[6] A. Hajnal, *Proof of a conjecture of S. Ruziewicz*, Fund. Math. 50 (1961), 123–128.
[7] S. Mercourakis, *The structure of K-analytic spaces and their relation with the Corson compact spaces* (Thesis University of Athens 1984) (In Greek).
[8] S. Piccard, *Sur un problème de M. Ruziewicz de la théorie des relations*, Fund. Math. 29 (1937), 5–9.
[9] H. Rosenthal, *The heredity problem for weakly compactly generated Banach spaces*, Compositio Math. 28 (1974), 83–111.
[10] M. Talagrand, *Sur une conjecture de H. H. Corson*, Bull. Sci. Math. (Série 2) 99 (1975), 211–212.
[11] M. Talagrand, *Espaces de Banach faiblement K-analytiques*, Ann. of Math. 110 (1979), 407–438.
[12] L. Vašak, *On one generalization of weakly compactly generated Banach spaces*, Studia Math. 70 (1980), 11–19.

MATHEMATICAL INSTITUTE
ATHENS UNIVERSITY
Solonos 57
Athens 27

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Automorphisms of the Loeb algebra

by

David Ross (Iowa City)

Abstract. Let $(\Omega, L(\Omega), L(\mu))$ be a uniform hyperfinite probability space in a sufficiently saturated nonstandard model of analysis. We prove: 1. Every automorphism of the measure algebra over Ω is induced by an invertible point transformation. 2. Some automorphisms are *not* given by *internal* transformations. 3. The restriction of every automorphism to a small subalgebra *is* given by an internal transformation.

We discuss applications to ergodic theory and hyperfinite measure theory.

1. Introduction. Suppose T is an invertible transformation, measurable in both senses, of a probability space $(X, \mathfrak{B}, m)$. T induces a Boolean (σ-) automorphism $\Phi = \Phi_T$ of the measure algebra $[\![\mathfrak{B}]\!]$ associated with $(X, \mathfrak{B}, m)$. Considerations from Ergodic Theory motivate the converse question: When is a given automorphism Φ induced by a transformation T?

The answer in "always" for most common spaces (von Neumann [14], Choksi [4]). For those spaces $(X, \mathfrak{B}, m)$ and automorphisms Φ of $[\![\mathfrak{B}]\!]$ *not* induced by a transformation, some authors have asked weaker questions, for example (Panzone and Segovia [15]), whether Φ is induced by a transformation T of a thick subset of X.

We consider here the question when $(X, \mathfrak{B}, m)$ is the uniform hyperfinite probability space $(\Omega, L(\Omega), L(\mu))$ deeply investigated by Loeb [13], Anderson [1] and others. This space has a variety of "universality" properties (Anderson [1], Hoover [9], Keisler [12]) which allow questions about more general or common spaces to be reduced to questions about Ω. (For a further discussion, see Section 5.)

Our main result, Theorem 4.1, is that in the presence of sufficient saturation, every measure algebra automorphism is indeed given by a permutation of Ω.

Since in application the most useful transformations of Ω are the internal ones, we consider whether the transformation in Theorem 4.1 can always be taken to be internal. Theorem 4.3 gives a negative answer. However, the restriction of Φ to any sufficiently small subset of $[\![L(\Omega)]\!]$ *is* induced by an internal permutation; this is Theorem 4.4. (Another proof of Theorem 4.4, using Hall's "Marriage Lemma", appears in Ross [16].)

We give some applications of these results in Section 5. Proposition 5.1 shows that hyperfinite spaces can be "pushed down" to an arbitrary Radon space in a manner respecting a given embedding of the Radon space's measure algebra into $[\![L(\Omega)]\!]$. Proposition 5.2, which is motivated by a selection theorem due to Edgar ([6]; see also Fremlin [7]), reverses this situation, and gives sufficient conditions under which any complete atomless probability space can be "pushed up" into Ω. The final application uses Keisler's "probability logic" ([12], [9]) to lift a theorem about measure spaces in general (in this case, a measure-algebra ergodic theorem) to Ω, where—as a consequence of our main results—it admits an easy proof.

2. Notation. For $(X, \mathfrak{B}, m)$ a probability space, and $B \in \mathfrak{B}$, denote by $[\![B]\!]$ the equivalence class of B under the relation $A \sim B$ if $m(A \triangle B) = 0$. If $\mathfrak{A} \subseteq \mathfrak{B}$, let $[\![\mathfrak{A}]\!] = \{[\![B]\!]: B \in \mathfrak{A}\}$. The set $[\![\mathfrak{B}]\!]$ is a σ-complete Boolean measure algebra under the operations and measure inherited from $\mathfrak{B}$.

If $(Y, \mathfrak{A}, \nu)$ is another probability space, and $\Phi: [\![\mathfrak{B}]\!] \to [\![\mathfrak{A}]\!]$ is a Boolean homomorphism which preserves measure, then Φ is a *set-homomorphism*. (We will not always assume that the domain of a Boolean or set homomorphism is the whole algebra.) A set homomorphism of $[\![\mathfrak{B}]\!]$ onto itself is a *set-automorphism*.

A function $T: X \to Y$ is $\mathfrak{B}$-$\mathfrak{A}$ *measurable*, or simply *measurable*, if $T^{-1}(A) \in \mathfrak{B}$ for all $A \in \mathfrak{A}$. If $m(T^{-1}(A)) = \nu(A)$ for all A, T is *measure-preserving*. If T is a bijection from X onto itself such that both T and T^{-1} are measurable and measure-preserving, then T is a *point-automorphism*.

We assume familiarity with nonstandard analysis in general, and Loeb's hyperfinite measure construction in particular ([5], [13]). For the duration of the paper we will assume that the following is true of the nonstandard model of analysis in which we work:

I. Every infinite internal set has the same (external) cardinality. (Denote it by $\varkappa$.

II. The model is saturated.

Though Property (I) holds in any reasonably well-behaved model, the second property (which implies the first) is quite strong. Additional set-theoretic assumptions, e.g., the General Continuum Hypothesis, are required to build such a model. (Models satisfying Property (II) are called "*ultraenlargements*" in [17].) We discuss in Section 6 the extent to which this extra hypothesis may be weakened.

Let Ω be a hyperfinite set. We adopt throughout the convention that $\|A\|$ is the internal cardinality of an internal set A, while $\mathrm{card}(A)$ is the external, set-theoretic cardinality of A. For A an internal subset of Ω, that is, $A \in {}^*\mathcal{P}(\Omega)$, let $\mu(A) = \|A\|/\|\Omega\|$. Then $(\Omega, L(\Omega), L(\mu))$ is a (standard, external) probability space, where $L(\Omega)$ is the smallest σ-algebra containing ${}^*\mathcal{P}(\Omega)$, and $L(\mu)$ is a countably additive extension of $\mathrm{st} \circ \mu$.

Note that while the completion $(\Omega, \overline{L(\Omega)}, \overline{L(\mu)})$ of this space has the same measure algebra, $[\![L(\Omega)]\!] = [\![\overline{L(\Omega)}]\!]$, it has more point-automorphisms. Every transformation constructed in this paper preserves "Borel rank", and so is a point-automorphism for both spaces. (See Corollary 4.2 for a consequence of this.)

The following well-known fact is a consequence of ω_1-saturation.

PROPOSITION 2.1. *For every $B \in L(\Omega)$, there is an internal A with $[\![B]\!] = [\![A]\!]$.*

Finally, it will be convenient to agree on some notation concerning ordinal numbers. If α is an ordinal, finite, or infinite, make the following identifications and definitions:

(i) $\alpha = \{\beta: \beta < \alpha\}$,

(ii) $2^\alpha = \{\tau: \alpha \to 2\}$ = the set of sequences of 0's and 1's with order type α,

(iii) $2^{<\alpha} = \bigcup_{\beta<\alpha} 2^\beta$.

If $\tau_1, \tau_2 \in 2^\alpha$, write $\tau_1 \prec \tau_2$ if $\mathrm{Domain}(\tau_1)$ is a proper subset of $\mathrm{Domain}(\tau_2)$ and $\tau_1 = \tau_2 \restriction_{\mathrm{Domain}(\tau_1)}$. If $\tau \in 2^\alpha$, define $\tau^\frown 0$ (respectively, $\tau^\frown 1$) on $2^{\alpha+1}$ by appending a 0 (respectively, a 1) to the sequence τ.

3. Preliminary results. Every $\tau \in 2^n$, $n \in N$, gives rise to an internal function $\underset{\sim}{b} = \underset{\sim}{b}_\tau: {}^*\mathcal{P}(\Omega)^n \to {}^*\mathcal{P}(\Omega)$ defined by

$$\underset{\sim}{b}_\tau(A_1, \ldots, A_n) = \Big(\bigcap_{\tau^{-1}(0)} A_{i+1}\Big) \cap \Big(\bigcap_{\tau^{-1}(1)} A_{i+1}^c\Big).$$

Note that for fixed $A_1, \ldots, A_n \in {}^*\mathcal{P}(\Omega)$, the set $\{\underset{\sim}{b}_\tau(A_1, \ldots, A_n): \tau \in 2^n\}$ is an internal partition of Ω.

Call two sequences $\{A_i\}_{i<\alpha}$ and $\{B_i\}_{i<\alpha}$ from ${}^*\mathcal{P}(\Omega)$ *similar* if for all $n \in N$, $i_1 < i_2 < \ldots < i_n < \alpha$, and $\tau \in 2^n$, $\|\underset{\sim}{b}_\tau(A_1, \ldots, A_n)\| = \|\underset{\sim}{b}_\tau(B_1, \ldots, B_n)\|$.

LEMMA 3.1. *Suppose $\{A_i\}_{i<\varkappa}$ and $\{B_i\}_{i<\varkappa}$ are similar enumerations of ${}^*\mathcal{P}(\Omega)$. Then there is a point-automorphism T of Ω such that $T(A_i) = B_i$ for all $i < \varkappa$.*

Proof. Define T by $T(x) = y$, where $\{x\} = A_i$ and $\{y\} = B_i$. By similarity, T is well-defined and injective; since both sequences are enumerations, $\mathrm{Domain}(T) = \mathrm{Range}(T) = \Omega$. For $i < \varkappa$, and $x \in \Omega$, similarity guarantees that $x \in A_i$ if and only if $T(x) \in B_i$, so $T(A_i) = B_i$. ∎

LEMMA 3.2. *If $\alpha < \varkappa$, and $\{A_i\}_{i<\alpha}$ and $\{B_i\}_{i<\alpha}$ are similar, then there are internal *permutations F and G of Ω, and $A \in {}^*\mathcal{P}(\Omega)$, such that $F(A_i) = G(A_i) = B_i$, but $L(\mu)(F(A) \Delta G(A)) = 0$.*

Proof. By $\varkappa$-saturation it suffices to show that the following conditions on F, G, and A are finitely satisfiable:

(i) F and G are *permutations of Ω,

(ii) $F(A_i) = G(A_i) = B_i$, all $i < \alpha$,

(iii) $\mu(F(A) \Delta G(A)) < \varepsilon$, all $\varepsilon \in Q \cap (0, 1]$.

Let $n \in N$, $\varepsilon > 0$, and $i_1 < i_2 < \ldots < i_n < \alpha$. The sets $\{\underset{\sim}{b}_\tau(A_{i_1}, \ldots, A_{i_n}): \tau \in 2^n\}$ and $\{b_\tau(B_{i_1}, \ldots, B_{i_n}): \tau \in 2^n\}$ each partition Ω, as noted above. By internally refining these partitions if necessary, we obtain internal partitions $\{A^i\}_{i<m}$ and $\{B^i\}_{i<m}$, with $\|A^i\| = \|B^i\|$ either *even or equal to 1 for $i \leqslant m$. Divide each A^i (respectively, B^i), with $\|A^i\|$ *even, into two internal subsets A^i_+ and A^i_- (resp., B^i_+ and B^i_-) so that $\|A^i_+\| = \|A^i_-\| = \|B^i_+\| = \|B^i_-\| = \frac{1}{2}\|A^i\|$. Define F and G so that F maps A^i_+ (resp., A^i_-) bijectively onto B^i_+ (resp., B^i_-), and G maps A^i_+ (resp., A^i_-)

bijectively onto B^i_+ (resp., B^i_-). F and G should take A^i to B^i for those i with $\|A^i\| = 1$. Let $A = \bigcup_{i \leqslant m} A^i_+$. Evidently, F, G, and A satisfy (i)–(iii) for $\{i_1, \ldots, i_n\}$. ■

LEMMA 3.3. *Suppose* $\alpha < \varkappa$, $\{A_i\}_{i<\alpha}$ *and* $\{B_i\}_{i<\alpha}$ *are similar*, $A_\alpha \in {}^*\mathscr{P}(\Omega)$, *and* Φ *is a set-homomorphism of* $[\![\{A_i\}_{i\leqslant\alpha}]\!]$ *into* $[\![L(\Omega)]\!]$ *with* $\Phi([\![A_i]\!]) = [\![B_i]\!]$ *for all* $i < \alpha$. *Then there is a* $B_\alpha \in {}^*\mathscr{P}(\Omega)$ *such that* $[\![B_\alpha]\!] = \Phi([\![A_\alpha]\!])$, *and* $\{A_i\}_{i\leqslant\alpha}$ *and* $\{B_i\}_{i\leqslant\alpha}$ *are similar.*

Proof. Let $B \in {}^*\mathscr{P}(\Omega)$ with $[\![B]\!] = \Phi([\![A_\alpha]\!])$. It suffices to find an internal permutation F of Ω such that $F(A_i) = B_i$ for $i < \alpha$, and $\mu(F(A_\alpha)\triangle B) \approx 0$. We show that the following conditions are finitely satisfiable:

(i) F is a *permutation of Ω,

(ii) $F(A_i) = B_i$, all $i < \alpha$,

(iii) $\mu(F(A_\alpha)\triangle B) < \varepsilon$, all $\varepsilon \in Q \cap (0, 1]$.

Let $i_1 < \ldots < i_n < \alpha$, $\varepsilon \in Q \cap (0, 1]$. The sets $\{A^i\}_{i<m} = \{b_\tau(A_{i_1}, \ldots, A_{i_n})\colon \tau \in 2^n\}$ and $\{B^i\}_{i<m} = \{b_\tau(B_{i_1}, \ldots, B_{i_n})\colon \tau \in 2^n\}$ each internally partition Ω. For $i < m$, let $\varepsilon_i = \mu(A^i \cap A_\alpha) - \mu(B^i \cap B)$, and note $\varepsilon_i \approx 0$. If ε_i is positive (respectively, negative), choose $B^i_\alpha \subseteq B^i$ a superset (resp., subset) of $B^i \cap B$, with $\|B^i_\alpha\| = \|A^i \cap A_\alpha\|$; this can be done so that $\{B^i_\alpha\}_{i<m} \cup \{B^i \setminus B^i_\alpha\}_{i<m}$ partitions Ω. Define an internal permutation F of Ω so that $F(A^i \cap A_\alpha) = B^i_\alpha$ and $F(A^i \setminus A_\alpha) = B^i \setminus B^i_\alpha$ for $i < m$. This F satisfies (i) and (ii), and the calculation

$$\mu(F(A_\alpha)\triangle B) = \mu\big(\bigcup_{i<m} (B^i_\alpha \triangle B\big) = \sum_{i<m} \varepsilon_i \approx 0$$

verifies (iii). ■

4. Inducing automorphisms. We now can prove the main results.

THEOREM 4.1. *Let* Φ *be a set automorphism of* $[\![L(\Omega)]\!]$. *Then there is a point automorphism* T *such that* $\Phi = \Phi_T$.

Proof. We assume that ${}^*\mathscr{P}(\Omega)$ is endowed with some fixed well-ordering. Define a function $\varrho\colon \varkappa \to \{0, 1\}$ inductively so that $\varrho(\alpha) = 0$, when α is a limit ordinal; $\varrho(\alpha+1) = (\varrho(\alpha)+1)\bmod 2$ otherwise.

Inductively construct sequences $\{A_i\}_{i<\varkappa}$ and $\{B_i\}_{i<\varkappa}$ as follows.

If $\varrho(\alpha) = 0$, let A_α be the first element of ${}^*\mathscr{P}(\Omega)\setminus\{A_i\}_{i<\alpha}$, and let B_α as given in Lemma 3.3.

If $\varrho(\alpha) = 1$, let B_α be the first element of ${}^*\mathscr{P}(\Omega)\setminus\{B_i\}_{i<\alpha}$, and let A_α as given in Lemma 3.3, in this case reversing the roles of $\{A_i\}_{i<\alpha}$ and $\{B_i\}_{i<\alpha}$, and replacing Φ by Φ^{-1}.

By the construction, $\{A_i\}_{i<\varkappa}$ and $\{B_i\}_{i<\varkappa}$ are similar; by similarity and the construction, each sequence is an enumeration of ${}^*\mathscr{P}(\Omega)$. The conclusion follows from Lemma 3.1. ■

COROLLARY 4.2. *Let* T *be any* $L(\Omega)-L(\Omega)$ *or* $\overline{L(\Omega)}-\overline{L(\Omega)}$ *measurable point-automorphism of* Ω. *There is some point-automorphism* T' *of* Ω *which takes internal sets to internal sets and which induces the same set-automorphism as does* T.

THEOREM 4.3. *Not all set automorphisms of* $[\![L(\Omega)]\!]$ *are induced by internal point-automorphisms.*

Proof. Since $\operatorname{card}([\![L(\Omega)]\!]) = 2^\varkappa$, there are at most $2^\varkappa$ set-automorphisms; we show that there are in fact exactly $2^\varkappa$ such automorphisms. The theorem follows from the fact that Ω has only $\varkappa$ internal permutations.

As in Theorem 4.1, give ${}^*\mathscr{P}(\Omega)$ a well-ordering. Define a function $\varrho\colon \varkappa \to \{0, 1, 2\}$ by $\varrho(\alpha) = 0$, α a limit; $\varrho(\alpha+1) = (\varrho(\alpha)+1)\bmod 2$. Let

$$T = \{\tau \in 2^{<\varkappa}\colon \forall \alpha \in \operatorname{Domain}(\tau),\ \varrho(\alpha) \neq 2 \Rightarrow \tau(\alpha) = 0\}.$$

(T is a binary tree of height $\varkappa$ which has been "pruned" so that it only branches at every third node.) Inductively define $\{A_\sigma\}_{\sigma\in T}$ and $\{B_\sigma\}_{\sigma\in T}$ as follows.

Given $\tau \in T$, let $\alpha = \operatorname{Domain}(\tau)$. If $\varrho(\alpha) = 0$, let A_τ be the first element of ${}^*\mathscr{P}(\Omega)\setminus\{A_\sigma\colon \sigma \prec \tau\}$. Choose $B_\tau \in {}^*\mathscr{P}(\Omega)$ so that $\{A_\sigma\}_{\sigma\preceq\tau}$ and $\{B_\sigma\}_{\sigma\preceq\tau}$ are similar.

If $\varrho(\alpha) = 1$, let B_τ be the first element of ${}^*\mathscr{P}(\Omega)\setminus\{B_\sigma\}_{\sigma\preceq\tau}$, and choose $A_\tau \in {}^*\mathscr{P}(\Omega)$ so that $\{A_\sigma\}_{\sigma\preceq\tau}$ and $\{B_\sigma\}_{\sigma\preceq\tau}$ are similar. We also define $A_{\tau^\frown i}$, $B_{\tau^\frown i}$ at this stage, $i \in \{0, 1\}$. By Lemma 3.2, let A, F, G internal such that $F(A_\sigma) = G(A_\sigma) = B_\sigma$ for all $\sigma \preceq \tau$, and $L(\mu)(F(A) \triangle G(A)) = 0$. Put $A_{\tau^\frown 0} = A_{\tau^\frown 1} = A$, $B_{\tau^\frown 0} = F(A)$, and $B_{\tau^\frown 1} = G(A)$.

Let $T' = \{\tau \in 2^\varkappa\colon \forall \alpha \in \operatorname{Domain}(\tau),\ \tau\restriction_\alpha \in T\}$. Note $\operatorname{card}(T') = 2^\varkappa$. Suppose $\tau \in T'$. By the construction, $\{A_\sigma\}_{\sigma\prec\tau}$ and $\{B_\sigma\}_{\sigma\prec\tau}$ are similar enumerations of ${}^*\mathscr{P}(\Omega)$, so define a point automorphism T_τ, and thus a set automorphism $\Phi_\tau = \Phi_{T_\tau}$. It remains to show that the automorphisms $\{\Phi_\tau\}_{\tau\in T'}$ are distinct.

Let $\tau_1 \neq \tau_2 \in T'$, let α least with $\tau_1(\alpha) \neq \tau_2(\alpha)$. Note $\varrho(\alpha) = 2$. Let $\sigma = \tau_1\restriction_\alpha = \tau_2\restriction_\alpha$; by the last part of the construction $B_{\sigma^\frown 0} = B^c_{\sigma^\frown 1}$. Thus, Φ_{τ_1} and Φ_{τ_2} differ on $[\![A_{\sigma^\frown 0}]\!]$. ■

THEOREM 4.4. *Suppose* $\mathfrak{A} \subseteq L(\Omega)$, $\operatorname{card}(\mathfrak{A}) < \varkappa$, *and* Φ *is a set-homomorphism from* $[\![\mathfrak{A}]\!]$ *into* $[\![L(\Omega)]\!]$. *Then there is an internal permutation* T *of* Ω *such that* $[\![T(A)]\!] = \Phi([\![A]\!])$ *for all* $A \in \mathfrak{A}$.

Proof. Without loss of generality, $\mathfrak{A} \subseteq {}^*\mathscr{P}(\Omega)$. Let $\{A_i\}_{i<\alpha}$ be an enumeration of $\mathfrak{A}$. Use Lemma 3.3 to inductively define $\{B_i\}_{i<\alpha} \subseteq {}^*\mathscr{P}(\Omega)$ so that $\{A_i\}_{i<\alpha}$ and $\{B_i\}_{i<\alpha}$ are similar and $[\![B_i]\!] = \Phi([\![A_i]\!])$ for $i < \alpha$. By Lemma 3.1, there is an internal permutation T of Ω with $T(A_i) = B_i$ for all $i < \alpha$. ■

5. Related results, applications. Uniform hyperfinite spaces have the following "universality" properties:

1. (Anderson [1]) Every Radon probability space $(X, \mathfrak{B}, m)$, with $\operatorname{card}(\mathfrak{B}) < \varkappa$, is the image of $(\Omega, \overline{L(\Omega)}, \overline{L(\mu)})$ under a measure preserving transformation.

2. (Bernstein and Wattenberg [2], Henson [8]). If $(X, \mathfrak{B}, m)$ is an atomless probability space, then there is a uniform hyperfinite space $(\Omega, L(\Omega), L(\mu))$ with $X \subseteq \Omega \subseteq {}^*X$ and $\mu({}^*B \cap \Omega) \approx m(B)$ for all $B \in \mathfrak{B}$.

3. (Keisler [12]) Every standard probability space is L_{AP}-equivalent to a uniform hyperfinite space.

We consider results related to each of these properties.

Suppose first that $(X, \mathfrak{B}, m)$ is a Radon probability space, and that $\psi\colon \Omega \to X$ is measure-preserving. Let Φ be a set-homomorphism from $[\![\mathfrak{B}]\!]$ into $[\![L(\Omega)]\!]$. We show that the ψ can be chosen to respect Φ.

Note that ψ^{-1} embeds $\mathfrak{B}$ into $\overline{L(\Omega)}$, hence $[\![\mathfrak{B}]\!]$ into $[\![L(\Omega)]\!]$. Let Φ' be the induced set-homomorphism from $[\![\psi^{-1}(\mathfrak{B})]\!]$ to $[\![L(\Omega)]\!]$. By Theorem 4.4, there is an internal bijection T of Ω such that $[\![T^{-1}(A)]\!] = \Phi'([\![A]\!])$ for all $[\![A]\!] \in [\![\psi^{-1}(\mathfrak{B})]\!]$. The function $f = \tau \circ T$ suffices for the following proposition.

PROPOSITION 5.1. *Let $(X, \mathfrak{B}, m)$ be a Radon probability space,* $\mathrm{card}(\mathfrak{B}) < \varkappa$, *and let $\Phi\colon [\![\mathfrak{B}]\!] \to [\![L(\Omega)]\!]$ be a set-homomorphism. Then there is a measurable $f\colon \Omega \to X$ such that for all $A \in \mathfrak{B}$, $[\![f^{-1}(A)]\!] = \Phi([\![A]\!])$.*

Now suppose that $(X, \mathfrak{B}, m)$ is an arbitrary complete atomless probability space. By universality property (2), there is an isomorphism between $[\![\mathfrak{B}]\!]$ and $[\![\{{}^*B\colon B \in \mathfrak{B}\}]\!]$ which is given by the natural embedding of X into $\Omega \subseteq {}^*X$. The following generalization is a minor modification of a result of Edgar [6], in which the target space is Radon instead of hyperfinite.

PROPOSITION 5.2. *Let $(X, \mathfrak{B}, m)$ be a complete probability sp ace, $\Gamma \subseteq {}^*\mathscr{P}(\Omega$ with* $\mathrm{card}(\Gamma) < \varkappa$, *and let $\mathfrak{A}$ be the smallest σ-algebra containing Γ. If Φ is a Boolean σ-homomorphism from $\mathfrak{A}$ into $[\![\mathfrak{B}]\!]$ (or if Φ is a set-homomorphism from $[\![\mathfrak{A}]\!]$ into $[\![\mathfrak{B}]\!]$) then there is a $\mathfrak{B}$-$\mathfrak{A}$ measurable $f\colon X \to \Omega$ such that $[\![f^{-1}(A)]\!] = \Phi([\![A]\!])$ for all $A \in \mathfrak{A}$.*

Proof. We may assume that Γ is an algebra. Let θ be a *lifting* of $\mathfrak{B}$, that is, a σ-homomorphism from $[\![\mathfrak{B}]\!]$ to $\mathfrak{B}$ with $[\![\theta([\![B]\!])]\!] = [\![B]\!]$ for all $B \in \mathfrak{B}$ (see [10]). For $x \in X$ let $\Gamma_x = \{A \in \Gamma\colon x \in \theta(\Phi(A))\}$. Evidently Γ_x has the finite intersection property, so $\bigcap \Gamma_x \neq \emptyset$; let $f(x) \in \Gamma_x$.

It suffices to show that for $A \in \Gamma$, $f^{-1}(A) = \theta(\Phi(A))$. The reverse inclusion is immediate, so suppose $f(x) \in A$. Since Γ is an algebra, $\bigcap \Gamma_x \subseteq A$, but then by $\varkappa$-saturation $C \in \Gamma_x$ for some $C \subseteq A$, so $x \in \theta(\Phi(C)) \subseteq \theta(\Phi(A))$ as required. ∎

Our last application presupposes familiarity with the "Probability Logic" of Hoover and Keisler; see ([12]) for a discussion. We illustrate the following metamathematical principle:

Let ζ be a property of measure algebras, and suppose ζ holds for every space in which set-automorphisms are induced by point-automorphisms. Then ζ holds for all spaces.

This follows from Theorem 4.1, universality property (3), and the general rule that properties of measure algebras can be expressed in L_{AP}.

For example, suppose that $(X, \mathfrak{B}, m)$ is a probability space, Φ a set-automorphism of $\mathfrak{B}$, and $A \in \mathfrak{B}$. Let $\{A_i\}_{i \in Z}$ be a sequence from $\mathfrak{B}$ such that $[\![A_i]\!] = \Phi^i([\![A]\!])$. The statement

$$(\mathfrak{F}) \qquad \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \chi_{A_k}(x) = m(A) \text{ almost surely}$$

expresses an "algebra" form of Birkhoff's Ergodic Theorem ([3]) which in fact follows from the usual theorem in those spaces where Φ is given by a point-automorphism.

Let $\mathscr{L}$ be a probability logic including a countable set $\{\psi_i(x)\colon i \in Z\}$ of unary predicate symbols, and consider $(X, \mathfrak{B}, m)$ as a model for $\mathscr{L}$, where the interpretation of ψ_i in X is the set A_i. Then $\mathfrak{F}$ is expressible in $\mathscr{L}$ as

$$(\hat{\mathfrak{F}}) \qquad \bigwedge_{k\in N} \bigvee_{N\in N} \bigwedge_{n>N} \bigvee_{\tau\in \Lambda_{nk}} \Big(\bigwedge_{\tau(i)=1} \psi_i(x) \wedge \bigwedge_{\tau(i)=0} \neg\psi_i(x)\Big)$$

$$\text{where } \Lambda_{nk} = |\, \|\tau^{-1}(\{1\})\| - n\cdot m(A)| \leqslant n/k\,.$$

Now, let $(\Omega(\overline{L(\Omega)}, \overline{L(\mu)}) \equiv_{L_{AP}} (X, \mathfrak{B}, m)$ in this language. (We may assume $\mathrm{card}(\mathfrak{B}) < \varkappa$.) The elementary equivalence gives an embedding of $[\![\mathfrak{A}]\!]$ into $[\![L(\Omega)]\!]$, where $\mathfrak{A}$ is the σ-algebra generated by $\{A_i\}_{i \in Z}$, and so Theorem 4.4 guarantees that the image of $\Phi \upharpoonright_{[\mathfrak{A}]}$ in $[\![L(\mu)]\!]$ is given by an internal permutation of Ω. Thus, by the regular ergodic theorem applied to Ω (better: by a minor modification of Kamae [11]), the sentence $\hat{\mathfrak{F}}$ holds in $(\Omega, \overline{L(\Omega)}, \overline{L(\mu)})$, hence in $(X, \mathfrak{B}, m)$.

6. Discussion. As noted in Section 2, the hypothesis on $\varkappa$-saturation is extremely strong. It seems, however, to be necessary for Theorem 4.1 (and thus Corollary 4.2), though we can't prove that this is indeed the case.

On the other hand, for Lemma 3.2 and Lemma 3.3, and thus Theorem 4.4 and the results in Section 5, the number of internal sets plays no significant role, and it suffices to assume that $(\Omega, L(\Omega), L(\mu))$ is $\varkappa$-saturated, where $\varkappa$ may have no relation to $\mathrm{card}(\Omega)$.

Theorem 4.3 will hold in any *special* model of nonstandard analysis, and thus is independent of special set-theoretic axioms: for a regular cardinal $\varkappa$, if $\varkappa = \bigcup_{\lambda<\varkappa} 2^\lambda$, then a model satisfying (I) and (II) can be constructed. Otherwise, build a special model of cardinality $\varkappa$, find $\lambda < \varkappa$ with $2^\lambda > \varkappa$, and apply the construction of Theorem 4.3 to a λ-saturated model in the specializing chain.

We conclude with some remarks on the theorem of Edgar mentioned above. That theorem is that if Φ is a σ-homomorphism from a compact topological space's Borel algebra to $[\![\mathfrak{B}]\!]$, where $(X, \mathfrak{B}, m)$ is any complete probability space, then there is a measurable $f\colon X \to Y$, where Y is the compact space, such that $[\![f^{-1}(A)]\!] = \Phi(A)$, for all $A \subseteq Y$ open. Fremlin [7] points out that this is true as well for Φ a set-homomorphism from $[\![\mathfrak{A}]\!]$ to $[\![\mathfrak{B}]\!]$, where $(Y, \mathfrak{A}, \nu)$ is a Radon probability space.

This result follows easily from universality property (1) and Proposition 5.2; unfortunately, nothing much is gained here, since the proof of this latter proposition is almost identical to Edgar's. However, an affirmative answer to the following question will yield a simple, "lifting-free" proof of both Proposition 5.2 and Edgar's Theorem.

QUESTION. *Suppose in Lemma 3.3 that $\mathfrak{A}$ and Γ are as in Proposition 5.2, that $B_i \in \Gamma$ for all $i < \alpha$, and that* $\mathrm{Range}(\Phi) \subseteq [\![\mathfrak{A}]\!]$. *Can B_α be chosen in Γ?*

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References

[1] R. M. Anderson, *Star-finite representations of measure spaces*, Trans. Amer. Math. Soc. 271 (1982), 667–687.

[2] A. Bernstein and F. Wattenberg, *Nonstandard measure theory*, in *Applications of model theory to algebra, analysis, and probability* (Holt, Rinehart, and Winston, New York 1969), 171–185.

[3] G. Birkhoff, *Proof of the ergodic theorem*, Proc. Nat. Acad. Sci. U.S.A. 17 (1931), 656–660.

[4] J. R. Choksi, *Measurable transformations on compact groups*, Trans. Amer. Math. Soc. 184 (1973), 101–124.

[5] N. Cutland, *Nonstandard measure theory and its applications*, Bull. London Math. Soc. 15 (1983), 529–589.

[6] G. A. Edgar, *Measurable weak selections*, Illinois J. Math. 20 (1976), 630–646.

[7] D. H. Fremlin, *Measurable functions and almost continuous functions*, Manus. Math. 33 (1981), 387–405.

[8] C. W. Henson, *On the nonstandard representation of measures*, Trans. Amer. Math. Soc. 172 (1972), 437–466.

[9] D. Hoover, *Probability logic*, Annals of Math. Logic 14 (1978), 287–313.

[10] A. and C. Ionescu Tulcea, *Topics in the theory of lifting* (Springer-Verlag, New York 1969).

[11] T. Kamae, *A simple proof of the ergodic theorem using nonstandard analysis*, Israel J. Math. 42 (1982), 284–290.

[12] H. J. Keisler, *Probability quantifiers*, in *Abstract model theory and logics of mathematical concepts* (Ed., J. Barwise and S. Feferman, Springer-Verlag, to appear), ch. 14.

[13] P. A. Loeb, *An introduction to nonstandard analysis and hyperfinite probability theory*, in *Probabilistic analysis and related topics*, Vol. 2 (Ed., A. T. Bharucha-Reid, Academic Press, New York 1979), 105–142.

[14] J. von Neumann, *Einige Sätze über meßbare Abbildungen*, Ann. of Math. 33 (1932), 574–586.

[15] R. Panzone and C. Segovia, *Measurable transformations on compact spaces and o. n. systems on compact groups*, Union Math. Arg. Revista 22 (1964), 83–102.

[16] D. Ross, *Measurable transformations in saturated models of analysis* (unpublished Ph. D. thesis, 1983).

[17] K. Stroyan and J. M. Bayod, *Foundations of infinitesimal stochastic analysis* (North Holland, to appear).

DEPARTMENT OF MATHEMATICS
THE UNIVERSITY OF IOWA
Iowa City, IA 52242

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On characterizations of classes of metrizable spaces that have transfinite dimension

by

Yasunao Hattori (Osaka)

Abstract. We are concerned with a characterization of two classes of infinite-dimensional spaces. First, we characterize the class of metrizable spaces which have large transfinite dimension, in terms of partitions, a special base and a dimension-raising mapping. Second, we give a characterization of the class of metrizable spaces which have strong large transfinite dimension, in terms of a dimension-raising mapping and a special refinement.

1. Introduction. In this paper we are concerned with a characterization of two classes of metrizable spaces of transfinite dimension. We say that a metrizable space is countable-dimensional if it can be expressed as the union of countably many zero-dimensional subsets (in the sense of dim or equivalently of Ind). We have been inspired by the following interesting theorem, obtained by J. Nagata [11] and K. Nagami and J. H. Roberts [10], which characterizes the class of countable-dimensional metrizable spaces.

THEOREM A. *For a metrizable space X, the following conditions are equivalent:*

(a) *X is countable-dimensional.*

(b) *For every sequence $\{(A_i, B_i): i \in N\}$ of pairs of disjoint closed sets of X, there is a sequence $\{L_i: i \in N\}$ of closed sets such that each L_i is a partition between A_i and B_i in X and the family $\{L_i: i \in N\}$ is point finite.*

(c) *X has a σ-discrete base $\mathscr{B}$ such that the family $\{\mathrm{Bd}\,B: B \in \mathscr{B}\}$ is point finite.*

(d) *There are a metrizable space Z and a closed continuous mapping f of Z onto X such that $\dim Z \leqslant 0$ and $f^{-1}(x)$ consists of at most finitely many points, for each point $x \in X$.*

In [5], R. Engelking and R. Pol characterized the class of metrizable spaces of large transfinite dimension by use of a strongly point finite family (see § 2 for the definition) of partitions. But the concept of strong point finiteness cannot characterize this class in terms of a σ-discrete base. In Section 2 we characterize this class in terms of partitions and of a σ-discrete base simultaneously by use of a new concept of "point finiteness". A characterization of the same class in terms of a dimension-raising closed continuous mapping from a zero-dimensional metrizable