and x and y are endpoints of some component of the complement of C_m, where m is taken as a minimum. Then the first m letters of $\gamma(x)$ must be $v_1 v_2 v_3 \ldots v_{m-1} 0$ and those of $\gamma(y)$ must be $v_1 v_2 v_3 \ldots v_{m-1} 1$. (The first $m-1$ co-ordinates of $\gamma(x)$ and $\gamma(y)$ must co-incide.) Moreover, at the stage C_{m+1} in the construction of C the point x must belong to an even interval which forces y to belong to an odd interval. This is also true at all succeeding stages. Thus,

$$\gamma(x) = v_1 v_2 \ldots v_{m-1} 0111111 \ldots$$

and

$$\gamma(y) = v_1 v_2 \ldots v_{n-1} 10000 \ldots$$

The first $m-1$ letters are equal, the nth letters are 0, 1, respectively, and the kth letters are 1,0, respectively, for $k > n$.

Let $\sim$ denote the decomposition whose classes consist of points or pairs of complementary endpoints. Now $\sim$ is a closed relation on C and $C/\sim$ is an arc. Under γ the decomposition $\sim$ is carried into another decomposition $\approx$ defined on V^ω which is also closed. Thus, $V^\omega/\approx$ is an arc.

Now the decomposition $\approx$ extends to all of V^∞ by taking classes as singletons outside of V^ω that is on V^*.

The decomposition $\approx$ defined now on V^∞, is also a congruence. (One need only note that V^ω is composed of left zeros and if W_1 and W_2 are $\approx$ equivalent words in V^ω and $v^* \in V^*$ then clearly $v^* W_1 \approx v^* W_2 \in V^\omega$.)

Thus, it follows that $V^\infty/\approx$ is one dimensional, consisting of a discrete, dense (free) semigroup whose boundary is an arc of left zeros.

Here is an alternative brief description of the semigroup discussed above: Let N_∞ denote the one point compactification of $N =$ the natural numbers. Let Y denote the cartesian product $I \times N_\infty$ where I is the unit interval. Let Y_1 denote the space obtained from Y by removing all open middle thirds from all components. Let Y_2 be obtained from Y_1 by removing all middle thirds from all components except those whose second co-ordinate is 1. Obtain Y_{k+1} from Y_k by removing all middle thirds from all components except those whose second coordinate is $\leqslant k$.

Let Y_0 denote the common part of the Y_k and let Y_0^j denote that subset of Y_0 whose second coordinate is j. Note that Y_0^∞ is the usual cantor set c as above. Now for $n < \infty$ let T be a component of Y_0^n. Then T corresponds to a unique word $w = w_1 w_2 \ldots w_n$ as follows: w_i is 0 or 1 as T is an odd or even component of Y_0^n (from left to right as usual) and in general w_i is 0 or 1 as the projections of T to Y_0^i is in an odd or even component.

In this way the midpoints of the components and words in $\{0, 1\}$ are in one to one correspondence.

Thus, V^∞ can be viewed as a closed subspace of Y_0.

Thus, as we see, the compact zero dimensional semigroup V^∞ admits a dimension raising continuous homomorphism. It is of course a monoid and removing the identity (which is an isolated point) provides a 2 generator semigroup. The resulting semigroup $V^\infty \setminus \{1\}$ is a continuous homomorphic image of the zero dimensional compactification of the free semigroup on two generators. Thus one has the following:

COROLLARY. *Let F_2 denote the free semigroup on two generators and let $\tilde{F}_2$ denote its zero dimensional compactification. $\tilde{F}_2$ admits a dimension raising homomorphism.*

Proof of Theorem B. Let A be a compact finitely generated abelian semigroup and let $x_1, x_2, \ldots, x_n$ be the generators. Note that $\mathcal{Q}\langle x_i \rangle =$ the closure of the semigroup generated by x_i, has a countable number of $\mathcal{H}$-classes. This is clear from the well known structure of any compact monothetic semigroup. (See, for example, [6].) Now $\mathcal{Q}\langle x_1, x_2\rangle$ = the closure of the semigroup generated by x_1 and x_2 is just the set product $(\mathcal{Q}\langle x_1\rangle)(\mathcal{Q}\langle x_2\rangle)$ which has again a countable of $\mathcal{H}$-classes. By induction then, A has but a countable number of $\mathcal{H}$-classes.

Now if α is any continuous homomorphism defined on A then α cut down to any $\mathcal{H}$-class H is topologically equivalent to a homomorphism defined on the Schützenberger group of H, (See [3]). In particular, $\alpha(H)$ is again zero dimensional. Then, by the classical sum theorem of dimension theory, $\alpha(H)$ must be zero dimensional.

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DEPARTMENT OF MATHEMATICS
THE PENNSYLVANIA STATE UNIVERSITY
215 McAllister Building
University Park, Pennsylvania 16802
U. S. A.

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On saturated ideals and $P_\varkappa\lambda$

by

C. A. Johnson (Keele)

Abstract. We present some results concerning saturated ideals on $P_\varkappa\lambda$. In particular, we prove that if $\lambda^{<\varkappa} = \lambda$ and $\varkappa$ is ethereal or λ-Shelah then $NS_{\varkappa\lambda}$, the ideal of non-stationary subsets of $P_\varkappa\lambda$ fails to be λ^+-saturated. Indeed, in the former case $\diamondsuit_{\varkappa\lambda}$ holds.

In this paper we present some results concerning saturated ideals on $P_\varkappa\lambda$. In § 1 we generalise some well-known properties of saturated ideals on $\varkappa$ to the $P_\varkappa\lambda$ context. For instance, we show that if $\varkappa = \mu^+$ then $P_\varkappa\lambda$ carries no λ-saturated ideals, that certain restrictions of $NS_{\varkappa\lambda}$, the ideal of non-stationary subsets of $P_\varkappa\lambda$ cannot be λ^+-saturated and that saturation is related to the GCH and a closure property of the generic ultrapower.

Our main results concerning $NS_{\varkappa\lambda}$ appear in §§ 2 and 3. In [1], Baumgartner, Taylor and Wagon introduced the notion of an M-ideal, and used in to prove (for instance) that if $\varkappa$ is weakly compact then the ideal of non-stationary subsets of $\varkappa$ is not $\varkappa^+$-saturated. § 2 contains analogous results for ideals on $P_\varkappa\lambda$: If $\varkappa$ is ethereal or λ-Shelah (and $\lambda^{<\varkappa} = \lambda$) then $NS_{\varkappa\lambda}$ is not λ^+-saturated.

In [10], Ketonen proved that if $2^{<\varkappa} = \varkappa$ and $\varkappa$ is ethereal then $\diamondsuit_\varkappa$ holds. In § 3 we adapt his argument to show that if $\lambda^{<\varkappa} = \lambda$ and $\varkappa$ is ethereal then $\diamondsuit_{\varkappa\lambda}$ holds.

Our set-theoretical notation and terminology is standard. Throughout $\varkappa$ will denote a regular uncountable cardinal and λ a cardinal $\geqslant \varkappa$. $P_\varkappa\lambda = \{x \subset \lambda \mid |x| < \varkappa\}$ and $\lambda^{<\varkappa}$ is the cardinality of this set. For $x \in P_\varkappa\lambda$, $\hat{x} = \{y \in P_\varkappa\lambda \mid x \subset y\}$, $\varkappa_x = \varkappa \cap x$ and $\bar{x}$ denotes the order type of x. For $A \subseteq P_\varkappa\lambda$,

$$[A]^2_< = \{(x, y) \in A^2 \mid x \subset y \text{ and } |x| < |\varkappa \cap y|\}.$$

A is said to be *unbounded* iff $(\forall x \in P_\varkappa\lambda)$ $(A \cap \hat{x} \neq \emptyset)$ and $I_{\varkappa\lambda}$ denotes the ideal of not unbounded subsets of $P_\varkappa\lambda$.

Throughout, I will denote a proper, $\varkappa$-complete ideal on $P_\varkappa\lambda$ extending $I_{\varkappa\lambda}$ and I^* the filter dual to I. If $A \in I^+$ $(= \{X \subseteq P_\varkappa\lambda \mid X \notin I\})$ then $I|A$ is the ideal on $P_\varkappa\lambda$ given by $I|A = \{X \subseteq P_\varkappa\lambda \mid X \cap A \in I\}$.

Clearly, all these concepts could be similarly defined for $P_\varkappa X$ where X is any set of ordinals of cardinality $\geqslant \varkappa$.