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DRUKARNIA UNIWERSYTETU JAGIELLOŃSKIEGO W KRAKOWIE

Whitney continua of graphs admit all homotopy types of compact connected ANRs

by

Hisao Kato (Hiroshima)

Abstract. In this paper, it is proved that if X is a compact connected ANR, then there exist a graph G and a Whitney map ω for $C(G)$ such that for some $t \in (0, \omega(G))$, $\omega^{-1}(t)$ is homotopy equivalent to X.

1. Introduction. By a *compactum* we mean a compact metric space. A *continuum* is a connected compactum. Let $C(X)$ denote the hyperspace of nonempty subcontinua of the continuum X. $C(X)$ is metrized with the Hausdorff metric (see e.g. [4] and [8]). One of the most convenient tools in order to study the structure of $C(X)$ is a monotone map $\omega\colon C(X) \to [0, \omega(X)]$ defined by H. Whitney [13]. A map $\omega\colon C(X) \to [0, \omega(X)]$ is said to be a *Whitney map* for $C(X)$ provided that

(i) $\omega(\{x\}) = 0$ for $x \in X$, and

(ii) $\omega(A) < \omega(B)$ whenever $A, B \in C(X)$ and $A \subsetneq B$.

The continua $\{\omega^{-1}(t)\}$ are called the *Whitney continua* of X. We may think of the map ω as measuring the size of a continuum. Note that $\omega^{-1}(0)$ is homeomorphic to X and $\omega^{-1}(\omega(X)) = \{X\}$. Naturally, we are interested in the structure of $\omega^{-1}(t)$ $(0 < t < \omega(X))$. In [10], J. T. Rogers proved that for any continuum X and any Whitney map ω for $C(X)$, there is an induced injection $r^*\colon H^1(\omega^{-1}(t)) \to H^1(X)$, where $H^n(X)$ denotes the n-th Čech cohomology. Also, it is proved that if $H^1(X)$ is finitely generated, then for any Whitney map ω for $C(X)$ and some $t_0 \in (0, \omega(X))$, $H^1(\omega^{-1}(t)) \cong H^1(X)$ for $0 \leqslant t \leqslant t_0$ (see [10], [5] and [2]). In [9], A. Petrus showed that there is a Whitney map ω for $C(D)$ such that $H^2(\omega^{-1}(t)) \neq 0$ for some $t > 0$, where D is a 2-cell. This example shows that there is no injection $H^n(\omega^{-1}(t)) \to H^n(X)$ for $n \geqslant 2$. In [3, (2.6)], we showed that there is a graph (1-dimensional connected polyhedron) $G(n)$ such that for any Whitney map ω for $C(G(n))$, there is no injection $H^n(\omega^{-1}(t)) \cong Z \to H^n(G(n)) = 0$ $(n \geqslant 2)$ for some $t \in (0, \omega(G(n)))$. In fact, $\omega^{-1}(t)$ is homotopy equivalent to the n-sphere S^n $(n \geqslant 2)$. In [1], R. Duda has carried out an incredibly detailed study of hyperspaces of graphs. Also, see [3] for some results on Whitney continua of curves.

The aim of this paper is to prove that for any compact connected ANR X, there are a graph G and a Whitney map ω for $C(G)$ such that for some $t>0$, $\omega^{-1}(t)$ is homotopy equivalent to X, i.e., Whitney continua of graphs admit all homotopy types of compact connected ANRs.

2. Preliminaries. In this section, we list some facts which will be needed in the sequel.

The following fact is well known (see e.g. R. Brown, *Elements of modern topology*, McGraw-Hill, London 1968, p. 240).

(2.1) *Let* $\underline{X}=\{X_i\}_{i=1,2,\dots,n}$ *and* $\underline{Y}=\{Y_i\}_{i=1,2,\dots,n}$ *be finite families of compact* ANR*s such that* $\bigcap_{i\in E} X_i$ *and* $\bigcap_{i\in E} Y_i$ *are empty sets or* ANR*s for any subset* E *of* $\{1,2,\dots,n\}$. *Let* $f\colon \bigcup_{i=1}^n X_i\to\bigcup_{i=1}^n Y_i$ *be a map such that* $f(X_i)\subset Y_i$ *for each* $i=1,2,\dots,n$. *If* $f|\bigcap_{i\in E}X_i\colon \bigcap_{i\in E}X_i\to\bigcap_{i\in E}Y_i$ *is a homotopy equivalence for any subset* E *of* $\{1,2,\dots,n\}$, *then* f *is a homotopy equivalence.*

Let $(P,<)$ be a partially ordered space. Then a map $\omega\colon P\to[0,\infty)$ is said to be a *Whitney map* if

(i) $\omega(p)=0$ for $p\in\operatorname{Min}P$,
(ii) $\omega(p)<\omega(q)$ for $p<q$, and
(iii) $\omega(p)=\omega(q)$ for $p,q\in\operatorname{Max}P$.

Thus a Whitney map ω for $C(X)$ is a Whitney map in the above sense for $C(X)$ ordered by inclusion.

We need the following facts:

(2.2) (L. E. Ward, Jr. [11]). *Let* P *be a compact metric partially ordered space such that* $\operatorname{Min}P$ *and* $\operatorname{Max}P$ *are disjoint closed sets and let* Q *be a closed subset of* P *such that* $\operatorname{Min}Q\subset\operatorname{Min}P$ *and* $\operatorname{Max}Q\subset\operatorname{Max}P$. *Then a Whitney map for* Q *can be extended to a Whitney map for* P.

(2.3) (M. Lynch [6]). *Let* X *be any continuum and* $A\in C(X)$. *Then for any Whitney map* ω *for* $C(X)$ *and any* $t\in[\omega(A),\omega(X)]$, *the set*

$$C(A,\omega,t)=\{B\in\omega^{-1}(t)|\ B\supset A\}$$

is an AR.

3. Main theorem. First, we prove the following theorem.

(3.1) THEOREM. *Let* K *be a finite simplicial complex and* $|K|=P$. *Assume that* P *is connected. Then there exists a Whitney map* ω *for* $C(|K^1|)$ *such that for some* $t\in(0,\omega(|K^1|))$, $\omega^{-1}(t)$ *is homotopy equivalent to* P, *where* K^1 *denotes the* 1*-skeleton of* K.

Proof. Let K^n denote the n-skeleton of K. Consider the sets

$$\mathcal{S}=\{|L|\mid L \text{ is a subcomplex of } K^1 \text{ and } |L| \text{ is connected}\},$$
$$\mathcal{A}_1=\{|L|\in\mathcal{S}|\ |L| \text{ is contained in some simplex of } K\},$$
$$\mathcal{A}_2=\{|L|\in\mathcal{S}|\ |L| \text{ is not contained in any simplex of } K\}.$$

Then $\mathcal{S}=\mathcal{A}_1\cup\mathcal{A}_2$. Let $G=|K^1|$. Set $\mathcal{A}=F_1(G)\cup\mathcal{S}$ where $F_1(G)=\{\{x\}|\ x\in G\}$. Since $\mathcal{S}$ is a finite set, we can easily see that there is a map $\omega'\colon\mathcal{A}\to[0,\infty)$ such that

(i) $\omega'(B)=0$ for $B=\{x\}\in F_1(G)$,
(ii) $\omega'(B)<1$ for $B\in\mathcal{A}_1$,
(iii) $\omega'(B)>1$ for $B\in\mathcal{A}_2$, and
(iv) if $A,B\in\mathcal{A}$ and $A\subsetneqq B$, then $\omega'(A)<\omega'(B)$.

By (2.2), there is a Whitney map $\omega\colon C(G)\to[0,\omega(G)]$ which is an extension of ω'.

Now, we shall prove that $\omega^{-1}(1)$ is homotopy equivalent to P. For any $A\in C(G)$, consider the set

$$C(A,\omega,1)=\{B\in\omega^{-1}(1)|\ B\supset A\}.$$

By (2.3), if $\omega(A)\leqslant 1$, then $C(A,\omega,1)$ is a nonempty AR. If $\langle V,W\rangle\in K^1$, then $\omega(\langle V,W\rangle)<1$. Hence we have

$$\omega^{-1}(1)=\bigcup\{C(V,\omega,1)|\ V\in K^0\}.$$

Let $\{V_0,V_1,\dots,V_k\}\subset K^0$. Assume that $\langle V_0,V_1,\dots,V_k\rangle\in K$. Then

$$\bigcup\{\langle V_i,V_j\rangle|\ i,j\in\{0,1,\dots,k\}\}\in\mathcal{A}_1.$$

Hence we have

$$\bigcap_{i=0}^k C(V_i,\omega,1)\supset C(\bigcup\{\langle V_i,V_j\rangle|\ i,j\in\{0,1,\dots,k\}\},\omega,1)\neq\emptyset.$$

Conversely, we shall show that if $\bigcap_{i=0}^k C(V_i,\omega,1)\neq\emptyset$, then $\langle V_0,V_1,\dots,V_k\rangle\in K$. Let $A\in\bigcap_{i=0}^k C(V_i,\omega,1)$. Choose a connected subcomplex L of K^1 such that $|L|\subset A$ and $|L|\supset\{V_0,V_1,\dots,V_k\}$. Since $\omega(|L|)\leqslant\omega(A)=1$, $|L|$ is contained in some simplex of K, which implies that $\langle V_0,V_1,\dots,V_k\rangle\in K$. Hence we conclude that for any $\{V_0,V_1,\dots,V_k\}\subset K^0$, $\langle V_0,V_1,\dots,V_k\rangle\in K$ if and only if $\bigcap_{i=0}^k C(V_i,\omega,1)\neq\emptyset$.

Next, we shall show that if $\langle V_0,V_1,\dots,V_k\rangle\in K$, then $\bigcap_{i=1}^k C(V_i,\omega,1)$ is an AR. Let $\mathcal{L}=\{L\}$ be the family of subcomplexes L of K^1 such that $|L|\in\mathcal{A}_1$ and $|L|\supset\{V_0,V_1,\dots,V_k\}$. Then we can easily see that

$$\bigcap_{i=0}^k C(V_i,\omega,1)=\bigcup_{L\in\mathcal{L}}C(|L|,\omega,1).$$

Let $L_0\in\mathcal{L}$ be such that $|L_0|\subset\langle V_0,V_1,\dots,V_k\rangle\in K$. For each $L\in\mathcal{L}$, consider the set $A(L)=C(|L_0|\ \omega,1)\cup C(|L|,\omega,1)$. Note that $\bigcap_{i=0}^k C(V_i,\omega,1)=\bigcup_{L\in\mathcal{L}}A(L)$ and $A(L)$ is an AR. In fact, since $|L_0|\cup|L|\in\mathcal{A}_1$, $C(|L_0|,\omega,1)\cap C(|L|,\omega,1)=C(|L_0|\cup|L|,\omega,1)$ is an AR which implies that $A(L)$ is an AR. We shall show that if $\mathscr{P}$ is a subset of $\mathscr{S}$, then $\bigcap_{L\in\mathscr{P}} A(L)$ is an AR. In fact, we have

$$\begin{aligned}\bigcap_{L\in\mathscr{P}} A(L) &= \bigcap_{L\in\mathscr{P}} (C(|L_0|, \omega, 1)\cup C(|L|, \omega, 1))\\ &= C(|L_0|, \omega, 1)\cup\big(\bigcap_{L\in\mathscr{P}} C(|L|, \omega, 1)\big) \neq \emptyset .\end{aligned}$$

If $\bigcap_{L\in\mathscr{P}} C(|L|, \omega, 1) = C(\bigcup_{L\in\mathscr{P}} |L|, \omega, 1) \neq \emptyset$, then $\bigcup_{L\in\mathscr{P}} |L|$ is contained in some simplex σ of K; hence $|L_0|\cup\bigcup_{L\in\mathscr{P}} |L| \subset \sigma$. This implies that $\omega(|L_0|\cup\bigcup_{L\in\mathscr{P}} |L|) \leqslant 1$. Hence $C(|L_0|, \omega, 1)\cap\bigcap_{L\in\mathscr{P}} C(|L|, \omega, 1) = C(|L_0|\cup\bigcup_{L\in\mathscr{P}} |L|, \omega, 1)$ is an AR, which implies that $\bigcap_{L\in\mathscr{P}} A(L)$ is an AR. If $\bigcap_{L\in\mathscr{P}} C(|L|, \omega, 1) = \emptyset$ then $\bigcap_{L\in\mathscr{P}} A(L) = C(|L_0|, \omega, 1)$ is an AR. Consequently, we conclude that for any subset $\mathscr{P}$ of $\mathscr{S}$, $\bigcap_{L\in\mathscr{P}} A(L)$ is a nonempty AR. Hence we see that $\bigcap_{i=0}^{k} C(V_i, \omega, 1) = \bigcup_{L\in\mathscr{P}} A(L)$ is an AR.

Consider the barycentric subdivision $\mathrm{Sd}K$ of K and the decomposition $\{\mathrm{St}(V, \mathrm{Sd}K)|\ V\in K^0\}$, where the symbol $\mathrm{St}(V, \mathrm{Sd}K)$ denotes the closed star. We have proved that for any subset $\{V_0, V_1, \ldots, V_k\}$ of K^0, the following statements (a), (b) and (c) are equivalent.

(a) $\bigcap_{i=0}^{k} \mathrm{St}(V_i, \mathrm{Sd}K) \neq \emptyset$.

(b) $\langle V_0, V_1, \ldots, V_k\rangle \in K$.

(c) $\bigcap_{i=0}^{k} C(V_i, \omega, 1) \neq \emptyset$.

Moreover, $\bigcap_{i=0}^{k} \mathrm{St}(V_i, \mathrm{Sd}K)$ and $\bigcap_{i=0}^{k} C(V_i, \omega, 1)$ are ARs. Hence we can construct a map $f\colon P = \bigcup_{V\in K^0} \mathrm{St}(V, \mathrm{Sd}K) \to \omega^{-1}(1) = \bigcup_{V\in K^0} C(V, \omega, 1)$ such that

$$f(\mathrm{St}(V, \mathrm{Sd}K)) \subset C(V, \omega, 1)$$

for each $V\in K^0$. By (2.1), f is a homotopy equivalence. Hence $\omega^{-1}(1)$ is homotopy equivalent to P. This completes the proof.

In [12], J. E. West proved the following theorem.

(3.2) THEOREM (J. E. West [12]). *Every compact* ANR *has the homotopy type of a compact polyhedron.*

Hence we have

(3.3) THEOREM. *For every compact connected* ANR X, *there exist a graph G and a Whitney map ω for $C(G)$ such that for some $t\in(0, \omega(G))$ $\omega^{-1}(t)$ is homotopy equivalent to X.*

(3.4) THEOREM. *Let K be a finite simplicial complex and $P = |K|$. Assume that P is connected and $\dim P = n$. Then there is a Whitney map ω for $C(|K^1|)$ such that for some positive numbers $t_1 < t_2 < \ldots < t_n < \omega(|K^1|)$, $\omega^{-1}(t_i)$ $(i = 1, 2, \ldots, n)$ is homotopy equivalent to $|K^i|$, where K^i denotes the i-skeleton of K.*

Outline of proof. The proof is similar to the proof of (3.1). Consider the set

$$\mathscr{S} = \{|L|\mid L \text{ is a subcomplex of } K^1 \text{ and } |L| \text{ is connected}\} .$$

For each $j = 0, 1, \ldots, n$, consider the set

$$\mathscr{A}_j = \{|L|\in\mathscr{S}\mid |L| \text{ is contained in some } j\text{-simplex of } K\} .$$

Also, let

$$\mathscr{B} = \{|L|\in\mathscr{S}\mid |L| \text{ is not contained in any simplex of } K\} .$$

Choose real numbers t_i $(i = 1, 2, \ldots, n)$ such that $0 = t_0 < t_1 < t_2 < \ldots < t_n$. Since $\mathscr{S}$ is a finite set, we can define a map $\omega'\colon \mathscr{S}\cup F_1(|K^1|) \to [0, \infty)$ such that

(i) $\omega'(\{x\}) = 0$, for $\{x\}\in F_1(|K^1|)$,
(ii) $t_{j-1} < \omega'(A) < t_j$, for $A\in\mathscr{A}_j - \mathscr{A}_{j-1}$ $(j = 1, 2, \ldots, n)$,
(iii) $\omega'(A) > t_n$, for $A\in\mathscr{B}$,
(iv) if $A, B\in F_1(|K^1|)$ and $A\subsetneq B$, then $\omega(A) < \omega(B)$.

Then there is a Whitney map ω for $C(|K^1|)$ which is an extension of ω (see (2.2)).

Assume that $|L|\in\mathscr{S}$. Note that $\omega(|L|)\leqslant t_i$ $(i = 1, 2, \ldots, n)$ if and only if $|L|$ is contained in some simplex of K^i. In the same way as in the proof of (3.1), we see that $\omega^{-1}(t_i)$ is homotopy equivalent to $|K^i|$ $(i = 1, 2, \ldots, n)$.

(3.5) Remark. In [7], Lynch proved that if X is a compact connected 1-dimensional ANR and ω is any Whitney map for $C(X)$, then $\omega^{-1}(t)$ is an ANR for each $0\leqslant t\leqslant\omega(X)$. Also, in [3] we proved that if G is a graph and ω is any Whitney map for $C(G)$, then $\omega^{-1}(t)$ is a compact polyhedron for each $0\leqslant t\leqslant\omega(G)$.

For a neighborhood of $t = 0$, we have the following

(3.6) PROPOSITION (cf. [3, (2.3)]). *Let X be a compact connected 1-dimensional* ANR *and let ω be any Whitney map for $C(X)$. If X contains a simple closed curve, assume that $t_0 = \min\{\omega(S)|\ S$ is a simple closed curve in $X\}$. Otherwise, assume that $t_0 = \omega(X)$. Then $\omega^{-1}(t)$ is homotopy equivalent to X for each $0\leqslant t < t_0$.*

Proof. Since X is a Peano continuum, X admits a convex metric ϱ. The proof is similar to the proof of [3, (2.1) and (2.3)].

We close this paper with the following questions.

QUESTION 1. *Let G be a graph and let ω be any Whitney map for $C(G)$. What is the homotopy type of the Whitney continuum $\omega^{-1}(t)$? How many homotopy types do Whitney continua $\omega^{-1}(t)$ admit?*

QUESTION 2. *Let X be a continuum. Does there exist a curve (1-dimensional continuum) X' such that X' admits a Whitney map ω for $C(X')$ with the property that for some $t\geqslant 0$, $\omega^{-1}(t)$ has the same shape as X, i.e., $\mathrm{Sh}\,\omega^{-1}(t) = \mathrm{Sh}\,X$?*

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Sur la quasi-continuité et la quasi-continuité approximative

par

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Abstract. We prove that every cliquish (d cliquish) function is the limit of a sequence of quasi-continuous (d quasicontinuous) functions (d denotes the density topology).

Soient R l'ensemble des nombres réels et T une topologie dans R.

Une fonction f: $R \to R$ est dite T *quasi-continue* (T *cliquish*) au point $x \in R$ lorsqu'il existe pour tout nombre $\varepsilon > 0$ et pour tout entourage $U \in T$ du point x un ensemble nonvide $V \subset U$, $V \in T$ tel que

$$|f(t)-f(x)| < \varepsilon$$

pour tout $t \in V$ ($\operatorname{osc}_V f \leqslant \varepsilon$).

Étant fixés l'ensemble mesurable (au sens de Lebesgue) $A \subset R$ et le point $x \in R$, la limite supérieure

$$\limsup_{h \to 0^+} \frac{m(A \cap [x-h, x+h])}{2h}$$

est dite la *densité supérieure* de l'ensemble A au point x. S'il existe la limite

$$\lim_{h \to 0} \frac{m(A \cap [x-h, x+h])}{2h} = 1,$$

x est dit un point de densité de l'ensemble A.

La famille composée de l'ensemble vide et de tous les ensembles $A \subset R$ tels que tout point $x \in A$ est un point de densité de l'ensemble A est une topologie. Cette topologie est dite la *topologie de densité* ([5])

Désignons par T_e la topologie euclidienne et par T_d la topologie de densité dans R. Si K est une famille de fonctions f: $R \to R$, alors $B(K)$ désigne la famille de toutes les fonctions f de la forme $f = \lim_{n \to \infty} f_n$, où toutes les fonctions $f_n \in K$ ($n = 1, 2, \ldots$).

Désignons par Q (Q_d) la famille de toutes les fonctions T_e quasi-continues (T_d quasi-continues) et par P (P_d) la famille de toutes les fonctions T_e cliquish (T_d cliquish).