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Strongly discrete subsets in ω^*

by

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Abstract. We prove that the statement: "$\bar{D} = \beta D$ for each strongly discrete subset $D \subseteq \omega^*$ with $|D| = \omega_1$" is consistent with ZFC+MA. We also give an example of a *B*-ideal over ω which cannot be extended to a *P*-point.

0. It is well known that if D is a countable discrete subset of the remainder $\omega^* = \beta[\omega]\setminus\omega$, ($\beta[\omega]$ = the Stone–Čech compactification of the discrete space ω), then the closure $\bar{D}$ in ω^* is (homeomorphic to) the space $\beta[\omega]$, or equivalently, D is C^*-embedded in ω^*.

In this paper we turn our attention to discrete sets $D \subseteq \omega^*$ of cardinality ω_1. Under the consistent assumption $2^{\omega_0} = 2^{\omega_1}$, the space $\beta[\omega_1]$ (the Stone–Čech compactification of a discrete space of cardinality ω_1) can be embedded into ω^*. Hence we may ask whether $\bar{D} = \beta D$ for discrete D with $|D| = \omega_1$.

Balcar, Simon and Vojtáš [1981] constructed a discrete set $D \subseteq \omega^*$, $|D| = \omega_1$, having the following property: there is a point $x \in \omega^*$ such that each neighbourhood of x contains all but countably many points of D. Obviously, $\bar{D} \neq \beta D$ for such a D. Hence we shall consider strongly discrete D in the following sense: there is a family of pairwise disjoint closed-open neighbourhoods, each containing a single point of D. Note that each countable discrete set D is strongly discrete.

The main result of this paper is the following

THEOREM. *Assuming the consistency of the Zermelo–Fraenkel set theory* ZFC, *there is a model of* ZFC *plus Martin's Axiom in which the closure $\bar{D}$ of each strongly discrete set $D \subseteq \omega^*$, $|D| = \omega_1$, is homeomorphic to βD (i.e. D is C^*-embedded in ω^*). In addition, $2^{\omega_0} = \omega_2$ and $\beta[\omega_1]$ is not a continuous image of ω^*.*

It can be proved, that the theorem fails in a model obtained by adding ω_2 Cohen reals.

1. We represent $\beta[\omega]$ as the space of all ultrafilters over ω with the Stone topology. The remainder $\omega^* = \beta[\omega]\setminus\omega$ consists then of all nonprincipal ultrafilters. The basic open-closed neighbourhoods are of the form $A^* = \bar{A} \cap \omega^*$, for an $A \subseteq \omega$, and A^* consists of all nonprincipal ultrafilters containing the set A. Let $D = \{F_\alpha : \alpha < \omega_1\}$ be a strongly discrete set of cardinality ω_1. According to the Taimanov Theorem (Engelking [1968]) in order that $\bar{D} = \beta D$ it is sufficient that, for an arbitrary $E \subseteq \omega_1$, the parts

(T) $$\{F_\alpha\colon \alpha \in E\} \quad \text{and} \quad \{F_\alpha\colon \alpha \notin E\}$$

can be separated by open-closed subsets of ω^*.

Since, by our assumption, D is strongly discrete there are almost disjoint sets $A_\alpha \subseteq \omega$ such that $A_\alpha \in F_\alpha$ for all $\alpha < \omega_1$. Consider the following forcing notion $\boldsymbol{P}_E\{A_\alpha\}$: the conditions are pairs $p = \langle s_p, t_p\rangle$, where s_p, t_p are finite functions;

$$\mathrm{Dm}(s_p) \subseteq E, \quad \mathrm{Dm}(t_p) \subseteq \omega_1 \setminus E, \quad \mathrm{Rg}(s_p), \mathrm{Rg}(t_p) \subseteq \omega$$

and

(*) $$\bigcup_{\alpha \in \mathrm{Dm}(s_p)} [A_\alpha \setminus s_p(\alpha)] \cap \bigcup_{\beta \in \mathrm{Dm}(t_p)} [A_\beta \setminus t_p(\beta)] = \emptyset .$$

The ordering on $\boldsymbol{P}_E\{A_\alpha\}$ is defined as inverse inclusion. Note that since the A_α's are almost disjoint, for arbitrary finite domains

$$a = \{\alpha_1, \ldots, \alpha_n\} \subseteq E \quad \text{and} \quad b = \{\beta_1, \ldots, \beta_m\} \subseteq \omega_1 \setminus E ,$$

any s, t on a, b, respectively, form a condition $p = \langle s, t\rangle$ if only the values of s, t are large enough.

Let $G \subseteq \boldsymbol{P}_E\{A_\alpha\}$ be a generic filter and let

$$s_G = \bigcup \{s\colon \exists t\ [\langle s, t\rangle \in G]\} ,$$

$$t_E = \bigcup \{t\colon \exists s\ [\langle s, t\rangle \in G]\} .$$

Obviously, $s_G\colon E \to \omega$ and $t_G\colon \omega_1 \setminus E \to \omega$. Now, if $A = \bigcup_{\alpha \in E} [A_\alpha \setminus s_G(\alpha)]$ and $B = \bigcup_{\beta \notin E} [A_\beta \setminus t_G(\beta)]$, then for all $\alpha \in E$ we have $A_\alpha \subseteq_* A$ ($X \subseteq_* Y$ denotes "almost inclusion", i.e. $X \setminus Y$ is finite) and $A_\beta \subseteq_* B$ for $\beta \notin E$ and $A \cap B = \emptyset$. Thus the open-closed neighbourhoods A^*, B^* will separate the sets $\bigcup_{\alpha \in E} A_\alpha^*$ and $\bigcup_{\beta \notin E} A_\beta^*$ in any model containing A, B. After a long enough iteration of separating forcings $\boldsymbol{P}_E\{A_\alpha\}$ we obtain an extension $V[G]$ of a ground model V in which property (T) holds for all strongly discrete sets $D \in V[G]$ and hence our theorem will be valid in $V[G]$.

It is obvious that our iteration should not collapse ω_1, and even more: at each stage we have to force with a forcing satisfying the c.c.-condition. Generally, a forcing $\boldsymbol{P}_E\{A_\alpha\}$ need not satisfy the c.c.-condition: let $\{A_\alpha\colon \alpha < \omega_1\}$ be an almost disjoint family with the property: for each $\alpha < \omega_1$ and $k \in \omega$ the set

$$\{\beta < \alpha\colon \max A_\beta \cap A_\alpha < k\}$$

is finite.

If both E and $\omega_1 \setminus E$ are uncountable, then clearly $\boldsymbol{P}_E\{A_\alpha\}$ collapses ω_1. Nevertheless, for any such $\boldsymbol{P}_E\{A_\alpha\}$ we shall find, in the course of iteration, an improvement, i.e. a forcing $\boldsymbol{P}_E\{B_\alpha\}$ with $B_\alpha \subseteq A_\alpha$ which satisfies the c.c.-condition.

2. We shall investigate uncountable antichains in $\boldsymbol{P}_E\{A_\alpha\}$. For a condition $p = \langle s_p, t_p\rangle$

$$K_p = \bigcup_{\alpha \in \mathrm{Dm}(s_p)} [A_\alpha \setminus s_p(\alpha)]$$

and

$$L_p = \bigcup_{\beta \in \mathrm{Dm}(t_p)} [A_\beta \setminus t_p(\beta)] .$$

Thus $K_p \cap L_p = \emptyset$. For conditions p, q define

$$p * q = (K_p \cup K_q) \cap (L_p \cup L_q) = (K_p \cap L_q) \cup (L_p \cap K_q) .$$

If $s_p \cup s_q$ and $t_p \cup t_q$ are functions, then p, q are incompatible if and only if $p * q \neq \emptyset$.

Suppose that $\boldsymbol{P}_E\{A_\alpha\}$ contains an uncountable antichain. Using the Δ-system lemma and after some thinning out we can assume that there is an antichain $C = \{p_\alpha\colon \alpha < \omega_1\}$ satisfying the following properties:

$p_\alpha * p_\beta \neq \emptyset$ for all $\alpha < \beta < \omega_1$;
$\max \mathrm{Dm}(s_{p_\alpha}) < \min \mathrm{Dm}(s_{p_\beta})$ for $\alpha < \beta$;
$\max \mathrm{Dm}(t_{p_\alpha}) < \min \mathrm{Dm}(t_{p_\beta})$;
the domains of all the s_{p_α}'s are of the same length: $|\mathrm{Dm}(s_{p_\alpha})| = |\mathrm{Dm}(s_{p_\beta})|$, for all $\alpha < \beta$ and the same for the t_{p_α}'s;
if $\mathrm{Dm}(s_{p_\alpha}) = \{\gamma_1^\alpha, \ldots, \gamma_n^\alpha\}$, then $s_{p_\alpha}(\gamma_i^\alpha) = s_{p_\beta}(\gamma_i^\beta)$ for $i = 1, \ldots, n$ and all $\alpha < \beta$ and similarly for the t_{p_α}'s.

The following lemma states the fundamental property of uncountable antichains.

Lemma. *Let $C = \{p_\alpha\colon \alpha < \omega_1\}$ be an uncountable antichain in $P_E\{A_\alpha\}$ as described above. Then there exist a tree T on $[\omega]^{<\omega} \setminus \{\emptyset\}$, an ordinal $\gamma < \omega_1$ and a function $q\colon T \to \{p_\alpha\colon \alpha < \gamma\}$ such that for each $\alpha \geqslant \gamma$ there is a branch $e = \langle e_n\colon n \in \omega\rangle$ of T such that (1) $e_n = p_\alpha * q(e|n)$ for each $n \in \omega$ and (2) the family $\{e_n\colon n \in \omega\}$ is pairwise disjoint.*

Proof. For $e_0 \in [\omega]^{<\omega} \setminus \{\emptyset\}$ define

$$X(e_0) = \{\alpha > 0\colon p_0 * p_\alpha = e_0\}$$

and let $S = \{e_0\colon |X(e_0)| = \omega_1\}$. Some of the sets $X(e_0)$ can be countable or finite, so take $\gamma_0 = \sup\{\sup X(e_0)\colon e_0 \in S\}$ and write $Y(e_0) = X(e_0) \setminus \gamma_0$. We have

$$\omega_1 \setminus \gamma_0 = \bigcup \{Y(e_0)\colon e_0 \in S\} .$$

Let $q(e_0) = p_{\alpha(e_0)}$, where $\alpha(e_0) = \inf Y(e_0)$. Now, repeat the process for each $q(e_0)$, $e_0 \in S$: let

$$X(e_0, e_1) = \{\alpha \in Y(e_0)\colon p_\alpha * q(e_0) = e_1\}$$

and

$$S(e_0) = \{e_0\colon |X(e_0, e_1)| = \omega_1\} .$$

If $\gamma_1 = \sup\{\sup X(e_0, e_1)\colon e_0 \in S \text{ and } e_1 \in S(e_0)\}$ and $Y(e_0, e_1) = X(e_0, e_1)\setminus\gamma_1$, then we have

$$Y(e_0)\setminus\gamma_1 = \bigcup\{Y(e_0, e_1)\colon e_1 \in S(e_0)\} \quad \text{for } e_0 \in S.$$

Let $\alpha(e_0, e_1) = \inf Y(e_0, e_1)$ and write $q(e_0, e_1) = p_{\alpha(e_0, e_1)}$. Continuing in this way we obtain a sequence $\gamma_0 < \gamma_1 < \gamma_2 < \ldots$ of countable ordinals, a sequence $Y = \omega_1$, $Y(e_0)$, $Y(e_0, e_1)$, ... of uncountable subsets of ω_1, and subsets S, $S(e_0)$, $S(e_0, e_1)$, ... of $[\omega]^{<\omega}\setminus\{\emptyset\}$ such that

$$\omega_1\setminus\gamma_0 = \bigcup\{Y(e_0)\colon e_0 \in S\}$$

and for each $n \in \omega$

$$Y(e_0, \ldots, e_n)\setminus\gamma_{n+1} = \bigcup\{Y(e_0, \ldots, e_{n+1})\colon e_{n+1} \in S(e_0, \ldots, e_n)\}.$$

Define $T = \{\langle e_0, \ldots, e_n\rangle\colon e_0 \in S, \ldots, e_n \in S(e_0, \ldots, e_{n-1})\}$ and $\gamma = \sup\{\gamma_n\colon n \in \omega\}$. If $\alpha \geqslant \gamma$, then from the construction there are sets $e_0 \in S$, $e_1 \in S(e_0)$, $e_2 \in S(e_0, e_1)$, ... such that for all $n \in \omega$, $\alpha \in Y(e_0, \ldots, e_n)$. Thus $e = \langle e_n\colon n \in \omega\rangle$ is a branch of T and

$$p_\alpha * q(e_0, \ldots, e_n) = e_{n+1}.$$

Also $e_0 = p_0 * p_\alpha$, so if we assume in addition $q(\emptyset) = \emptyset$, then $e_n = p_\alpha * q(e|n)$ also holds for $n = 0$.

It remains to show that the finite nonempty sets e_n are pairwise disjoint. Observe first that we have

$$e_n = q(e|n) * p_\alpha = q(e|n) * q(e|m) \quad \text{for } m > n.$$

Assume inductively that $e_0, \ldots, e_{n-1}$ are pairwise disjoint. We have

$$e_n = (K_{q(e|n)} \cap L_{p_\alpha}) \cup (L_{q(e|n)} \cap K_{p_\alpha}).$$

Now, since $\bigcup_{i<n} L_{q(e|i)} \cap K_{p_\alpha} \subseteq K_{p_\alpha}$ and $K_{q(e|n)} \cap L_{p_\alpha} \subseteq L_{p_\alpha}$ and $K_{p_\alpha} \cap L_{p_\alpha} = \emptyset$ we have $(K_{q(e|n)} \cap L_{p_\alpha}) \cap (\bigcup_{i<n} L_{q(e|i)} \cap K_{p_\alpha}) = \emptyset$. And since $K_{q(e|n)} \cap L_{p_\alpha} \subseteq K_{q(e|n)}$ and $\bigcup_{i<n} K_{q(e|i)} \cap L_{p_\alpha} \subseteq L_{q(e|n)}$ and $K_{q(e|n)} \cap L_{q(e|n)} = \emptyset$, we have $(K_{q(e|n)} \cap L_{p_\alpha}) \cap (\bigcup_{i<n} K_{q(e|i)} \cap L_{p_\alpha}) = \emptyset$. It follows that $(K_{q(e|n)} \cap L_{p_\alpha}) \cap (e_0 \cup \ldots \cup e_{n-1}) = \emptyset$. Symmetrically, $L_{q(e|n)} \cap K_{p_\alpha}$ is disjoint from $e_0, \ldots, e_{n-1}$ and the proof is complete.

Remark. Let $\bar{\gamma}$ exceed the domains of all the conditions p_α, $\alpha < \gamma$. Then the branch $e = \langle e_n\colon n \in \omega\rangle$ corresponding to any p_α with $\alpha \geqslant \gamma$ can be defined from the following parameters: T, q and the sets $\{A_\beta\colon \beta < \bar{\gamma}\}$; p_α and the sets

$$\{A_\beta\colon \beta \in \mathrm{Dm}(s_{p_\alpha}) \cup \mathrm{Dm}(t_{p_\alpha})\}.$$

3. As was remarked earlier, the separating forcings $\boldsymbol{P}_E\{A_\alpha\}$, which we intend to iterate, need not satisfy the c.c.-condition. Here we prove that any such $\boldsymbol{P}_E\{A_\alpha\}$ has an improvement; in fact, an improvement will be produced in the course of iteration in at most ω_1 steps. Hence we consider below a finite support iteration

$\boldsymbol{P} = \sum_{\alpha<\omega_1} \boldsymbol{P}_\alpha$, which should be understood as a fragment of length ω_1 of our "real" iteration described in Section 4.

Lemma. *Let $\boldsymbol{P} = \sum_{\alpha<\omega_1} \boldsymbol{P}_\alpha$ be a finite support iteration of nontrivial forcings satisfying the c.c.-condition and let $G \subseteq \boldsymbol{P}$ be a generic filter over V. If an almost disjoint family $\{A_\alpha\colon \alpha < \omega_1\}$ and a set $E \subseteq \omega_1$ are in V, then for each family $\{F_\alpha\colon \alpha < \omega_1\}$ of ultrafilters in $V[G]$ such that $A_\alpha \in F_\alpha$, there are sets $B_\alpha \subseteq A_\alpha$ in $V[G]$, $B_\alpha \in F_\alpha$, such that $\boldsymbol{P}_E\{B_\alpha\}$ is an improvement of $\boldsymbol{P}_E\{A_\alpha\}$ (i.e. $\boldsymbol{P}_E\{B_\alpha\}$ satisfies the c.c.-condition in $V[G]$).*

Proof. It is well known that a finite support iteration of nontrivial forcings adds a Cohen set $c \subseteq \omega$ in each sequence of ω steps. Thus let λ_α be an increasing enumeration of all countable limit ordinals and let $c_\alpha \in V[G_{\lambda_{\alpha+1}}]$ be a Cohen set over $V[G_{\lambda_\alpha}]$. Define

$$B_\alpha = \begin{cases} A_\alpha \cap c_\alpha & \text{if } c_\alpha \in F_\alpha, \\ A_\alpha \setminus c_\alpha & \text{if } c_\alpha \notin F_\alpha. \end{cases}$$

Thus $B_\alpha \subseteq A_\alpha$ and $B_\alpha \in F_\alpha$ for each $\alpha < \omega_1$. It remains to show that $\boldsymbol{P}_E\{B_\alpha\} \in V[G]$ satisfies the c.c.-condition $V[G]$. Assume, on the contrary that there is an uncountable antichain $\{p_\alpha\colon \alpha < \omega_1\}$ in $\boldsymbol{P}_E\{B_\alpha\}$ (with the properties described in Section 2). Usually, a condition from $\boldsymbol{P}_E\{B_\alpha\}$ need not be in $\boldsymbol{P}_E\{A_\alpha\}$ but since the domains of the conditions in the antichain are pairwise disjoint, for each $\alpha < \omega_1$ there is a $k_\alpha \in \omega$ such that if

$$\tilde{A}_\gamma = (B_\gamma \cap k_\alpha) \cup (A_\gamma \setminus k_\alpha) \quad \text{for } \gamma \in \mathrm{Dm}(s_{p_\alpha}) \cup \mathrm{Dm}(t_{p_\alpha})$$

and

$$\tilde{A}_\gamma = B_\gamma \quad \text{for } \gamma \notin \bigcup_{\alpha<\omega_1} \mathrm{Dm}(s_{p_\alpha}) \cup \mathrm{Dm}(t_{p_\alpha})$$

then each p_α is a condition in $\boldsymbol{P}_E\{\tilde{A}_\gamma\}$. Moreover, $\{p_\alpha\colon \alpha < \omega_1\}$ is then an antichain in $\boldsymbol{P}_E\{\tilde{A}_\gamma\}$ because $p_\alpha * p_\beta$ calculated in $\boldsymbol{P}_E\{B_\gamma\}$ is a subset of $p_\alpha * p_\beta$ calculated in $\boldsymbol{P}_E\{\tilde{A}_\gamma\}$, and hence $p_\alpha * p_\beta \neq \emptyset$ in $\boldsymbol{P}_E\{\tilde{A}_\gamma\}$. We now apply the Lemma of Section 2 to the antichain $\{p_\alpha\colon \alpha < \omega_1\}$ in $\boldsymbol{P}_E\{A_\alpha\}$ and take a $\beta < \omega_1$ such that T, γ, q, $\{A_\xi\colon \xi < \bar{\gamma}\}$ (cf. Remark at the end of Section 2) are all in $V[G_\beta]$. Fix an $\alpha > \beta$ and let $\gamma_1 < \ldots < \gamma_r$ enumerate $\mathrm{Dm}(s_{p_\alpha}) \cup \mathrm{Dm}(t_{p_\alpha})$. Since p_α and $A_{\gamma_1}, \ldots, A_{\gamma_r}$ are in V, we infer that the branch $e = \langle e_n\colon n \in \omega\rangle$ corresponding to p_α belongs to $V[G_\beta]$. Since $\beta < \alpha \leqslant \gamma_1 \leqslant \lambda_{\gamma_1}$, the set c_{γ_1} is a Cohen set over $V[G_\beta]$, and hence the set

$$S_{\gamma_1} e = \{e_n\colon e_n \subseteq S_{\gamma_1}\}$$

is infinite, where S_γ denotes c_γ if $c_\gamma \in F_\gamma$ and $\omega\setminus c_\gamma$ otherwise. After r steps we obtain an infinite family $S_{\gamma_r} \ldots S_{\gamma_1} e \subseteq \{e_n\colon n \in \omega\}$. Choose an e_n from $S_{\gamma_r} \ldots S_{\gamma_1} e$. By the definition of the B_γ's we see that

$$B_\gamma \subseteq A_\gamma \setminus e_n \quad \text{for } \gamma = \gamma_1, \ldots, \gamma_r,$$

and hence $p_\alpha * q(e|n)$ in $\boldsymbol{P}_E\{B_\gamma\}$ is disjoint from e_n. On the other hand, $p_\alpha * q(e|n)$ in $P_E\{B_\gamma\}$ is a subset of $p_\alpha * q(e|n)$ in $\boldsymbol{P}_E\{\tilde{A}_\gamma\}$, i.e. a subset of e_n. Hence $p_\alpha * q(e|n) = \emptyset$, which means that p_α and $q(e|n)$ are compatible, a contradiction. The proof of the lemma is complete.

4. Now, we can finish the proof of our main theorem. We begin with $V = L$ and shall use the following principle $\diamondsuit$: there is a sequence $\langle S_\alpha\colon \alpha < \omega_2$ and $\operatorname{cf}(\alpha) = \omega_1\rangle$ such that for each $X \subseteq \omega_2$, the set $\{\alpha\colon X \cap \alpha = S_\alpha\}$ is stationary. Let H be the family of all sets (in V) of hereditary power $< \omega_2$ and $f\colon \omega_2 \to H$ a bijection. Set $H_\alpha = f[\alpha]$ for each α, and $T_\alpha = f[S_\alpha]$ whenever $\operatorname{cf}(\alpha) = \omega_1$. Then, for each $Y \subseteq H$, the set $\{\alpha\colon Y \cap H_\alpha = T_\alpha\}$ is stationary.

Each forcing P_α defined below is of cardinality $\leqslant \omega_1$, and hence P_α-names under consideration can be regarded as elements of the set H.

Let $P_0 =$ the Cohen forcing and $P_\alpha = \sum_{\beta<\alpha} P_\beta$ (the direct limit), for each limit $\alpha < \omega_2$. If $\operatorname{cf}(\alpha) = \omega_1$ we look at T_α and if $P_\alpha \Vdash$ "T_α satisfies the c.c.-condition" then let $P_{\alpha+1} = P_\alpha * T_\alpha$; and if T_α is a disjoint union $T_\alpha = a \cup e \cup d$ of P_α-names of an almost disjoint family, of a subset of ω_1 and of a strongly discrete set of ultrafilters, respectively, and $P_\alpha \Vdash$ "there is an improvement of $P_e\{a\}$ w.r.t. d" then let $P_{\alpha+1} = P_\alpha * Q$, where Q is a P_α-name of such an improvement. Finally, let $P_{\alpha+1} = P_\alpha$ in each remaining case.

Let $P = \sum_{\alpha<\omega_2} P_\alpha$ and let $G \subseteq P$ be a generic filter.

Obviously, in $V[G]$ Martin's Axiom plus $2^{\omega_0} = \omega_2$ hold true. Now, let $D = \{F_\alpha\colon \alpha < \omega_1\} \in V[G]$ be a strongly discrete sequence of ultrafilters. Fix an almost disjoint family $\{A_\alpha\colon \alpha < \omega_1\}$ with $A_\alpha \in F_\alpha$ and a subset $E \subseteq \omega_1$. Then, for some $\beta < \omega_2$, both $\{A_\alpha\colon \alpha < \omega_1\}$ and E are in $V[G_\beta]$. The restricted sequence $D|\gamma = \{F_\alpha \cap V[G_\gamma]\colon \alpha < \omega_1\}$ need not belong to $V[G_\gamma]$ but it does for many γ's: the set $\{\gamma < \omega_2\colon D|\gamma \in V[G_\gamma]\}$ is ω_1-normal (i.e. it is unbounded in ω_2 and closed under ω_1-limits).

To see this let us encode D as

$$D = \{\langle \alpha, x\rangle\colon \alpha < \omega_1 \text{ and } x \in F_\alpha\}.$$

Then the restrictions are of the form

$$D|\gamma = \{\langle \alpha, x\rangle \in D\colon x \in V[G_\gamma]\}.$$

We choose a canonical P-name $\underset{\sim}{D}$ for D which consists of pairs $\langle\langle \alpha, x\rangle^{P}, p\rangle$, where $\underset{\sim}{x}$ is a canonical name for $x \subseteq \omega$, $p \in P$ and

$$\underset{\sim}{D}(\alpha, x) = \{p \in P\colon \langle\langle \alpha, \underset{\sim}{x}\rangle^{P}, p\rangle \in \underset{\sim}{D}\}$$

is an antichain. Define the subnames $\underset{\sim}{D}|\gamma$:

$$\underset{\sim}{D}|\gamma = \{\langle\langle \alpha, \underset{\sim}{x}\rangle^{P}, p\rangle \in \underset{\sim}{D}\colon \underset{\sim}{x} \in V^{P_\gamma} \text{ and } \underset{\sim}{D}(\alpha, x) \subseteq P_\gamma\}.$$

Then $\underset{\sim}{D}|\gamma$ is a P_γ-name. The set

$$C_1 = \{\gamma < \omega_2\colon \forall x, \alpha\ [x \in V^{P_\gamma} \to \underset{\sim}{D}(\alpha, x) \subseteq P_\gamma]\}$$

is ω_1-normal and for $\gamma \in C_1$ we have

$$(\underset{\sim}{D}|\gamma)[G_\gamma] = D|\gamma,$$

and thus $D|\gamma \in V[G_\gamma]$ for each $\gamma \in C_1$. Note that then $D|\gamma$ is a strongly discrete sequence of ultrafilters in $V[G_\gamma]$.

Now we take P_β-names a and e for $\{A_\alpha\colon \alpha < \omega_1\}$ and E, respectively. Since a and e are in H and $\underset{\sim}{D} \subseteq H$, it is easy to check that the set

$$C_2 = \{\gamma < \omega_2\colon (a \cup e \cup \underset{\sim}{D}) \cap H_\gamma = a \cup e \cup (\underset{\sim}{D}|\gamma)\}$$

is ω_1-normal. Let $C = C_1 \cap C_2$. Applying the principle $\diamondsuit$ and the lemma of the preceding section, we can find a large enough $\gamma \in C$, with $\operatorname{cf}(\gamma) = \omega_1$, for which $T_\gamma = a \cup e \cup (\underset{\sim}{D}|\gamma)$ and $P_\gamma \Vdash$ "there is an improvement of $P_e\{a\}$ w.r.t. $\underset{\sim}{D}|\gamma$". Hence $P_{\gamma+1} = P_\gamma * Q$ for a P_γ-name Q of such an improvement. Thus in $V[G_{\gamma+1}]$ there are sets $A, B \subseteq \omega$ such that in $V[G]$ we have

$$\{F_\alpha\colon \alpha \in E\} \subseteq A^*, \quad \{F_\alpha\colon \alpha \notin E\} \subseteq B^* \quad \text{and} \quad A^* \cap B^* = \emptyset,$$

which finishes the proof of the theorem.

If we wish to conclude, in addition, that $\beta\omega_1$ is not a continuous image of ω^* we have to combine our forcing with that in Frankiewicz [1985]. Let B denote the Boolean algebra contained in $P(\omega_1)$ generated by countable subsets of ω_1. We add one more case in the iteration: if $\operatorname{cf}(\alpha) = \omega_1$ and $P_\alpha \Vdash$ "T_α is an embedding of B into $P(\omega)/\mathrm{fin}$" then let $P_{\alpha+1} = P_\alpha * Q$, where Q is a P_α-name of a c.c.c. forcing making the gap $T_\alpha(L)$, for an L in B, indestructible.

5. We conclude the paper with some simple remarks on B-ideals. We include these remarks here since the method used below is very similar to that in Section 2.

A nonprincipal ideal J over ω is called a ***B*-*ideal*** if the following holds: whenever the sets A_n are in J and

$$\min_n A_n \to \infty$$

then, for some infinite $z \subseteq \omega$, $\bigcup\{A_n\colon n \in Z\} \in J$.

Burzyk [198•] uses such ideals to construct certain normed linear spaces. Observe that (the dual of) a P-point is a B-ideal. Indeed, if $A_n \in J$, then there is an $A \in J$ such that $A_n \subseteq_* A$ for each $n \in \omega$. Writing $e_n = A_n \setminus A$ we have, for any $Z \subseteq \omega$,

$$\bigcup\{A_n\colon n \in Z\} \subseteq A \cup \bigcup\{e_n\colon n \in Z\}.$$

If $\min_n A_n \to \infty$, then we can find a $Z \subseteq \omega$ such that $\{e_n\colon n \in Z\}$ is a disjoint family. For any partition $Z = Z_0 \cup Z_1$, the sets

$$\bigcup\{e_n\colon n \in Z_0\} \quad \text{and} \quad \bigcup\{e_n\colon n \in Z_1\}$$

are disjoint and hence one of them is in J; denote it by Y. Then we have

$$\bigcup\{A_n\colon n \in Y\} \subseteq A \cup \bigcup\{e_n\colon n \in Y\} \in J$$

and hence $\{A_n\colon n \in Y\}$ is in J.

Thus the existence of B-ideals follows, for example, from the Continuum Hypothesis. Now, it is easy to see that each B-ideal is a P-ideal (but not necessarily maximal). Indeed, suppose that $A_n \in J$, where J is a B-ideal. We may assume that the sequence is increasing. If $B_n = A_n \setminus n$, then the B_n's are in J and $\min_n B_n \to \infty$, and hence, for some Z, $B = \bigcup \{B_n\colon n \in Z\}$ is in J. But $A_n \subseteq_* B_n \subseteq B$ for each $n \in \omega$, and hence J is a P-ideal.

Finally, we prove the following

Proposition. *Assuming* CH, *there is a B-ideal, and hence a P-ideal, which cannot be extended to a P-point. In particular, there are nonmaximal B-ideals.*

Proof. The Balcar–Frankiewicz–Mills Theorem shows that the space $G(2^\omega)$ (the Gleason space of the Cantor set) can be embedded into ω^* as a closed P-set X. Hence the family

$$F = \{A \subseteq \omega\colon X \subseteq A^*\}$$

is a P-filter. If F were extendible to a P-point p then, since $\{p\} = \bigcap \{A^*\colon A \in p\}$ and $A \cap X \neq \emptyset$ for each $A \in p$, we would have $p \in X$, which is impossible, because X is separable and without isolated points. The dual $J = \{\omega \setminus A\colon A \in F\}$ is then a P-ideal not extendible to a P-point and, in fact, it is a B-ideal: suppose that $A_n \in J$ and $\min_n A_n \to \infty$. Let $A \in J$ almost contain each A_n and let $e_n = A_n \setminus A$. There is an infinite $Z \subseteq \omega$ such that $\{e_n\colon n \in Z\}$ is a disjoint family. It is possible to form 2^ω almost disjoint subunions $\bigcup \{e_n\colon n \in Z_\alpha\}$, for almost disjoint $Z_\alpha \subseteq Z$. One of them is in J, for otherwise we would have 2^ω nonempty open-closed disjoint subsets of X, which is impossible as $G(2^\omega)$ has countable cellularity.

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Nielsen reduction in free groups with operators

by

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Abstract. The Nielsen method is generalized to an equivariant situation, in which the variables of a free group are freely permuted by an operator group G. Critical elements $W = A \cdot x(A)^{-1}$, $x \in G$ occur, which are analysed in detail. An equivariant Grushko–Neumann Theorem is deduced and applications to low-dimensional CW-complexes are given.

§ I. Introduction. Let G be an arbitrary group, $F(a_1, \dots, a_n)$ a free group of finite rank, and let $\bar{F}$ be the normal closure of F in $G * F$. $\bar{F}$ is freely generated by the xa_ix^{-1}, $x \in G$, with G operating on $\bar{F}$ by conjugation. Alternatively we may think of $\bar{F}$ as a free group with basis $x(a_i)$ ($\mathrel{\widehat{=}} xa_ix^{-1}$), $x \in G$, which is freely permuted by G. The *length* of an element W of $\bar{F}$ is understood to be the length with respect to the (in general infinite) basis $x(a_i)$ and is denoted by $|W|$.

If $W_1, \dots, W_m$ are finitely many elements of $\bar{F}$, then we denote by $\mathrm{Gp}(W_1, \dots, W_m)$ the subgroup of $\bar{F}$ generated by the W_i; by $\overline{\mathrm{Gp}(W_1, \dots, W_m)}$ we denote the smallest G-invariant subgroup of $\bar{F}$ containing the W_i, i.e. the subgroup, which is generated by all $x(W_i)$, $x \in G$. $(W_1, \dots, W_m)$ is called a *G-generating system* of $\overline{\mathrm{Gp}(W_1, \dots, W_m)}$. *A* G-generating system is called (G-) *free* or a (G-) *basis* of $\overline{\mathrm{Gp}(W_1, \dots, W_m)}$, if the $x(W_i)$, $x \in G$, $i = 1, \dots, m$ are free in the ordinary sense. If a G-invariant subgroup of $\bar{F}$ has a G-basis, then this subgroup is said to be G-free.

$\mathrm{Gp}(W_1, \dots, W_m)$ remains unchanged if the m-tuple $(W_1, \dots, W_m)$ is subject to *Nielsen transformations* (NT), i.e. a finite sequence of the following elementary transformations:

(1)
(i) $W_i \to W_i^{-1}$ for some i (*inversion*),
(ii) $W_i \to W_iW_j$, $i \neq j$ (*multiplication*),
(iii) *deletion* of some W_i, where $W_i = 1$.

For $\overline{\mathrm{Gp}(W_1, \dots, W_m)}$ we may enlarge this list by

(2) (iv) $W_i \to x(W_i)$ for some i, $x \in G$ ((G-) *conjugation*).