Man erhält solche Beispiele für jeden Drehnenner $m = p^x p'^y \bar{m}$, für den es Zahlen a gibt mit der Eigenschaft:

$$a \equiv +\text{quadr. Rest} \bmod p^x,$$
$$a \equiv -\text{quadr. Rest} \bmod p'^y,$$
$$a \not\equiv \pm\text{quadr. Rest} \bmod \bar{m}, \text{ falls } \bar{m} > 1,$$

z.B. für $m = 21 = 7 \cdot 3$:

Nach Satz 2 gilt: $L_{21}(1) \times L_{21}(1) \simeq L_{21}(8) \times L_{21}(1)$,

$$\text{denn } 8 \equiv 1 \bmod 7,$$
$$\text{und } 8 \equiv -1 \bmod 3,$$

aber die Kongruenzklasse 8 (mod 21) ist nicht enthalten in der Menge

$$\{\pm\text{quadr. Reste } \bmod 21\} = \{1, 4, 5, 16, 17, 20\}.$$

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Dual properties within graph theory

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Abstract. Graph-theoretic duality lacks many of the nice features of matroid duality. In particular, equivalent statements need not dualize into equivalent statements, and so properties such as being eulerian cannot strictly be said to have graph-theoretic duals. Lacking natural examples of this for the traditional circuit/cutset duality, an alternative duality between vertices and minimal spanning sets of edges is discussed. In this context, completeness can be characterized in two natural ways, which dualize in simple but quite nonequivalent ways.

1. Introduction. Graph theory is one of the most easily accessible, yet widely applicable areas of discrete mathematics. Within graph theory, duality has become so central a concept that it has been cited as being a major justification for even calling graph theory a "theory." Yet misconceptions abound concerning the uses and limitations of graph-theoretic duality, largely due to confusion with a more general notion called matroidal duality, which was partially created to "explain" the graph-theoretic variety. Despite the frequency of reference to (and, at times, invocation of) duality, surprisingly little attention has been paid to the phenomenon itself (except for a few papers primarily interested in its applicability to electrical engineering, such as [8], [9], [12], and [4]).

Section 2 describes this duality, emphasizing its seldom-realized limitation that, within graph theory itself, concepts do not have dual concepts. This definitely conflicts with the way graph theorists commonly think and talk. It is rather only specific syntactical formulations of concepts which can be dualized. The difficulty is that statements which are equivalent for all graphs can dualize in nonequivalent ways. Unfortunately, the only known examples of this behavior are quite contrived. Hence, in Section 3 we introduce an alternative concept of duality, which has almost exactly the same logical structure as the traditional duality, but in which these limitations are nearer the surface and so can be naturally illustrated. Moreover, this alternative duality has the advantage over matroid-based duality of being built around the common notion of vertex. Precisely because it lacks an enveloping self-dual context such as matroid theory, this vertex-based duality can serve as a laboratory for studying the problematic nature of traditional duality.

2. The nature of duality. While many "dualities" have been observed in graph theory (see, for instance, [2]), our concern is with the traditional circuit/cutset (or cycle/cocycle) duality of Whitney, as is emphasized in [15]. (We follow the notation and terminology of that text, although the fundamental concepts which we employ are common to almost every presentation of graph theory.)

The simplest approach to duality begins with plane graphs. Interchanging the roles of vertices with faces while essentially preserving the role of edges produces new graphs which are dual to the originals; see [15, Section 15]. In so doing, other objects become interchanged; e.g., circuits with cutsets (i.e., with minimal disconnecting sets of edges) and spanning trees with complements of spanning trees. Certain properties of graphs are preserved and called *self-dual*, such as 2-connectedness; other pairs of properties become interchanged, such as being eulerian with bipartiteness. If connectivity is assumed, then the dual of a dual is isomorphic to the original. If at least 3-connectedness is assumed, then isomorphic graphs have isomorphic duals, corresponding to the well-known duality of polyhedra.

To be maximally useful, more conventions are needed. For instance, if isthmuses are to be allowed, then we must also allow their dual structures: loops. Similarly, allowing vertices of degree two can necessitate allowing multiple edges. In other words, if we allow cutsets of sizes one and two, then our notion of graph should also allow circuits of sizes one and two. If nonplanar graphs are to be allowed, then we must also allow their dual structures, but it is well-known that these dual structures cannot be graphs.

Thus attempts to approach duality by means of dual structures require more general sorts of structures and so a more general theory. The theory of matroids [14] has become the traditional answer to this need. (We suppose, but do not directly use, some basic knowledge of matroid theory as in [15] or [16]; [13] provides a comprehensive treatment.) Although faces no longer have any meaning, and vertices have no general matroidal counterpart, Whitney showed that interchanging the matroidal counterparts of circuits and cutsets still allows all matroids to dualize into matroids, with all graphs interpretable as special (graphic) matroids and with matroidal duality agreeing with the geometric duality for planar graphs; see [15, section 32]. Assuming at least 3-connectedness again insures that duality preserves isomorphism.

The duality of eulerian and bipartite disappears, however, in the matroidal generalization: the dual (matroid) of an eulerian matroid need not be bipartite. If we further assume that the matroidal counterparts of circuits and cutsets always intersect evenly, as they do in graphs, then we are in the context of binary matroids and the eulerian/bipartite duality is restored. But the point is, the duality of these two properties depends upon the context in which we are generalizing graphs. Moreover, this structural approach does not directly discuss in what sense (if any) eulerian and bipartite are duals within the theory of graphs itself despite the common tendency among graph theorists to think of them as duals.

The approach using dual structures to dualize concepts such as eulerian thus

fails, due to the family of graphs not being closed under dualization. It is natural to switch instead to a syntactical approach, looking at statements about graphs, rather than at graphs themselves. Given a statement mentioning only edges, circuits, cutsets, and basic set-theoretic notions such as membership and cardinality, we can produce a dual statement by simply interchanging all mentions of "circuits" with mentions of "cutsets." For instance, the characterization "every cutset is even" of eulerian dualizes to the "every circuit is even" characterization of bipartite. This approach is very attractive to graph theorists, promising to transform concepts into dual concepts and theorems into dual theorems, just as in projective geometry; there is a clear hope of "something for nothing." But while, with due caution, this all happens in projective geometry (and in matroid theory), we shall see that graph-theoretic duality is a very different matter.

Caution is always needed in using duality whenever statements involve notions other than the basic fixed and interchanged elements: edges and circuits/cutsets in our case. For instance, "subgraph" cannot be left unchanged, since it involves edge deletion, which matroid theory demands is dual to edge contraction. Thus it is natural to look at other notions as abbreviations of their definitions in terms of the primitive notions. For instance, an isthmus can be defined as an edge which is in no circuit, and so mentions of isthmuses could be replaced by mentions of edges which are in no circuits and so dualized to edges which are in no cutsets; i.e., to loops. But isthmuses can be characterized in other ways; e.g., as singleton cutsets. What if isthmuses could be characterized in terms of edges, circuits and cutsets in such a way that this characterization dualizes to something other than loops? Or, more realistically, what if a graph property such as eulerian could become dualized in nonequivalent ways? To justify a "replacement" technique for defined terms, we would have to know that equivalent statements (i.e., statements equivalent for all graphs, although perhaps not for all matroids) dualize into equivalent statements, and so that dualization is well-defined. But, in fact, duality runs into just this sort of trouble.

It is easier to look at the broader question of whether every theorem dualizes into a theorem. The answer is "no"; as a counterexample, suppose $\varkappa$ is a statement asserting the existence of exactly ten edges forming 37 circuits and 15 cutsets precisely as in K_5 (the complete graph on five vertices), and let $\varkappa^*$ be the dual statement. Then $\varkappa$ contains more than enough to characterize being isomorphic to K_5 and, since no graph is dual to K_5, $\neg\varkappa^*$ (the negation of $\varkappa^*$) is a theorem, yet $(\neg\varkappa^*)^*$ ($\equiv \neg\varkappa$) is not a theorem. While working graph theorists rightly object to the artificiality of this example, it illustrates how more natural theorems may well to carry over to dual theorems. There is, of course, no logical boundary separating artificial from natural examples.

This situation is very different from that in projective geometry or matroid theory where there are self-dual axiomatizations which force theorems to dualize into theorems. There is no self-dual axiomatization of the edge/ciruit/cutset structure of graph theory, unless we are willing to extend the theory into matroids or perhaps to restrict it to the plane. Things stop sort of complete chaos, however: while the dual of a theorem need not be a theorem, it will at least still be true of all planar graphs and so cannot be inconsistent.

As a special case, equivalent statements do not necessarily dualize into equivalent statements. Hence, in the nonself-dual setting of graph theory, instead of dualizing a concept such as eulerian, we can only dualize a specific formulation. If a concept can be characterized by a statement entirely in terms of edges, circuits and cutsets (and it is an interesting question which concepts can be so characterized), then it is trivial to find the dual statement. But it is quite possible that the concept could be characterized in a second way (although the equivalence of the two would not hold in the more general, and so weaker, theory of matroids) which would have a different dual statement. It is not clear how to illustrate this phenomenon for graph theory without imposing major artificiality (e.g., conjuncting statements with the valid statement $\neg\varkappa^*$ used earlier). It would be very interesting to find a natural example, but since such an example would have to somehow involve nonplanarity, there is most likely no simple natural example. We shall be able to give very simple, natural examples for the alternative duality discussed in Section 3.

We are not meaning to suggest that duality is at all useless, but only to emphasize that it is usually matroidal or planar duality which is really being used, rather than graph-theoretic duality. Yet the dual of a theorem is at least a good candidate for being a theorem: to show that it is, we can either show that the original theorem holds more broadly—say for all binary matroids—or show that the steps of the original proof can each be modified so as also to prove the dual; see [12]. But, either way, the second theorem does not follow automatically from the mere syntactical formulation of the original.

3. A vertex-based duality. The matroids based on circuits and on cutsets are not the only matroids definable on the edges of a graph; see [11]. Their predominance is at least partly due to the naturalness of the concept of circuit. Vertices are equally natural, but are pointedly absent in matroid theory. Thus it would be desirable to consider a notion of duality which interchanges vertices with something. Vertex/face polyhedral duality is one possibility; another is the vertex/edge duality of Ore [10] which, although it has led to the important topic of line graphs, is not sustainable as an actual "duality." But it is Ore's book which takes the first steps toward what can be viewed as another duality based on vertices.

We can justify calling this a "duality" within an even more general context than matroid theory. Following Woodall [17], two families $\mathscr{F}$ and $\mathscr{F}'$ of sets of edges of a graph are said to be *Menger duals* of each other if and only if the members of $\mathscr{F}'$ are precisely the minimal sets of edges which have at least one edge in common with each member of $\mathscr{F}$. Matroid duality can be viewed as a special case of Menger duality by focusing on a distinguished edge.

In our alternative duality, vertices are viewed as sets of edges having a common endpoint, with vertices of degree one forbidden since they would, as edge sets, be contained in other vertices. The objects (Menger) dual to vertices are minimal spanning sets of edges (called "minimal covering graphs" by Ore and "minimal line covers" in [3]); we call them *minimal covers*. Thus, vertices are minimal edge sets which meet every minimal cover at least once, and minimal covers are minimal edge sets which meet every vertex at least once. This duality is particularly intriguing because of the local nature of vertices versus the global nature of minimal covers.

In addition to forbidding vertices of degree one, we assume that all graphs discussed are connected (preventing awkward multiplication of the number of minimal covers) and that there are neither loops nor multiple edges. Theorem 13.2.1 of [10] characterizes minimal covers as spanning forests of diameter less than or equal to two; i.e., as spanning families of vertex-disjoint star subgraphs.

Completeness can be characterized as "every two vertices contain a common edge", which is easily seen to be equivalent to "every vertex contains a minimal cover." (This sort of syntactic transformation is not only common to, but is the logical essence of both traditional duality and all Menger dualities; see [6, 7].) These two characterizations of completeness happen to have graph-theoretically equivalent duals, which are the subject of the following theorem.

THEOREM [10, Theorem 13.2.3]: *For any graph G, the following are equivalent.*

(i) *Every two minimal covers of G contain a common edge.*

(ii) *Every minimal cover of G contains a vertex.*

(iii) *G is simply an odd circuit graph.*

It is important to realize that this theorem does not mean that odd circuit graphs are somehow "duals" of complete graphs: as in Section 2, the concept of completeness will not have a dual within graph theory. This corresponds to graphs not having "dual graphs." (While you could start with a graph G and imagine forming a new graph having vertices corresponding to the minimal covers of G, it is unclear what the edges would be and how the process could be made idempotent.) Conceivably, our context could be enlarged (much as matroid theory enlarges graph theory) so that graphs could be vertex/minimal-cover duals of some sort of nongraph structure. But it is not clear what this extension beyond graph theory would involve, except that it should be closed under duality. This means that the glaring disparateness between vertices and minimal covers would have to be resolved; e.g., the cognates of edges would have to be capable of being in several of the cognates of vertices, just as edges can be in many minimal covers. Thus this extension beyond graphs might involve hypergraphs.

The best way to see that being an odd circuit graph is not the dual of being complete is to see that a characterization of completeness can be dualized into something else—in fact, completeness can easily be characterized in a self-dual manner, thus dualizing into itself. This follows from the following proposition, observing that dualizing clauses (i) and (ii) trivally characterizes completeness.

PROPOSITION. *A graph G is complete if and only if there is a family $\mathscr{S}$ of minimal covers of G such that:*

(i) *every edge of G is in exactly two minimal covers from $\mathscr{S}$, and*
(ii) *every two minimal covers from $\mathscr{S}$ contain exactly one edge of G.*

Proof. If G is complete, take $\mathscr{S}$ to be the family of all vertices of G. Conversely, suppose there is such an $\mathscr{S}$ and consider the graph G' having vertices which correspond to the minimal covers in $\mathscr{S}$, with two vertices of G' adjacent if and only if the corresponding minimal covers intersect. By (i) and (ii), the edges of G' correspond to the edges of G, and so the completeness of G' implies the same of G.

One interesting question suggested by this self-dual characterization of completeness within this alternative duality is whether every concept has a self-dual characterization. The same question is, of course, much more interesting for the traditional circuit/cutset duality. Coming from a completely different point-of-view, [5] characterizes those graph-theoretic concepts which have self-dual characterizations within the narrower context of planar graphs.

While there are many questions which this alternative duality suggests for study, we conclude by merely mentioning an intriguing possibility. Woodalls's notion of Menger duality itself has a dual, which is called Konig duality. (Konig duality is obtained from Menger duality by replacing "minimal" with "maximal" and "at least" with "at most"; see [17] or [6].) This is especially nice in our vertex-based setting: the Konig dual of vertices are maximal matchings (i.e., maximally independent sets of edges). Hence the theorem of Gallai [1] is relevant: The sum of the minimum number of edges in a minimal cover with the maximum number of edges in a maximal matching—i.e., the sum of the extreme sizes of the Menger and Konig duals of vertices—always equals the number of vertices.

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