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On irreducibility and indecomposability of continua

by

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Abstract. Kuratowski (1927) showed that in metric continua their points of indecomposability are always points of irreducibility. The aim of this paper is to exhibit a general form of those Hausdorff continua for which the result of Kuratowski does not hold.

1. Introduction and preliminaries. In this paper X will always be a *Hausdorff continuum*, shortly a $\mathcal{T}_2$-*continuum*, i.e. a connected and compact topological space which satisfies the $\mathcal{T}_2$-axiom of separability. A point a of X is said to be a *point of indecomposability* of X if there is no decomposition of X into two proper subcontinua which both contain a, i.e. for every two subcontinua K_1 and K_2 of X

(1.1) $a \in K_1 \cap K_2$ and $K_1 \cup K_2 = X$ imply $K_1 = X$ or $K_2 = X$.

A point a of X is said to be a *point of irreducibility* of X if there is $b \in X$ such that no proper subcontinuum of X contains both a and b, i.e for every subcontinuum K of X

(1.2) $a \in K$ and $b \in K$ imply $K = X$;

X is then said to be *irreducible between* a *and* b.

Directly by the above two definitions, every point of irreducibility is a point of indecomposability, and the converse assertion:

(1.3) Every point of indecomposability is a point of irreducibility

has been proved for metric continua in [10] (Théorème XIX, p. 270). In connection with some fixed point theorems [4], [12], [13] and [15], the assertion (1.3) has been proved for hereditarily decomposable $\mathcal{T}_2$-continua in [14] (Theorem 1, p. 52, where in fact no axiom of separability is used). In [2], a $\mathcal{T}_2$-continuum has been constructed which is indecomposable but not irreducible, i.e. its every point is a point of indecomposability but no point is a point of irreducibility, so that the above assertion (1.3) is not true for an arbitrary $\mathcal{T}_2$-continuum X.

In the present paper, we shall characterize those $\mathcal{T}_2$-continua X which have this singularity, i.e. such X that

(1.4) There exists a point of indecomposability of X which is not a point of irreducibility of X;

we shall then say that X has the *Kuratowski singularity*. This characterization, proved in Chapter 3, is as follows: X is a union of two subcontinua $I \neq X$ and $C = \overline{X-I}$ such that I is irreducible between some point $a \in I$ and every point $b \in I \cap C$, every composant C_p of C (i.e. the union of all proper subcontinua of C which contain $p \in C$) meets I and C is indecomposable in X. We call C *indecomposable in* X (a notion extracted from the proofs of the fixed point theorems in [5], [6] and [8]) if for every two subcontinua $K_1, K_2 \subset X$

(1.5) $C \subset K_1 \cup K_2$ implies $C \subset K_1$ or $C \subset K_2$.

which is equivalent to saying that for every subcontinuum $K \subset X$

(1.6) $\mathrm{Int}_C K \neq \emptyset$ implies $C \subset K$;

namely (1.6) obviously implies (1.5), and (1.5) implies (1.6) by applying the *decomposition theorem* (cf. [11], p. 133): If C is a subcontinuum of a $\mathcal{T}_2$-continuum X and $X-C$ is disconnected, then there exists a decomposition of X into two proper subcontinua which both contain C.

We shall also prove in this paper that the Kuratowski singularity cannot be obtained when a metric continuum is taken for C in the above description. To this end, we shall give a common generalization of the results of [10] and [14] mentioned above (Remark 2 and Corollary 1 at the end of Chapters 3 and 4).

In the final Chapter 5, we include an improvement of the theorems of [3] and [17] on the connectedness of the set of all points of indecomposability of a $\mathcal{T}_2$-continuum (Corollaries 2–4).

Some results of this paper were communicated to the Topology Semester at the Banach Center in Warszawa on June 28, 1984.

2a. On points of indecomposability. Given a point $a \in X$ and a subcontinuum $C \subset X$, we shall say that X *irreducibly contains* a *and* C if X is minimal with respect to the property:

$$a \in X \quad \text{and} \quad C \subset X.$$

THEOREM 1. *A point* $a \in X$ *is a point of indecomposability of* X *if and only if there exists a subcontinuum* $C \subset X$ *such that*

(2.1) X *irreducibly contains* a *and* C;

(2.2) C *is indecomposable in* X;

and then either $C = \overline{\mathrm{Int}\, C}$ *or* C *reduces to one point.*

Proof. Sufficiency. Suppose (2.1) and (2.2) hold and take two subcontinua $K_1, K_2 \subset X$ such that $a \in K_1 \cap K_2$ and $K_1 \cup K_2 = X$. Then $C \subset K_1$ or $C \subset K_2$ in view of (1.5).

If $C \subset K_1$, then by (2.1) we have $K_1 = X$.

If $C - K_1 \neq \emptyset$, then $\mathrm{Int}_C K_2 \neq \emptyset$ and hence $C \subset K_2$ in view of (1.6) so that by (2.1) we have $K_2 = X$.

Necessity. Suppose that a is a point of indecomposability of X, i.e. (1.1) holds.

Let $C_\tau \subset X$, where $\tau \in \mathcal{T}$, be a nested family of nonempty subcontinua such that for every $\tau \in \mathcal{T}$

(2.3) X irreducibly contains a and C_τ.

Take a subcontinuum $K \subset X$ which contains a and $\bigcap_{\tau \in \mathcal{T}} C_\tau$. Then

(2.4) $\emptyset \neq \bigcap_{\tau \in \mathcal{T}} C_\tau \subset K$

and hence $K \cup C_\tau$ is a subcontinuum of X which contains a and C_τ for every $\tau \in \mathcal{T}$. By (2.3), $K \cup C_\tau = X$ for all $\tau \in \mathcal{T}$ and therefore $K \cup \bigcap_{\tau \in \mathcal{T}} C_\tau = X$. It follows from (2.4) that $K = X$, i.e. X irreducibly contains a and $\bigcap_{\tau \in \mathcal{T}} C_\tau$.

It follows from the Kuratowski–Zorn lemma that there exists a subcontinuum $C \subset X$ such that

(2.5) C is minimal with respect to the property (2.1);

in particular, C satisfies (2.1).

Before proving (2.2), we shall show that C is indecomposable.

Let C_1 and C_2 be subcontinua of C such that $C = C_1 \cup C_2$ and take, by using the Kuratowski–Zorn lemma, two subcontinua $K_1, K_2 \subset X$ such that for $i = 1$ and $i = 2$

(2.6) K_i irreducibly contains a and C_i.

Then $K_1 \cup K_2$ is a subcontinuum of X which contains a and C, and by (2.1) we have $K_1 \cup K_2 = X$. Since $a \in K_1 \cap K_2$ in view of (2.6), and a is a point of indecomposability of X by assumption, it follows from (1.1) that $K_1 = X$ or $K_2 = X$. Thus, by (2.6), X irreducibly contains a and C_i for $i = 1$ or $i = 2$. Hence, by (2.5), we have $C_1 = C$ or $C_2 = C$.

We have thus proved that

(2.7) C is indecomposable.

Now we can prove (2.2).

If $a \in C$, then by (2.1) we have $C = X$ and hence C is obviously indecomposable in X by (2.7).

If

(2.8) $a \in X - C$,

then by (2.1) and the decomposition theorem $X - C$ is connected and thus

(2.9) $\overline{X-C}$ is a subcontinuum of X.

Case 1. $\overline{X-C} \neq X$.

Since a is a point of indecomposability of X by assumption, by (2.8), (2.9) and the decomposition theorem $X - \overline{X-C}$ is connected. It follows from (2.7) that

(2.10) $\overline{X-\overline{X-C}} = C$.

Let K be a subcontinuum of X such that

(2.11) $\mathrm{Int}_C K \neq \emptyset$.

If $K\subset C$, then by (2.7) we have $C=K$, so that (1.6) implies (2.2).

If $K-C\neq\emptyset$, then $\overline{X-C}\cup K$ is a subcontinuum of X which contains a by (2.8) and (2.9). Hence $X-(\overline{X-C}\cup K)$ is connected by (2.1) and the decomposition theorem, and since this is an open and nowhere dense subset of C in view of (2.11), it follows from (2.7) that it is empty, i.e. $X=\overline{X-C}\cup K$. Therefore by (2.10) we have $C\subset K$. Thus, in view of (1.6) and (2.11), we have verified that (2.2) holds.

By (2.10), we have $\operatorname{Int}\overline{C}=C$ in case 1.

Case 2. $\overline{X-C}=X$.

First, it will be proved that

(2.12) X is irreducible between a and every $p\in C$.

To this end, take a subcontinuum K of X which contains a and p. Then $K\cup C$ is a subcontinuum of X which contains a and C and therefore, by (2.1), we have $K\cup C=X$. Hence $K=X$ by the equality $\overline{X-C}=X$ which proves (2.12).

By (2.12) and (2.5), C reduces to one point.

2b. On the irreducibility of a union of two continua. Given two subcontinua A and B of X and a point $p\in X$, we shall say that A is *irreducible between p and B* if A is minimal with respect to the property:

$$p\in A \quad \text{and} \quad A\cap B\neq\emptyset.$$

LEMMA 1. *If $p\in A-B$ and A is irreducible between p and B, then $\overline{A-B}=A$ and A is a unique subcontinuum of $A\cup B$ which is irreducible between p and B.*

Proof. Since $A-B$ is connected by the decomposition theorem, the formulas

$$p\in\overline{A-B} \quad \text{and} \quad \overline{A-B}\cap B\neq\emptyset$$

imply, by the irreducibility of A between p and B, that $\overline{A-B}=A$.

Further, if K is a subcontinuum of $A\cup B$ which is irreducible between p and B, then by the proved part of the lemma (with K taken for A) we have $K=\overline{K-B}$. Since $K-B\subset A-B$, it follows that $K\subset A$. Hence, by the irreducibility of A between p and B, we have $K=A$.

THEOREM 2. *Let C be an indecomposable subcontinuum of X such that X irreducibly contains a and C and let*

$$a\in X-C \quad \text{and} \quad \overline{X-C}\neq X.$$

Then a is a point of irreducibility of X if and only if $\overline{X-C}\cap C_p=\emptyset$ for some $p\in C$.

Proof. Sufficiency. Suppose that $\overline{X-C}\cap C_p=\emptyset$ for some $p\in C$ and let K be a subcontinuum of X which contains a and p. Since X irreducibly contains a and C, $\overline{X-C}$ is a subcontinuum of X by the decomposition theorem. By Lemma 1, C is a unique subcontinuum of X which is irreducible between p and $\overline{X-C}$. Hence $C\subset K$ and therefore $K=X$ because X irreducibly contains a and C by assumption.

Necessity. Let X be irreducible between a and some $p\in X$. Since $\overline{X-C}$ is a subcontinuum of X and $a\in\overline{X-C}\neq X$, we have $p\in X-\overline{X-C}$.

Suppose, on the contrary, that $\overline{X-C}\cap C_p\neq\emptyset$. Then there exists a proper subcontinuum L of C which contains p and meets the subcontinuum $\overline{X-C}$ of X. Then L is nowhere dense in C by the indecomposability of C, so that $\overline{X-C}\cup L$ is a proper subcontinuum of X which contains a and p, a contradiction.

Remark 1. The following is an extension to $\mathcal{T}_2$-continua of a theorem of [7] (p. 314, Theorem A): *A decomposable $\mathcal{T}_2$-continuum X is irreducible between its two points p and q if and only if there exist two subcontinua A and B of X such that $X=A\cup B$ and $A_p\cap B=\emptyset=A\cap B_q$.*

Sufficiency. If the above equalities hold, then by applying Lemma 1, A is uniquely irreducible between p and B and B is uniquely irreducible between q and A. Thus for every subcontinuum $K\subset X$ which contains p and q we have $A\subset K$ and $B\subset K$, and hence $K=X$.

Necessity. If X is decomposable and irreducible between p and q, then there exists a proper subcontinuum L of X such that $p\in\operatorname{Int}L$. Define

$$B=\overline{X-L} \quad \text{and} \quad A=\overline{X-B}$$

so that A and B are subcontinua of X by the irreducibility of X and the decomposition theorem. Also, A is irreducible between p and B, i.e. $A_p\cap B=\emptyset$. Analogously, by the irreducibility of X, we have $L\cap B_q=\emptyset$, and since $A\subset L$ by the definition of A and B, it follows that $A\cap B_q=\emptyset$.

3. A characterization of the $\mathcal{T}_2$-continua which have the Kuratowski singularity. We need one more

LEMMA 2. *Let C be a subcontinuum of a $\mathcal{T}_2$-continuum X and let $a\in X-C$. Then X irreducibly contains a and C if and only if $\overline{X-C}$ is a subcontinuum of X which is irreducible between a and C; and then $\overline{X-C}$ is irreducible between a and every point of $\overline{X-C}\cap C$.*

Proof. Sufficiency. If $\overline{X-C}$ is a subcontinuum irreducible between a and C, then by Lemma 1, $\overline{X-C}$ is a unique subcontinuum of X which is irreducible between a and C. Thus for every subcontinuum $K\subset X$ such that $a\in K$ and $C\subset K$ we have $\overline{X-C}\subset K$ and hence $K=X$.

Necessity. Suppose that X irreducibly contains a and C so that the set $X-C$ is connected by the decomposition theorem. Then $\overline{X-C}$ is a subcontinuum of X which satisfies the conditions

$$a\in\overline{X-C} \quad \text{and} \quad \overline{X-C}\cap C\neq\emptyset.$$

If, on the contrary, there exists a proper subcontinuum L of $\overline{X-C}$ such that

$$a\in L \quad \text{and} \quad L\cap C\neq\emptyset,$$

then there exists an open subset of X which is disjoint with L and meets C. Since $L \cap C \subset \overline{X-C} \cap C$ are boundary subsets of $\overline{X-C}$, it follows that $L \cup C$ is a proper subcontinuum of X which contains a and C, a contradiction.

Theorems 1 and 2, in view of Lemmas 1 and 2, directly imply

THEOREM 3. *A $\mathcal{T}_2$-continuum X has the Kuratowski singularity if and only if X is a union of two subcontinua*

$$I \neq X \quad \textit{and} \quad C = \overline{X-I}$$

such that the following three conditions are satisfied:

(3.1) *I is irreducible between some $a \in I$ and every $b \in I \cap C$,*

(3.2) *$I \cap C_p \neq \emptyset$ for every $p \in C$,*

(3.3) *C is indecomposable in X.*

We now give three examples illustrating the above description of the Kuratowski singularity.

EXAMPLE 1. Let C be an indecomposable $\mathcal{T}_2$-continuum with exactly one composant, as constructed in [2]. Then (3.1)–(3.3) are fulfilled with I reduced to a point of C.

EXAMPLE 2. Let C be an indecomposable $\mathcal{T}_2$-continuum with exactly two composants (as constructed in [2]) and let A be any arc having only its end points p and q in common with C. Take a homeomorphic image I of the $\sin 1/x$ — curve having exactly the arc A for the continuum of condensation and let a be the end point of I which is opposite to A. Define

$$X = I \cup C \quad \text{so that } I \cap C = \{p, q\}.$$

Then the conditions (3.1)–(3.3) are satisfied (only (3.3) needs a longer proof which consists in showing that $K \cap C$ has exactly two components for every subcontinuum K of X such that $\emptyset \neq K \cap C \neq C$, and then $\mathrm{Int}_C K = \emptyset$ which verifies (1.6)).

EXAMPLE 3. Let C be the simplest Knaster indecomposable continuum on the Cartesian plane (as described in [11], p. 204, Example 1) and let A denote the unit arc of the x-axis, so that the intersection $A \cap C$ is the Cantor set. Take again a homeomorphic image of the $\sin 1/x$ — curve having exactly A for the continuum of condensation and let a be the end point of I which is opposite to A. Define

$$X = I \cup C \quad \text{so that } I \cap C = A \cap C.$$

Then X does not have the Kuratowski singularity, because the condition (3.3) is obviously not satisfied.

Remark 2. The Kuratowski singularity cannot be realized when any indecomposable metric continuum C is taken in the description given in Theorem 3, as will follow from Corollary 1 at the end of Chapter 4. The problem whether the Kuratowski singularity can be obtained with any indecomposable $\mathcal{T}_2$-continuum C having infinitely many composants remains open.

4. Points of unique irreducibility. We need yet another generalization of irreducibility (which appears implicitly in papers [4], [5], [6], [12], [13] and [15], and also in Chapter 2a of this paper). Namely, a point p of a subcontinuum C of a $\mathcal{T}_2$-continuum X is said to be a *point of unique irreducibility* of C in X if there is $q \in C$ such that C is a unique subcontinuum of X which is irreducible between p and q.

LEMMA 3. *Let C be an indecomposable subcontinuum of X. If C is uniquely irreducible between p and q in X, then for all*

$$p' \in C_p \quad \textit{and} \quad q' \in C_q$$

C is uniquely irreducible between p' and q' in X.

Proof. Take subcontinua $K \subset C_p$ and $L \subset C_q$ so that

(4.1) $C = \overline{C-(K \cup L)}$,

(4.2) $p, p' \in K$ and $q, q' \in L$,

and let M denote an arbitrary subcontinuum of X such that

(4.3) $p', q' \in M$.

Then $K \cup M \cup L$ is a subcontinuum of X which contains p and q by (4.2) and (4.3). By assumption, $C \subset K \cup M \cup L$ and hence $C-(K \cup L) \subset M$. By (4.1), $C \subset M$.

LEMMA 4. *Let C be an indecomposable subcontinuum of X and $p \in C$. If p is a point of unique irreducibility of C in X, then C is indecomposable in X*

Proof. Let C be uniquely irreducible between p and q. Since for every subcontinuum K of X with $\mathrm{Int}_C K \neq \emptyset$ we have $C_p \cap K \neq \emptyset \neq K \cap C_q$ (cf. [11], p. 209, Theorem 2 and its proof valid also for $\mathcal{T}_2$-continua), by applying Lemma 3 with arbitrary $p' \in C_p \cap K$ and $q' \in K \cap C_q$, we have $C \subset K$.

Consider now the converse implication:

(4.4) If C is indecomposable in X, then there exists a point of unique irreducibility of C in X,

which appears in the proof of [5] and in a theorem of [9] (p. 680, Theorem 10). The assertion (4.4) is not true in general as the Example 2 above shows. Theorem 3, in view of Lemmas 1–3, directly implies

THEOREM 4. *A $\mathcal{T}_2$-continuum X has the Kuratowski singularity if and only if X irreducibly contains some point a and a subcontinuum C with* $\mathrm{Int}\, C \neq \emptyset$ *such that* (4.4) *is not satisfied.*

However, (4.4) holds when C is metrizable, as will be shown in the next theorem which yields a generalization of a crucial argument of [6] through a modification of the proof of [16].

THEOREM 5. *Let C be a metrizable subcontinuum of a $\mathcal{T}_2$-continuum X. Then C is indecomposable in X if and only if C is indecomposable and contains a point of unique irreducibility in X (and then every point of C has this property).*

Proof. Sufficiency follows directly from Lemma 4.

Necessity. Let $a \in C$ and let

$$B(p_1),\, B(p_2),\, \ldots$$

be a sequence of open balls in C such that for every $n = 1, 2, \ldots$,

(4.5) $p_n \in B(p_n) \subset C - \{a\}$,

(4.6) the balls $B(p_n)$ are a base of $C - \{a\}$.

For every $n = 1, 2, \ldots$, let K_n denote the component of a in $X - B(p_n)$ so that

(4.7) $K_n \subset X - B(p_n)$ for all $n = 1, 2, \ldots$

Let $C(a, X)$ denote the set of all $p \in C$ such that there exists a subcontinuum L of X satisfying

(4.8) L is irreducible between a and p

and $L \neq C$. Then, in view of (4.8) and the unique irreducibility of C in X, we have

(4.9) $C - L \neq \emptyset$.

With this notation, the following equality holds true:

(4.10) $C(a, X) = \bigcup_{n=1}^{\infty} K_n \cap C(a, X)$.

Indeed, if $p \in C(a, X)$, then there exists a subcontinuum $L \subset X$ which is irreducible between a and p, and then, in view of (4.6), (4.8) and (4.9), we have $L \cap B(p_n) = \emptyset$ for some n. It follows, according to (4.7), that $p \in K_n$.

Now, in order to apply the Baire category theorem, we shall verify that

(4.11) $\mathrm{Int}_C(K_n \cap C(a, X)) = \emptyset$ for all $n = 1, 2, \ldots$

But contradicting (4.11), we have $\mathrm{Int}_C K_m \neq \emptyset$ for some m, and then $C \subset K_m$ by the indecomposability of C in X. Hence $B(p_m) \cap C \subset B(p_m) \cap K_m$, and thus, by (4.7), we have $B(p_m) \cap C = \emptyset$, a contradiction to (4.5).

Theorems 4 and 5 directly imply a common generalization of the theorems of [10] and [14] mentioned in the introduction (cf. [10], p. 270, Théorème XIX and [14], p. 52, Theorem 1):

Corollary 1. *If every indecomposable subcontinuum of a $\mathscr{T}_2$-continuum X is metrizable, then every point of indecomposability of X is a point of irreducibility of X.*

5. Further corollaries. Following [3] and [17], we shall denote by E_X the set of all points of indecomposability of a $\mathscr{T}_2$-continuum X. Theorems 1 and 3, in view of Lemma 2 and some results of [10], imply

Theorem 6. *E_X is connected and*

(5.1) $\emptyset \neq E_X \neq X$

if and only if X has the Kuratowski singularity and X is decomposable.

Proof. Sufficiency. Assume that, according to Theorem 3, there exists a decomposition

(5.2) $I \cup C = X$

into subcontinua I and C such that the conditions (3.1)–(3.3) are satisfied for some $a \in E_X$ and all $b \in I \cap C$. We have

(5.3) $a \in I - C$

because X is decomposable by assumption.

For every $p \in C$, there exists a decomposition of X into two proper subcontinua: $K_1 = C \neq X$ by (5.2) and (5.3), and $K_2 = K \cup I$ where K is any proper subcontinuum of C which contains p and meets I by (3.2). It follows that

(5.4) $E_X \subset I$

by (5.2) and the definition (1.1) of the point of indecomposability.

If, on the contrary, $E_X \cap I_b \neq \emptyset$, then there exists $q \in E_X$ such that

$$q \in I_b$$

and then we have a decomposition of X into two proper subcontinua: $K_1 = I \neq X$, because of the decomposability of X, and $K_2 = L \cup C$ where L is any proper subcontinuum of I which contains q and b, by the definition of the composant I_b.

If follows, in view of (5.4), that $E_X \subset I - I_b$.

To prove the converse inclusion, let $a' \in I - I_b$. Since $I - I_b = I - I_{b'}$ for every $b' \in I \cap C$, it follows that X irreducibly contains a' and C in view of Lemma 2. By Theorem 1, according to (3.3), we have $a' \in E_X$.

Consequently, $E_X = I - I_b$, and the set $I - I_b$ is connected (cf. [11], p. 210, Theorem 3 and its proof valid also for $\mathscr{T}_2$-continua).

Necessity. Suppose, on the contrary, that the Kuratowski singularity does not occur so that for every $a \in E_X$ the $\mathscr{T}_2$-continuum X is irreducible between a and some $b \in X$. Since then we have $E_X = (X - X_b) \cup (X - X_a)$ by the irreducibility of X, the connectedness of E_X implies that X is indecomposable, a contradiction.

Theorems 3 and 6 immediately imply the following improvements of the results of [17] (p. 208–210, Theorems 1 and 2):

Corollary 2. *E_X is nonempty and connected if and only if either X is indecomposable or there exists a decomposition of X into two proper subcontinua I and $C = \overline{X - I}$ which satisfy the conditions (3.1)–(3.3), and then $E_X = I - I_b$ for all $b \in I \cap C$.*

Corollary 3. *E_X is disconnected if and only if X is decomposable and irreducible between two points.*

To prove the next corollary, let us note the following

Lemma 5. *Let a $\mathscr{T}_2$-continuum X be a union of two subcontinua $I \subset X$ and $C = \overline{X - I}$. Then*

(i) *C is indecomposable in X if and only if the quotient space X/I is an indecomposable $\mathscr{T}_2$-continuum.*

(ii) *$I \cap C_p \neq \emptyset$ for all $p \in C$ if and only if the $\mathscr{T}_2$-continuum X/I has exactly one composant.*

Proof. (i) follows from the obvious fact that, considering the natural function f from X onto X/I, for every subcontinuum $K \subset X$ the continuum $f(K)$ has nonempty interior in X/I if and only if K has nonempty interior in C.

(ii) Necessity is obvious.

Sufficiency. Suppose, on the contrary, that $C_p \cap I = \emptyset$ for some $p \in C-I$. Then it suffices to show that the composant of $P = \{p\} = \{f(p)\}$ in X/I is disjoint with $\{I\}$.

But otherwise there is a proper subcontinuum K of X/I which contains the elements P and I of X/I. Then the continuum $f^{-1}(K)$ has empty interior $\mathrm{Int}_C f^{-1}(K)$, and on the other hand by Lemma 1, $C \subset f^{-1}(K)$, a contradiction.

Theorem 3 reformulated by using Lemma 5 yields the following improvement of [3] (p. 191, Main Theorem):

COROLLARY 4. *E_X is connected and $\emptyset \neq E_X \neq X$ if and only if there exists a proper (and not reduced to one point) subcontinuum $I \subset X$ which is irreducible between some $a \in I$ and every $b \in I \cap \overline{X-I}$, and the quotient space X/I is an indecomposable $\mathscr{T}_2$-continuum with exactly one composant.*

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