[3] G. Debs, *Quelques propriétés des espaces α-favorables et applications aux convexes compacts*, Ann. Inst. Fourier XXX, 2 (1980), 29–43.
[4] W. G. Fleissner and K. Kunen, *Barely Baire spaces*, Fund. Math. 101 (1978), 229–240.
[5] W. Hurewicz, *Relativ Perfekte Teile von Punktmengen und Mengen* (A), Fund. Math. 12 (1928), 78–109.
[6] M. R. Krom, *Infinite games and special Baire space extensions*, Pacific J. Math. 55 (1974), 483–487.
[7] D. Preiss, *Metric spaces in which Prokhorov's Theorem is not valid*, Z. Wahrscheinlichkeits. Ver. Geb. (1973), 109–116.
[8] Saint Raymond, *Caractérisation des espaces polonais*, Séminaire d'Initiation à l'Analyse, Univ. Paris 6, (1971–73), Exposé 5.

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Nonexistence of local expansions on certain continua

by

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Abstract. It is shown that no path-connected continuum without any simple closed curve or closed connected manifold with finite fundamental group admits a local expansion, and that no path-connected continuum with finite fundamental group or tree-like continuum admits an open local expansion. It is also shown that no compact connected manifold with boundary admits a local expansion into itself with respect to any connected metric, and that every open local expansion on a Peano continuum is a local expansion with respect to some connected metric. Hence compact connected manifolds with boundary admit no open local expansions, i.e., no local expansions which are boundary-preserving.

All considered spaces are assumed to be metric. A continuous function $f\colon (X, d) \to (X, d)$ is said to be a *local expansion* provided that for each $x \in X$, there is an open set U containing x and a real number $M > 1$ so that if $y, z \in U$, then $d(f(y), f(z)) \geqslant Md(y, z)$. We say that a metric space X *admits a local expansion* if there exist a metric d that is equivalent to the original one given on X, and a mapping $f\colon X \to X$ satisfying the conditions of the above definition.

THEOREM 1. *No local expansion of a continuum onto itself can be a homeomorphism.*

Proof. Let $f\colon (X, d) \to (X, d)$ be a surjective local expansion, and suppose f is a homeomorphism. Consider the surjective mapping $g = f^{-1}\colon (X, d) \to (X, d)$. By the compactness of X, there are positive numbers δ and M, with $M < 1$, so that if $d(x, y) \leqslant \delta$ then $d(g(x), g(y)) \leqslant Md(x, y)$. Let $\mathcal{U} = \{U_1, U_2, \ldots, U_n\}$ be a finite cover of X with mesh $\mathcal{U} < \delta$, and let x and y be any points of X. Then, by the connectedness of X, there is a chain of open sets from x to y, chosen from $U_1, U_2, \ldots, U_n$. Evidently $d(x, y) \leqslant n\delta$. Considering the images of links of this chain under g we have $d(g(x), g(y)) \leqslant Mn\delta$ and, more generally, $d(g^k(x), g^k(y)) \leqslant M^k n\delta$ for each positive integer k. Now choosing k so large that $M^k n < 1$, we see that $\operatorname{diam} g^k(X) < \delta \leqslant \operatorname{diam} X$. So g is not surjective, a contradiction.

COROLLARY 1. *No local expansion of a continuum into itself can be an imbedding.*

Proof. Let $f\colon X \to X$ be a local expansion, and suppose f is an imbedding. Then with $Y = \lim_{n\to\infty} f^n(X)$, Y is a subcontinuum such that $f|Y\colon Y \to Y$ is a homeomorphism, and $f|Y$ is a local expansion, contradicting Theorem 1. (Y must be nondegenerate, since otherwise $f|f^n(X)\colon f^n(X) \to f^{n+1}(X)$ is a global expansion for all large n, which is clearly impossible.)

The following corollary implies an affirmative answer to Problem 5.1 in [1, p. 201].

COROLLARY 2. *No path-connected continuum containing no simple closed curve admits a local expansion.*

Proof. Since every local expansion from a path-connected continuum onto a continuum which contains no simple closed curve is a homeomorphism (see Corollary 4.2 of [1, p. 195]), the assertion holds by Corollary 1.

COROLLARY 3. *No tree-like continuum admits an open local expansion.*

Proof. Since every open local expansion from a continuum onto a tree-like continuum is a homeomorphism (see Proposition 3.8 of [1, p. 193]), the assertion holds by Theorem 1.

LEMMA. *If X is a path-connected continuum with finite fundamental group, then every covering projection $f\colon X \to X$ is a homeomorphism.*

Proof. Choose $p \in X$, and consider the homomorphism $\varphi\colon \pi(X, p) \to \pi(X, f(p))$ induced by f. Note that φ is injective. Then since $\pi(X)$ is finite, φ must be an isomorphism, which in turn implies that f is a homeomorphism.

THEOREM 2. *No path-connected continuum with finite fundamental group admits an open local expansion.*

Proof. Any open local expansion would be a covering projection, and therefore a homeomorphism by the above lemma. But this is impossible by Theorem 1.

COROLLARY 4. *No closed connected manifold with finite fundamental group admits a local expansion.*

Proof. By invariance of domain, each local expansion on a closed manifold must be open.

Thus, for example, no n-sphere S^n or n-dimensional projective space P^n, $n \geq 2$, admits a local expansion. However, for any integer $k > 1$, a mapping f defined on the unit sphere S^1 by $f(z) = z^k$ is a local expansion. So we have

COROLLARY 5. *The n-sphere S^n (projective space P^n) admits a local expansion if and only if $n = 1$.*

Thus, a torus $T = S^1 \times S^1 \times \ldots \times S^1$ admits a local expansion, because the mapping $f_1 \times f_2 \times \ldots \times f_k\colon X_1 \times X_2 \times \ldots \times X_k \to Y_1 \times Y_2 \times \ldots \times Y_k$ is a local expansion if and only if $f_i\colon X_i \to Y_i$ is for every $i = 1, 2, \ldots, k$.

PROBLEM. Characterize all closed connected manifolds which admit local expansions.

A metric on a Peano continuum is said to be a *connected metric* if each metric ball $N(x, \varepsilon)$ is connected.

THEOREM 3. *Let X be a Peano continuum with a connected metric d, and $f\colon (X, d) \to (X, d)$ a local expansion. Then for every nonempty proper open subset U of X such that $f|U\colon U \to X$ is an open map, $f(U) \setminus U \neq \emptyset$.*

Proof. Suppose f has expansion factor $E > 1$. For each n, the iterate f^n is a local expansion, with expansion factor E^n. If $f(U) \subset U$, then each f^n is open over U and $f^n(U) \subset U$. Thus it suffices to verify the theorem for any iterate f^n. In other words, we may assume that $E \geq 2$.

Choose $\varepsilon > 0$ such that for each $x \in X$, $f|N(x, \varepsilon)$ is a 2-expansion and such that for some $u \in U$, $N(u, \varepsilon) \subset U$. We claim that for each $x \in U$ such that $N(x, \varepsilon) \subset U$, $f(N(x, \varepsilon)) \supset N(f(x), 2\varepsilon)$. Since $f|U$ is open, $\mathrm{Bd} f(N(x, \varepsilon)) \subset f(\mathrm{Bd} N(x, \varepsilon))$, and since $f|N(x, \varepsilon)$ is a 2-expansion, $f(\mathrm{Bd} N(x, \varepsilon))$ is disjoint from $N(f(x), 2\varepsilon)$. Then since the ball $N(f(x), 2\varepsilon)$ is connected, we must have $N(f(x), 2\varepsilon) \subset \mathrm{int} f(N(x, \varepsilon))$. For each $x \in U$, let $n(x)$ denote the least integer n such that there exists an ε-chain $x_0 = x, x_1, \ldots, x_n$ in X between x and some $x_n \in X \setminus U$ (we require $d(x_{i-1}, x_i) < \varepsilon$ for each i). Choose $p \in U$ such that

$$n(p) = \max\{n(x)\colon x \in U\}.$$

Then by the choice of ε we have $n = n(p) > 1$.

Consider $q = f(p)$. If $q \in U$, then there exists an ε-chain $y_0 = q, y_1, \ldots, y_n \in X \setminus U$. Since $N(p, \varepsilon) \subset U$, we have $f(N(p, \varepsilon)) \supset N(q, 2\varepsilon)$, and hence there exists $x_1 \in N(p, \varepsilon)$ with $f(x_1) = y_2$. If $2 < n$, then $N(x_1, \varepsilon) \subset U$ and $f(N(x_1, \varepsilon)) \supset N(y_2, 2\varepsilon)$, thus there exists $x_2 \in N(x_1, \varepsilon)$ with $f(x_2) = y_4$ (or y_3). Continuing in this fashion we obtain an ε-chain $x_0 = p, x_1, \ldots, x_m$, where $2m$ is either n or $n+1$, with $f(x_m) = y_n$. Since $n > 1$, we have $m < n$; hence $x_m \in U$, while $f(x_m) = y_n \in X \setminus U$.

COROLLARY 6. *Let M be a compact connected manifold with boundary, and d a connected metric on M. Then (M, d) admits no local expansion into itself.*

Proof. Apply Theorem 3 with $U = M \setminus \partial M$.

Note. For any metric d on a Peano continuum, the topologically equivalent metric d^*, defined by

$$d^*(x, y) = \inf\{\mathrm{diam}_d K\colon K \text{ a subcontinuum containing } x \text{ and } y\},$$

is a connected metric.

PROPOSITION. *Every open local expansion on a Peano continuum is a local expansion with respect to a connected metric.*

Proof. Suppose $f\colon X \to X$ is an open local expansion, with respect to a metric d and with expansion factor $E > 1$. We assume for convenience that $E \leq 2$. Let d^* be the connected metric constructed from d as above, and consider $p \in X$. By hypothesis, there exists a neighborhood U of p such that $d(f(x), f(y)) \geq E d(x, y)$ for all $x, y \in U$. Since f is open, $f(U)$ is a neighborhood of $q = f(p)$, and there exists

$\varepsilon > 0$ such that $N^*(q, 5\varepsilon) = \{x \in X: d^*(x, q) < 5\varepsilon\} \subset f(U)$. Choose $0 < \delta < \varepsilon$ such that, for $V = N^*(p, \delta)$, we have $V \subset U$ and $f(V) \subset N^*(q, \varepsilon)$. We claim that for all $x, y \in V$, $d^*(f(x), f(y)) \geqslant Ed^*(x, y)$. For, suppose $d^*(f(x), f(y)) < Ed^*(x, y)$ for some $x, y \in V$. Then there exists a continuum K containing $f(x)$ and $f(y)$ such that $\operatorname{diam} K < Ed^*(x, y)$. Since $d^*(x, y) < 2\varepsilon$, $\operatorname{diam} K < 4\varepsilon$, and since $d^*(f(x), q) < \varepsilon$, this implies that $K \subset N^*(q, 5\varepsilon) \subset f(U)$. Thus, for $L = f^{-1}(K) \cap U$, the restriction $f|L\colon L \to K$ is a homeomorphism. It follows that L is a continuum containing x and y, with $\operatorname{diam} L \leqslant E^{-1} \cdot \operatorname{diam} K < d^*(x, y)$, a contradiction.

COROLLARY 7. *No compact connected manifold with boundary admits an open local expansion.*

QUESTION. Does there exist a local expansion for any compact connected manifold with boundary?

A final observation: if the definition of local expansion is relaxed by requiring only that, for some open cover $\mathscr{U}$ of X, $d(f(x), f(y)) > d(x, y)$ for all $x, y \in U \in \mathscr{U}$, $x \neq y$, then Theorems 1 and 2, and their corollaries, remain valid (but we do not see how to prove Theorem 3 in this setting).

References

[1] J. J. Charatonik and M. Kalota, *On local expansions*, Fund. Math. 117 (1983), 187–203.

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On infinite words and dimension raising homomorphisms

by

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Abstract. There exists a 2-generator compact zero dimensional semigroup which admits a continuous homomorphism onto a one dimensional semigroup. An abelian finitely generated compact zero dimensional semigroup admits no dimension raising homomorphisms.

It is well known that a compact topological group cannot admit dimension raising homomorphisms. Indeed, if G is such a group of dimension n, then any continuous homomorphism must decrease the dimension by that of the kernel.

It is also well known that a compact semigroup may admit dimension raising homomorphisms. See, for example, [1], [7], and [8]. The first example of a dimension raising homomorphism of a compact semigroup was observed by R. J. Koch.

From the nature of the various examples there is an understandable viewpoint that such homomorphisms are part of a theory that somehow is essentially abelian in nature. This is consistent, of course, with the fact that any pathology in the topological structure of a compact connected group is due to the abelian part. Similar considerations hold for compact connected monoids.

Playing a central role in such constructions are compact semigroups which are zero dimensional. Cone constructions will then easily yield appropriate higher dimensional examples. The earliest examples were of this sort.

Now among the compact zero dimensional semigroups, those which are (topologically) finitely generated would appear, as is the case for groups, to be more predictable. Indeed, it is the case that a compact finitely generated zero dimensional *abelian* semigroup admits no dimension raising homomorphisms.

It is, therefore, mildly surprising that a finitely generated compact semigroup may well admit such homomorphisms.

The semigroup in question is due to Boasson and Nivat and is of interest in the theory of languages [5]. It is studied there for entirely different reasons from those considered here.

The purpose of this note is to present the following two contrasting results:

THEOREM A. *There exists a two generator compact zero dimensional semigroup which admits a (continuous) dimension raising homomorphism. The semigroup may*