The structure of Eberlein, uniformly Eberlein and Talagrand compact spaces in $\Sigma(R^\Gamma)$

by

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Abstract. Well known compact spaces such as Eberlein, uniformly Eberlein and Talagrand compact spaces can be considered as subsets of $\Sigma(R^\Gamma)$. In this paper we give characterizations of these classes of sets distinguishing them by their structure inside $\Sigma(R^\Gamma)$. A consequence of these results is an easy proof of Talagrand’s example of a Talagrand compact space which is not Eberlein compact.

1. Introduction and preliminaries. Since the early sixties, various classes of compact spaces generalizing compact metric spaces have been studied by many authors. Thus, classes of Eberlein, uniformly Eberlein and Talagrand compact spaces were defined and distinguished by means of examples. However, there did not exist a characterization distinguishing them by their structure. Such characterizations are contained in the present paper.

A compact Hausdorff space is called Eberlein compact (E. C.) iff it is homeomorphic to a weakly compact subset of a Banach space. Amir and Lindenstrauss ([1]) proved that every E. C. space is homeomorphic to a weakly compact subset of $c_0(\Gamma)$ for some set $\Gamma$, where

$$c_0(\Gamma) = \{ f \in R^\Gamma : \text{for each } \varepsilon > 0 \text{ the set } \{ \gamma \in \Gamma : |f(\gamma)| > \varepsilon \} \text{ is finite} \}$$

(The weak topology on a weakly compact subset of $c_0(\Gamma)$ is exactly the topology of point-wise convergence.)

An E. C. space will be called uniformly Eberlein compact (U. E. C.) iff it is homeomorphic to a weakly compact subset of a Hilbert space. U. E. C. spaces have been introduced by Benyamini–Starbird in [4], where they give an example of an E. C. space which is not U. E. C.

A compact space $K$ is called Talagrand compact (T. C.) iff the Banach space $C(K)$ in its weak topology is $\aleph$-analytic.

S. Mercourakis proved in [7] that a compact space $K$ is T. C. iff it is homeomorphic to a compact subset of a space

$$c(\Gamma_\sigma : \sigma \in \Sigma) = \{ f \in R^\Gamma : f \text{ bounded and } f|_{\Gamma_\sigma} \in c_0(\Gamma_\sigma) \text{ for all } \sigma \in \Sigma \}$$
where $\Sigma = N^N$ and $\{\eta_x : \sigma \in \Sigma\}$ is a decomposition of a set $\Gamma$ with the property $\eta_x \in \Gamma_{\sigma}$ if $\sigma_1 < \sigma_2$. The space $\epsilon(\Gamma_{\sigma} : \sigma \in \Sigma)$ has the topology of pointwise convergence.

The classes of E.C., U.E.C., T.C. spaces have interesting stability properties such as being invariant under continuous images ([3], [11]). M. Talagrand proved ([10]) that E.C. spaces are T.C. spaces and Vaš (12) (implicitly) and Gulko ([5]) that every T.C. space is homeomorphic to a subset of $\Sigma(R^2)$ for some set $\Gamma$ where

$$\Sigma(R^2) = \{f : \Gamma \to \mathbb{R} : \text{the set } \{y \in \Gamma : f(y) \neq 0\} \text{ is at most countable}\}$$

Thus all these compact spaces may be considered as subsets of the space $\Sigma(R^2)$.

The present paper continues the study of E.C., U.E.C. and T.C. spaces separating these classes by means of their structure inside $\Sigma(R^2)$. In a previous paper by S. Argyros and the present author ([2]) we gave a characterization of U.E.C. sets in the Banach space $c_0(\Gamma)$. Extending the techniques of [2] to the present situation, we give a similar characterization of the structure of E.C. sets in the locally convex space $\Sigma(R^2)$ (Theorem 2.10). We also characterize the structure of E.C. and T.C. subsets of $\Sigma(R^2)$ (Theorems 2.9 and 2.11).

These characterizations separate the various classes of compact subsets of $\Sigma(R^2)$. Therefore imply a very simple proof of the properties of an example of Talagrand [11] of a T.C. space which is not E.C.

If $A$ is a set, card $A$ or $|A|$ will denote its cardinality, $\omega$ denotes the first infinite cardinal, $I_A$ denotes the characteristic function of $A$, $\epsilon(x) : I_A(x)$, and $\chi_A = x^{-1}I_A$, for $x \in R^2$ and $\epsilon \in \Sigma$.

Let $\mathcal{K}$ be a family of subsets of a set $\mathcal{K}$.

(i) The family $\mathcal{K}$ separates the points of $\mathcal{K}$ if for any $x, y \in \mathcal{K}$, $x \neq y$, there exists $\epsilon(x) \neq \epsilon(y)$.

(ii) The family $\mathcal{K}$ is point finite (point countable) if every $x \in K$ belongs to at most finitely (countable) many elements of $\mathcal{K}$.

(iii) The family $\mathcal{K}$ is $\epsilon$-finite ($\epsilon \in N$) if every $x \in K$ belongs to at most $\epsilon$ elements of $\mathcal{K}$.

PROPOSITION 1.1 ([9], [3]). A compact space $K$ is E.C. (U.E.C.) if there exists a family $\mathcal{K}$ of open $F_S$ subsets of $K$ with the properties:

(i) $\mathcal{K}$ separates the points of $\mathcal{K}$.

(ii) There exists a decomposition $\{\mathcal{K}_m : m \in N\}$ of $\mathcal{K}$ such that, for every $m \in N$, $\mathcal{K}_m$ is point finite (point countable) (2) $\mathcal{K}_m$ is $\epsilon$-finite ($\epsilon \in N$) for some $\epsilon \in N$.

PROPOSITION 1.2 [7]. A compact space $K$ is T.C. if there exists a point-countable family $\mathcal{K}$ of open $F_S$ subsets of $K$ with the properties:

(i) $\mathcal{K}$ separates the points of $\mathcal{K}$.

(ii) There exists a decomposition $\{\mathcal{K}_\sigma : \sigma \in \Sigma\}$ (2 = $N^N$) of $\mathcal{K}$ such that $\mathcal{K}_{\sigma_1} \subseteq \mathcal{K}_{\sigma_2}$ if $\sigma_1 < \sigma_2$ and the families $\mathcal{K}_\sigma$ are point-finite for every $\sigma \in \Sigma$.


2.1. DEFINITION. For each subset $K$ of $\Sigma(R^2)$, we denote by $\delta(K)$ the set

$$\delta(K) = \{f \in \Sigma(R^2) : \exists g \in K, |g| = |f|\}$$

2.2. PROPOSITION. Let $K$ be a compact subset of the space $\Sigma(R^2)$.

(a) If $K$ is E.C. then $\delta(K)$ is E.C.

(b) If $K$ is U.E.C. then $\delta(K)$ is U.E.C.

(c) If $K$ is T.C. then $\delta(K)$ is T.C.

The proof will be based on the following Lemmas 2.4, 2.5.

2.3. Notation. Consider a compact subset $K$ of $\Sigma(R^2)$ and an enumeration $(a_n, b_n) : n \in N$ of the open intervals with rational ends such that $0 \notin [a_n, b_n]$. For any $(y, a, b) \in \Gamma \times N$ put

$$V_{(y, a, b)} = \{f \in K : f(y) \in (a_n, b_n)\}.$$

It is easy to see that the family

$$\mathcal{K} = \{V_{(y, a, b)} : (y, a, b) \in \Gamma \times N\}$$

satisfies the following properties:

(i) $\mathcal{K}$ consists of open, $F_S$ subsets of $\Sigma(R^2)$.

(ii) $\mathcal{K}$ separates the points of $\mathcal{K}$.

(iii) $\mathcal{K}$ is a point-countable family.

2.4. LEMMA. For every weakly compact subset $K$ of the space $c_0(\Gamma)$ and every point countable family $\{V_t : t \in T\}$ of open $F_S$ subsets of $\Gamma$, such that $0 \notin V_t$ for all $t \in T$, there exists a family $\{U_t : t \in T\}$ of open $F_S$ subsets of $\Gamma$ with the following properties:

(i) For each $t \in T$ there exists $\delta \in \mathcal{F}$ such that $U_t \subseteq V_t$. $\delta$.

(ii) For each $t \in T$ there exists a countable $D_t \subseteq D$ such that $V_t = \bigcup\{U_t : D \subseteq D\}$.

(iii) There exists a decomposition $\{A_t : k \in N\}$ of $\mathcal{F}$ so that the family $\{U_t : t \in D\}$ is point finite for every $k \in N$.

Proof. For each $t \in T$ and $x \in V_t$ we can choose an open basic neighborhood $W_x$ of $x$ contained in $V_t$. Let

$$W_x = \{y \in K : \|y\xi_t - x\xi_t\| < e_x \text{ for } 1 \leq t \leq n_x\}$$

where $n_x \in N$, $\xi_t^t : 1 \leq t \leq n_x \subseteq A$ and $e_x$ is a rational number. Furthermore we can choose $e_x$ so that $|\xi_t| > \frac{e_x}{2}$ for every $\lambda \in M_x$ where $M_x = \{\xi_t^t : x \in L\} > 0$.

Each $V_t$ is weakly Lindelöf, hence there exists a countable subset $\{x_k : k \in N\}$ of $V_t$ so that $V_t = \bigcup_{k \in N} W_{x_k}$. Define $A = \bigcap_t X$ and $U = W'_{x_k}$ where $x = (x, t)$. It is easy to check that the family $\{U_t : t \in T\}$ satisfies properties (1) and (2).

For every finite subset $F$ of $A$ and $k \in N$ the set $L = \{t \in T : M_{x_k} = M\}$ is at most countable (for the proof see Lemma 1.4 of [2]).
Let \( \{W(M, r) : r \in \mathbb{N}\} \) be an enumeration of \( L \).
For every \((k, m, r) \in \mathbb{N}^3\) we define
\[
\Delta_{k, m, r} = \left\{ \delta = (t, k) \in \Delta : \frac{1}{m} > 1, M_{\delta} = W(M, r) \right\}.
\]
It remains to show that the family \( \{U_\delta : \delta \in \Delta(k, m, r)\} \) is point finite. Indeed, if there exists \( y \in \bigcap \{U_\delta : \delta \in \Delta\} \), \( \delta = (t, k) \) is a subset of \( \Delta(k, m, r) \), then
\[
|y(\delta)| > \frac{1}{2m}
\]
for every \( \lambda \in M \) where \( M = \bigcup_{\delta \in \Delta(k, m, r)} M_{\delta} \). But the set \( M \) is infinite, because the sets \( \{M_{\delta} : \delta \in \Delta\} \) are pairwise different. This is a contradiction and therefore condition (3) is also true.

2.5. Lemma. For every compact subset \( K \) of the space \( c_0(\Delta) \) and each point countable family \( \{V_\delta : \delta \in \Delta\} \) of open \( F_\alpha \) subsets of \( K \) such that \( 0 \notin V_\delta \) for all \( \delta \in \Delta \), there exists a point countable family \( \{U_\delta : \delta \in \Delta\} \) of open \( F_\alpha \) subsets of \( K \) with the properties (1), (2) of Lemma 2.4.

(4) There exists a decomposition \( \{A_\delta : \delta \in \Sigma\} \) of \( \Delta \), so that \( A_{\sigma_1} \subseteq A_{\sigma_2} \) if \( \sigma_1 < \sigma_2 \) and the family \( \{U_\delta : \delta \in A_{\sigma}\} \) is point finite for every \( \sigma \in \Sigma \).

Proof. As in the proof of the previous lemma, let \( \Delta = T \times N \) and \( U_\delta = W_{\delta} \)
for every \( \delta = (t, k) \in \Delta \). Obviously the family \( \{U_\delta : \delta \in \Delta\} \) is point countable, and has the properties (1), (2).

For every \( \sigma \in \Sigma \) and \((k, m, r) \in \mathbb{N}^3\) define the sets
\[
\Delta_{k, m, r}(\sigma) = \left\{ \delta = (t, k) \in \Delta : \frac{1}{m} > 1, M_{\delta} = W(M, r), \text{ and } M_{\delta} \subseteq A_{\sigma} \right\}.
\]

The family \( \{U_\delta : \delta \in \Delta_{k, m, r}(\sigma)\} \) is point finite. This is true because if there exists \( y \in \bigcap_{\delta \in \Delta(k, m, r)} \), \( \delta = (t, k) \) is an infinite subset of \( \Delta_{k, m, r}(\sigma) \), then |\( y(\delta) | > \frac{1}{m} \) for every \( \lambda \in \bigcup_{\delta \in \Delta(k, m, r)} M_{\delta} \). This is a contradiction because \( \bigcup_{\delta \in \Delta(k, m, r)} M_{\delta} \) is an infinite subset of \( A_{\sigma} \) and \( y |_{\delta} \in \varepsilon_0(\Delta) \).

Furthermore, it is clear that if \( \sigma_1 < \sigma_2 \) then \( \Delta_{k, m, r}(\sigma_1) \subseteq \Delta_{k, m, r}(\sigma_2) \).

For every \( \sigma = (\sigma(1), \sigma(2), \ldots) \in \Sigma \) denote by \( A_\sigma \) the set \( \bigcup_{\delta \in \Delta_{k, m, r}(\sigma)} \Delta_{k, m, r}(\sigma) \).

Obviously, this decomposition \( \{A_\sigma : \sigma \in \Sigma\} \) of \( \Delta \) satisfies property (4), and the proof of the lemma is complete.

Proof of Proposition 2.2. We can assume that the set \( K \) satisfies the property \( x(\gamma) > 0 \) for any \( x \in K, \gamma \in \Gamma \). (Otherwise we consider the continuous mapping \( G : K \to K_1 \) defined by \( G(x) = |x| \). By [3], [11], the set \( K_1 = G(K) \) is E. C., U. E. C., T. C., if \( K \) is E. C., U. E. C., T. C. respectively.) In the sequel
\[
\mathcal{K} = \{V_{(\gamma, n)} : (\gamma, n) \in \Gamma \times \mathbb{N}\}
\]
is the family described in 2.3.
If we set \( U_\delta = f^{-1}(A_\delta) \) for every \( \delta \in \mathcal{A} \), the families \( \{ V_{(\gamma,n)} : (\gamma,n) \in \Gamma \times N \} \) and \( \{ U_\delta : \delta \in \mathcal{A} \} \) satisfy properties (1), (2) and (4) of Lemma 2.5.

As in (a), we define, for every \( \delta \in \mathcal{A} \) and \( x \in U_\delta \), the open neighborhoods \( V_x \) of \( x \) and the corresponding open \( F_x \) subsets \( V_x^* \) of \( \mathcal{D}(K) \) for every \( x \in \{-1,1\}^\mathbb{N} \).

The set \( U_\delta \) is Lindelöf, therefore there exists a countable subset \( \{ x_{(\delta,n)} : \mu \in \mathbb{N} \} \) of \( U_\delta \) such that \( U_\delta = \bigcup_{\mu \in \mathbb{N}} V_{x_{(\delta,n)}}. \)

The family \( E = \{ V_{x_{(\delta,n)}}^* : (\delta, \mu) \in \mathcal{A} \times \mathbb{N}, x_{(\delta,n)} \in \{-1,1\}^\mathbb{N} \} \) separates the points of \( \mathcal{D}(K) \) and is point countable. Decompose \( E \) into sets \( E_{(\sigma,\epsilon)} = \{ V_{x_{(\delta,n)}}^* : \sigma \in \Sigma, \epsilon = (\sigma(1), \sigma(2), ..., \sigma(N)) \in \mathbb{N}^N \} \), where \( \Sigma = \{ (\sigma(1), \sigma(2), ..., \sigma(N)) : \sigma \in \Sigma \} \).

Recall that the family \( \{ U_\delta : \delta \in \mathcal{A} \} \) is point finite and if \( \sigma_1 < \sigma_2 \) then \( A_{\sigma_1} \cap A_{\sigma_2} = \emptyset. \)

It is therefore easy to check that the family \( E_{(\sigma,\epsilon)} \) is point finite and if \( \sigma_1 < \sigma_2 \) then \( E_{(\sigma_1,\epsilon)} \subseteq E_{(\sigma_2,\epsilon)}. \)

For every \( \sigma = (\sigma(1), \sigma(2), ..., \sigma(N)) \in \Sigma \) set \( E_\sigma = \bigcup_{\epsilon \in \mathbb{N}^N} E_{(\sigma,\epsilon)} \) where \( \sigma' = (\sigma(2), \sigma(3), ..., \sigma(N)) \in \Sigma \).

The families \( E_\sigma \) are point finite for any \( \sigma \in \Sigma \) and \( E_{\sigma_1} \subseteq E_{\sigma_2} \) if \( \sigma_1 < \sigma_2 \).

The fact that \( \mathcal{D}(K) \) is T.C. now follows by Proposition 1.2.

**Proposition 2.6.** Let \( K \) be a compact subset of \( \Sigma(R^N) \) and \( \co(K) \) its closed convex hull.

(a) If \( K \) is E.C. then \( \co(K) \) is E.C.

(b) If \( K \) is U.E.C. then \( \co(K) \) is U.E.C.

(c) If \( K \) is T.C. then \( \co(K) \) is T.C.

The proof is an immediate consequence of results of S. Mercourakis [7].

**Lemma 2.7.** Let \( K \) be a subset of \( \Sigma(R^N) \).

(a) If \( x \in \co(\mathcal{D}(K)) \) and \( A \) is a subset of \( \Gamma \) then \( x|_A \in \co(\mathcal{D}(K)) \).

(b) If \( L_1, L_2, ..., L_n \) are pairwise disjoint, finite subsets of \( \Gamma \), \( p_1, p_2, ..., p_n \) in \( \co(\mathcal{D}(K)) \), and \( a \leq a \leq b \), then \( a(L_1) + a + L_2 \in \co(\mathcal{D}(K)) \).

(c) If \( x \in \co(\mathcal{D}(K)) \) and \( x(\gamma) > 0 \) for every \( \gamma \in \supp(x) \) then \( x|_{\gamma \in \supp(x)} \in \co(\mathcal{D}(K)) \).

Proof. (a) and (b) are easy (see [2], Lemma 1.6).

The support of \( x \) is countable, say \( \supp(x) = \{ y_i : k \in \mathbb{N} \} \). If \( y = a(L)e \) then \( y \) is the pointwise limit of the sequence \( (y_n)_{n \geq 1} \) where \( y_n = a \sum_{k=1}^n e_k. \) According to (a) and (b), \( y \in \co(\mathcal{D}(K)). \)

The next combinatorial lemma is a special case of Hajnal’s theorem [6].

**Lemma 2.8.** Let \( m \in \mathbb{N} \) and \( \{ A_i : i \in I \} \) be a countable family (card \( I = \omega \) of subsets of \( \Gamma \) with \( \text{card } A_i \leq m \) for \( i \in I \). Then there exists an infinite subset \( I_1 \) of \( I \) such that if \( i_1 \neq i_2 \) are elements of \( I_1 \), then \( i_1 \neq A_i \).

We are now able to give the characterization of the structure of Eberlein compact subsets of \( \Sigma(R^N) \).

**Theorem 2.9.** Let \( K \) be a compact subset of \( \Sigma(R^N) \). The following are equivalent:

1. The set \( K \) is E.C.

2. For every \( \epsilon > 0 \) there exists a decomposition \( \{ V_{(\gamma,n)} : m \in \mathbb{N} \} \) of \( \Gamma \), such that for every \( \gamma \in K \) and \( m \in \mathbb{N} \)

\[
\text{card } \{ (\gamma,m) : |x(\gamma)| > \epsilon \} < \omega.
\]

Proof. (2=1) By compactness of \( K \), for every \( \gamma \in \Gamma \) there exists an \( n_\gamma \in \mathbb{N} \) such that \( |x(\gamma)| < M_\gamma \) for every \( x \in K \).

Let \( V_{(\gamma,n)} = \\{ x \in K : |x(\gamma)| < \frac{n}{n+1} \} \)

\( j \in A_\gamma = \{ j \in \mathbb{Z} : -nM_\gamma + 1 \leq j \leq -1 \text{ or } 3 \leq j \leq nM_\gamma + 1 \}. \)

The sets \( V_{(\gamma,n)} \) are open \( F \) subsets of \( K \) and \( \bigcup_{j \in A_\gamma} V_{(\gamma,n)} = \bigcup_{j \in A_\gamma} \{ x \in K : |x(\gamma)| < \frac{n}{n+1} \}. \)

The family \( \{ V_{(\gamma,n)} : \gamma \in \Gamma, n \in \mathbb{N}, j \in A_\gamma \} \) separates the points of \( K \).

Let \( \{ V_{(\gamma,n)} : m \in \mathbb{N} \} \) be the decomposition of \( \Gamma \) corresponding to \( \epsilon = \frac{1}{n} \). Decompose \( K \) into the families \( \{ K_{(m,n),n} \} \) where, for every \( (m,n) \in \mathbb{N}^2 \),

\( K_{(m,n),n} = \{ V_{(\gamma,n)} : \gamma \in \Gamma_{(m,n)} \} \).

We shall prove that every \( K_{(m,n),n} \) is point finite. Suppose, to the contrary, that there exists \( x \in \bigcap_{i \in I} V_{(\gamma,n)} \) where \( \{ V_{(\gamma,n)} : i \in I \} \) is an infinite subset of \( K_{(m,n),n} \).

Then \( |x(\gamma)| > \frac{1}{n} \) for every \( \gamma \in \Gamma \). This is a contradiction, because \( \{ \gamma : \gamma \in \Gamma \} \) is an infinite subset of \( \Gamma_{(m,n)} \).

(1=2) Denote by \( K_i = \{ x \in K \} \). Using Propositions 2.2 and 2.6, the set \( K \) is E.C., so there exists a homeomorphism \( f : K \to L \) satisfying \( f(0) = 0 \) and \( |f(\gamma)| \leq n \) for every \( x \in K_i \).

For a fixed number \( \epsilon, 0 < \epsilon < 1 \) and any \( \gamma \in \Gamma \), define the open sets

\( V_\gamma = \{ x \in K : |x(\gamma)| > \frac{\epsilon}{n} \}. \)

Note that \( 0 \notin V_\gamma \) and \( d_\gamma = \frac{\epsilon}{2} \gamma \) belongs to \( V_\gamma \) by Lemma 2.7. Since \( V_\gamma \) is open there exists a basic neighborhood \( W_\gamma \) of \( f(d_\gamma) \) with \( W_\gamma \subseteq f(V_\gamma) \). Let

\[ W_\gamma = \{ y \in f(K_i) : |f(d_\gamma(y)) - y(d_\gamma(y))| < \epsilon \text{ for every } 1 \leq i \leq n \}. \]
where \( \{ \lambda_1^\gamma, \ldots, \lambda_k^\gamma \} \subseteq A \) and \( \epsilon_\gamma > 0 \) for every \( \gamma \in \Gamma \). Since \( 0 \notin W_\gamma \), there is \( \lambda_1^\gamma = \lambda_1 \) such that \( |f(d_\gamma)(\lambda_1)| > 0 \) and we can assume that \( 0 < \epsilon_\gamma < \frac{|f(d_\gamma)(\lambda_1)|}{2} \).

Hence it is easy to see that

\[
|y(\lambda_1)| > \frac{|f(d_\gamma)(\lambda_1)|}{2} > \epsilon_\gamma \quad \text{for all } y \in W_\gamma.
\]

For each \( \gamma \in \Gamma \) we can choose a finite subset \( S_\gamma \) of \( \Gamma \) and \( 0 < \epsilon_\gamma < \frac{\epsilon}{4} \) such that the set

\[
U_\gamma = \left\{ x \in K_1 : |x(\delta)| < \epsilon_\gamma \text{ for any } \delta \in S_\gamma - \{ \gamma \} \text{ and } \left| x(\gamma) - \frac{\epsilon}{2} \right| < \epsilon_\gamma \right\}
\]

is an open neighborhood of \( d_\gamma \), contained in the set \( f^{-1}(W_\gamma) \).

Notice that the set

\[
A_\gamma = \left\{ \delta \in \Gamma : |f(d_\gamma)(\lambda_1)| > 2\epsilon_\gamma \right\}
\]

is finite.

For every \( (n, m) \in \mathbb{N}^2 \) put

\[
\Gamma^{(n, m)}_\gamma = \left\{ \gamma \in \Gamma : |f(d_\gamma)(\lambda_1)| > \frac{1}{n} \text{ and } \text{card}(S_\gamma \cup A_\gamma) < m \right\}.
\]

The family \( \{ \Gamma^{(n, m)}_\gamma : (n, m) \in \mathbb{N}^2 \} \) is a decomposition of \( \Gamma \).

The proof of the theorem will be complete if we can prove the claim:

For any \( x \in K_1 \) and \( (n, m) \in \mathbb{N}^2 \)

\[
\text{card} \left\{ \gamma \in \Gamma^{(n, m)}_\gamma : |x(\gamma)| > \epsilon \right\} < \omega.
\]

Suppose, to the contrary, that there exists an \( x \in K_1 \) and \( (n, m) \in \mathbb{N}^2 \) such that the set \( A = \bigcap_{\gamma \in \Gamma^{(n, m)}_\gamma} |x(\gamma)| > \epsilon \right\} \) is infinite.

According to Lemma 2.8, there exists an infinite subset \( B \) of \( A \) such that for \( \gamma_1, \gamma_2 \) elements of \( B \) with \( \gamma_1 \neq \gamma_2 \), \( \gamma_1 \notin A_1 \cup S_1 \).

Let \( x_1 = \sum x_1 \). Then the functions \( x_1, x_1|_B \) and \( y = \frac{\epsilon}{\gamma} \sum y \neq B \) belong to the set \( K_1 \) according to Lemma 2.7.

Therefore \( y \in \bigcap_{B \neq B} U_\gamma \), because for every \( \gamma_1 \neq \gamma_2 \) in \( B \) holds \( \gamma_1 \notin S_\gamma \).

Consequently \( f(y) \in \bigcap_{B \neq B} W_\gamma \) and by (*)

\[
|f(y)(\lambda_1)| > \frac{|f(d_\gamma)(\lambda_1)|}{2} > \frac{1}{2n}
\]

for every \( \gamma \in B \).

This is a contradiction because for every \( \gamma_1 \neq \gamma_2 \) in \( B \), \( \lambda_1 \neq \gamma_3 \) \( (\gamma_1 \neq A_1) \) and \( f(y) \in c_0(A) \).

If \( K \) is U. E. C., the cardinality of the sets \( \{ \gamma \in \Gamma^{(n, m)} : |x(\gamma)| > \epsilon \} \) can be controlled. In fact, we have the following characterization:

**Theorem 2.10.** Let \( K \) be a compact subset of \( \Sigma(R^\Gamma) \). The following are equivalent:

1. The set \( K \) is U. E. C.
2. For every \( \epsilon > 0 \) there exists a decomposition \( \{ \Gamma^{(0)}_m : m \in \mathbb{N} \} \) of \( \Gamma \) and a sequence \( \{ k(m) : m \in \mathbb{N} \} \), such that for every \( x \in K \) and \( m \in \mathbb{N} \)

\[
\text{card} \left\{ \gamma \in \Gamma^{(0)}_m : |x(\gamma)| > \epsilon \right\} < k(m).
\]

**Proof.** \((\Rightarrow)\) As in Theorem 2.9 we define the family \( \mathcal{K} \) of open \( F_\alpha \) subsets of \( K \), which separate the points of \( K \). For \( (n, m, l) \in \mathbb{N}^2 \), let

\[
\mathcal{K}_{(n, m, l)} = \left\{ V^{(l)}_\gamma : \lambda \in \Gamma^{(0)}_m \text{ such that card} \left\{ \lambda \in \Gamma^{(0)}_m : |x(\gamma)| > \epsilon \right\} < l \right\}.
\]

Clearly \( \mathcal{K}_{(n, m, l)} \) is a decomposition of \( \mathcal{K} \). Further, \( \mathcal{K}_{(m, n, l)} \) is \( \mathcal{K}_{(n, m, l)} \) \( l \)-finite.

(The proof is analogous to the corresponding one in 2.9). Hence Proposition 1.1 implies that \( K \) is U. E. C.

\((\Rightarrow)\) Let \( K \) be the set \( \bar{c}_0(\mathcal{F}(\mathcal{K}) \cup \{ \gamma \in \Gamma \}) \).

Using the Propositions 2.2 and 2.6 the set \( K_1 \) is U. E. C. So there exists a homeomorphism \( f : K_1 \to I_2(A) \) satisfying \( f(0) = 0 \) and \( \|f(x)\| < 1 \) for every \( x \in K_1 \).

For a fixed \( 0 < \epsilon < 1 \) we define the decomposition \( \{ \Gamma^{(n, m)}_\gamma : (n, m) \in \mathbb{N}^2 \} \) of \( \Gamma \) as in the proof of Theorem 2.9. It now follows, using methods analogous to the ones in [2], (Theorem 1.7), that, for any \( x \in K_1 \)

\[
\text{card} \left\{ \gamma \in \Gamma^{(n, m)}_\gamma : |x(\gamma)| > \epsilon \right\} < (2n)^{2\omega}
\]

and the proof of the theorem is complete.

In the following theorem, we characterize T. C. subsets of \( \Sigma(R^\Gamma) \) as subsets of a space \( c(\Gamma_\alpha ; \sigma) \in \Sigma \), thus improving a result of Mercourakis [7], to the effect that T. C. spaces are homeomorphic to compact subsets of \( c(\Gamma_\alpha ; \sigma) \in \Sigma \).

**Theorem 2.11.** Let \( K \) be a compact subset of \( \Sigma(R^\Gamma) \). The following are equivalent:

1. The set \( K \) is T. C.
2. There exists a decomposition \( \{ \Gamma_\alpha : \sigma \in \Sigma \} \) of \( \Gamma \) such that \( \Gamma_\alpha \subseteq \Gamma_\beta \) if \( \sigma_\alpha < \sigma_\beta \) and \( \{ x_\alpha : x \in K \} \subseteq c_0(\Gamma_\alpha) \) for every \( \sigma \in \Sigma \).

**Proof.** \((\Rightarrow)\) \( K \) is a subset of \( c(\Gamma_\alpha ; \sigma) \in \Sigma \) so \( K \) is T. C.

\((\Rightarrow)\) If \( K_1 = \bar{c}_0(\mathcal{F}(\mathcal{K}) \cup \{ \gamma \in \Gamma \}) \) then \( K_1 \) is T. C. according to Propositions 2.2 and 2.6. So there exists an homeomorphism \( f : K_1 \to c(A) : \sigma \in \Sigma \) satisfying \( f(0) = 0 \) and \( \|f(x)\| < 1 \) for every \( x \in K_1 \).

As in the proof of Theorem 2.9, we define for fixed \( 0 < \epsilon < 1 \) and \( \lambda \in \Gamma \), the open sets \( V_\lambda \subseteq K_1, W_\lambda \subseteq f(V_\lambda) \), \( U_\lambda \subseteq f^{-1}(W_\lambda) \) and the finite subsets \( A_\lambda \) of \( \Gamma \).
For every $(n, m) \in \mathbb{N}^2$ and $\sigma \in \Sigma$, set

$$
\Gamma_{(n, m, \sigma)}^0 = \left\{ \gamma \in \Gamma: \lambda_\gamma \in A_\sigma, |f(d_\gamma)(\lambda_\gamma)| > \frac{1}{n} \text{ and } \operatorname{card}(\gamma \cup A_\sigma) < m \right\}.
$$

We claim that:

1. $\Gamma = \bigcup \{\Gamma_{(n, m, \sigma)}^0: (n, m) \in \mathbb{N}^2, \sigma \in \Sigma\}$;
2. If $c_1 < c_2$, then $\Gamma_{(n, m, \sigma)}^0 \subseteq \Gamma_{(n, m, \sigma)}^c$;
3. (card $\{\gamma \in \Gamma_{(n, m, \sigma)}^0: |x(\gamma)| > \varepsilon\} < \omega$;

for every $x \in K, (n, m) \in \mathbb{N}^2$ and $\sigma \in \Sigma$.

It is easy to check that conditions 1 and 2 are true. Suppose that condition 3 is false and that the set $A = \{\gamma \in \Gamma_{(n, m, \sigma)}^0: |x(\gamma)| > \varepsilon\}$ is infinite for some $x \in K$ and $(n, m, \sigma) \in \Sigma$. Then, by Lemma 2.8, we can choose an infinite subset $B$ of $A$ such that

1. $\gamma_1, \gamma_2$ elements of $B, \gamma_1 \notin A_{\sigma_1} \cup S_{\sigma_1}$. Therefore, the element $y = \sum_{\gamma_1 \in B} c_\gamma$ of $K_1$

satisfies the relation $y \in \bigcap_{\gamma \in B} U_{\gamma}$ and for every $\gamma_1 \neq \gamma_2$ in $B$,

Consequently

$$
f(\gamma) \notin \bigcap_{\gamma \in B} W_\gamma \text{ and therefore } |f(\gamma)(\lambda_\gamma)| > \frac{1}{2n},
$$

for every $\gamma \in B$. This is a contradiction because $\lambda_\gamma \in A_\sigma$ and $f(\gamma)|_{A_\sigma} \in c_0(A_\sigma)$. Therefore condition 3 holds.

For every $\sigma \in \Sigma$, set $\sigma^* = (s(3), s(4), \ldots) \in \Sigma$ and

$$
\Gamma_{(\sigma^* \times \mathbb{N})}^0 = \bigcup_{1 \leq m < \omega} \Gamma_{(\sigma, m, \sigma)}^0
$$

and

$$
\Gamma_{(\sigma_1, \sigma_2, \ldots)} = \bigcap_{n \in \mathbb{N}} \Gamma_{(\sigma_1, \sigma_2, \ldots)}^{(n)} \text{ for } (\sigma_1, \sigma_2, \ldots) \in \Sigma^N.
$$

We then have

4. $\Gamma = \bigcup \{\Gamma_{(\sigma_1, \sigma_2, \ldots)}: (\sigma_1, \sigma_2, \ldots) \in \Sigma^N\}$;
5. If $(\sigma_1, \sigma_2, \ldots) \neq (\sigma_1', \sigma_2', \ldots)$ then $\Gamma_{(\sigma_1, \sigma_2, \ldots)} \subseteq \Gamma_{(\sigma_1', \sigma_2', \ldots)}$;
6. card $\{\gamma \in \Gamma_{(\sigma_1, \sigma_2, \ldots)}: |x(\gamma)| > \varepsilon\} < \omega$

for every $x \in K, \varepsilon > 0$ and $(\sigma_1, \sigma_2, \ldots) \in \Sigma^N$.

Conditions 4 and 5 are consequences of 1 and 2. Condition 6 is true because for every $\varepsilon > 0$ there exists an $\eta \in \mathbb{N}$ such that $\varepsilon > \frac{1}{n}$. But, for all $x \in K$,

$$
\operatorname{card}\{\gamma \in \Gamma_{(\sigma_1, \sigma_2, \ldots)}: |x(\gamma)| > \varepsilon\} \leq \operatorname{card}\{\gamma \in \Gamma_{\sigma_1}^{(n)}: |x(\gamma)| > \frac{1}{n}\}.
$$

which is finite by condition 3 and the definition of $\Gamma_{\sigma_1}^{(n)}$. Hence

$x_{\Gamma_{(\sigma_1, \sigma_2, \ldots)}} \in c_0(\Gamma_{(\sigma_1, \sigma_2, \ldots)})$

and the proof of the theorem is complete.

After the characterization of E.C. sets described in Theorem 2.9, the question arises whether the decomposition of $\Gamma$ can be made independent of $\varepsilon$, as for T.C. sets (i.e., the $x_{\Gamma_{(\sigma_1, \sigma_2, \ldots)}} \in c_0(\Gamma_{(\sigma_1, \sigma_2, \ldots)})$

This is impossible in general, as shown by Example 2.15 below. However, if $K$ consists of characteristic functions of subsets of $\Gamma$, then the decomposition can easily be made independent of $\varepsilon$, by choosing $\varepsilon = \frac{1}{k}$ in Theorem 2.9. Thus, we have:

**Corollary 2.12.** Let $\Gamma$ be a nonempty set and $\mathcal{A}$ be a family of subsets of $\Gamma$ such that the set $K = \{K_x: S \in \mathcal{A}\}$ is a compact subset of $\Sigma(R^\omega)$. The set $K$ is E.C. if there exists a decomposition $\{\Gamma_m: m \in \mathbb{N}\}$ of $\Gamma$ satisfying the condition

$$
\operatorname{card}(X \cap \Gamma_m) < \omega
$$

for every $X \in \mathcal{A}$ and $m \in \mathbb{N}$.

For the case of U. E.C. subsets of $\Sigma(R^\omega)$ which consist of characteristic functions, we have from Theorem 2.10 the following corollary. This corollary was proved in [2] for the special case where $K$ consists of characteristic functions of finite subsets of $\Gamma$.

**Corollary 2.13.** Let $\mathcal{A}$ be a family of subsets of a set $\Gamma$ such that the set $K = \{K_x: S \in \mathcal{A}\}$ is a compact subset of $\Sigma(R^\omega)$. The set $K$ is U. E.C. if there exists a decomposition $\{\Gamma_m: m \in \mathbb{N}\}$ of $\Gamma$ and a sequence $(k(m))$ of natural numbers satisfying the condition

$$
\operatorname{card}(X \cap \Gamma_m) < k(m)
$$

for every $X \in \mathcal{A}$ and $m \in \mathbb{N}$.

The characterization of E.C. and T.C. subsets of $\Sigma(R^\omega)$ given by Theorems 2.9 and 2.11 gives a very simple proof that Talagrand’s example [11] is a T.C. space which is not E.C.}

**Example 2.14.** Denote the set $N^\omega$ by $\Sigma$ and define the families $\mathcal{B} = \{\sigma: \sigma \in \Sigma\}$ and, for every $n \in \mathbb{N}$,

$$
\mathcal{B}_n = \{S \subseteq \mathbb{N}: |S| \geq 2, \text{ for all } n \in \mathbb{N}, \sigma \in S, \sigma \in \mathcal{B}_n, \sigma(n) \neq \sigma(n+1)\}
$$

where $\sigma_n = (\sigma(1), \sigma(2), \ldots, \sigma(n)) \in N^\omega$. Let $\mathcal{A} = \mathcal{B} \cup \{\bigcup_{n \in \mathbb{N}} \mathcal{B}_n\}$ and $K = \{K_x: S \in \mathcal{A}\}$. It is easy to check that the set $K$ is a compact subset of $\Sigma(R^\omega)$.

For every $\sigma \in \Sigma$ define

$$
\Sigma_\sigma = \{\varsigma \in \Sigma: \varsigma < \sigma\}.
$$
Then \( \{ \sigma_n : \sigma \in \Sigma \} \) is a decomposition of \( \Sigma \) and \( \Sigma_n \subseteq \Sigma_n \) if \( \sigma_1 < \sigma_2 \). Also,

\[
x_{|x|} : x \in K \subseteq c_0(\Sigma_n)
\]

because

\[
\text{card}(S \cap \Sigma) < \sigma(n) + 1
\]

for every \( S \in \mathcal{A} \).

Hence \( K \) is T. C. by Theorem 2.11.

Assume \( K \) is E. C. Then from Corollary 2.12 there exists a decomposition \( \{ \Gamma_m : m \in N \} \) of \( \Sigma \) such that

\[
\text{card}(S \cap \Sigma) < \omega
\]

for every \( S \in \mathcal{A} \) and \( m \in N \).

From Baire's theorem there exists an \( m_0 \in N \) such that the closure of \( \Sigma_{m_0} \) has a non-empty interior. If \( x \in \Sigma_{m_0} \) we can find an open set \( V_x = \{ y \in \Sigma : y_{|x|} = x_{|x|} \} \) so that \( V_x \subseteq \Sigma_{m_0} \). Hence for every \( k \in N \) there exists \( \gamma_k \in \Sigma_k \) where

\[
V_k = \{ y \in \Sigma : y_{|x|} = x_{|x|} \text{ and } y(n_0 + 1) = k \}.
\]

The set \( S = \{ \gamma_k : k \in N \} \) belong to \( \mathcal{A} \) and is a subset of \( \Sigma_{m_0} \). Therefore

\[
\text{card}(S \cap \Sigma) = \omega
\]

This is a contradiction, hence \( K \) is not an E. C. space.

In the following example we describe a compact subset of a space \( \Sigma(R^3) \) which is E. C., in fact U. E. C., but there does not exist a decomposition \( \{ \Gamma_m : m \in N \} \) of \( \Gamma \) such that

\[
\{ x_{|x|} : x \in K \} \subseteq c_0(G_m)
\]

for every \( m \in N \).

EXAMPLE 2.15 (in cooperation with S. Argyros). As in Example 2.14 we define the family \( \mathcal{A} = \mathcal{B} \cup \bigcup_{m \in N} \mathcal{A}_m \) of subsets of \( \Gamma \). For every \( S \) in \( \bigcup_{m \in N} \mathcal{A}_m \) there exists only one \( n \in N \) such that \( S \in \mathcal{A}_n \). If \( S \in \mathcal{A}_n \) we define the function \( \psi_x = \frac{1}{n} I_n \). For every \( \sigma \in \Sigma \) and \( k \in N \) we define \( \psi_{x_k} = \frac{1}{k} I_k \).

Denote by

\[
K = \{ \psi_x : S \in \bigcup_{n \in N} \mathcal{A}_n \} \cup \{ \psi_x : k \in N, \sigma \in \Sigma \} \cup \{ 0 \}
\]

The set \( K \) is a compact subset of \( \Sigma(R^3) \). To prove this it is enough to show that if \( x_{|x|} \) is a net in \( K \) which converges to \( x \in [-1, 1]^2 \) then \( x \in K \). It is clear that

\[
x(\sigma) : \sigma \in \Sigma \subseteq \{x_n : n \in N \} \cup \{0 \}
\]

If \( x(\sigma_1) = \frac{1}{k} \) and \( x(\sigma_2) = \frac{1}{\lambda} \) for \( k \neq \lambda \) in \( N \) then there exists \( i \in I \) such that

\[
x(\sigma_1) = \frac{1}{k} \quad \text{and} \quad x(\sigma_2) = \frac{1}{\lambda}
\]

This is a contradiction, therefore \( x = 0 \) or \( x = \frac{1}{k} I_k \)

for some \( k \in N \) and \( S \subseteq \Sigma \).

Let \( \text{card}S \geq 2 \) and \( \sigma_1 \neq \sigma_2 \) in \( S \). Then there exists \( i \in I \) such that \( x_i(\sigma_1) = x_i(\sigma_2) = \frac{1}{k} \). Therefore \( x_i = \frac{1}{k} \psi_x\) where \( S_i \in \mathcal{A}_k \) and then \( \{ \sigma_1, \sigma_2 \} \in \mathcal{A}_k \). Hence \( x \in K \) and consequently \( K \) is compact.

We will prove that \( K \) is U. E. C. using Theorem 2.10. For every \( \varepsilon = \frac{1}{n} \) define the sets

\[
\{ \sigma \in \Sigma : \sigma \in K \} = \{ (\sigma_1, \ldots, \sigma_{n+1}) : (\sigma_1, \ldots, \sigma_{n+1}) \in N^{n+1} \}
\]

for every \( (k_1, \ldots, k_{n+1}) \in N^{n+1} \).

These sets form a decomposition of \( \Sigma \) and it is easy to check that

\[
\text{card} \{ \sigma \in \Sigma : x(\sigma) > \frac{1}{n} \} \leq 1
\]

for every \( x \in K \) and \( (k_1, \ldots, k_{n+1}) \in N^{n+1} \).

In the sequel we shall show that there doesn't exist a decomposition \( \{ \Sigma_m : m \in N \} \) of \( \Sigma \) such that

\[
\{ x_{|x|} : x \in K \} \subseteq c_0(\Sigma_m) \quad \text{for every} \quad m \in N.
\]

Let \( \{ \Sigma_m : m \in N \} \) be a decomposition of \( \Sigma \). According to Baire's theorem, there exists an \( n_0 \in N \) so that \( (\Sigma_m)^{n_0} \neq \emptyset \). In the same way as in Example 2.14 there exists \( n_0 \in N \) and \( S \in \mathcal{A}_{n_0} \) such that \( \text{card}(S \cap \Sigma_m) = \omega \).

If \( \psi_x = \frac{1}{n_0} I_1 \) and \( \varepsilon < \frac{1}{n_0} \), then

\[
\text{card} \{ \sigma \in \Sigma_m : |x(\sigma) - \varepsilon| = \text{card}(S \cap \Sigma_m) = \omega
\]

so \( \psi_x \) does not belong to \( c_0(\Sigma_m) \).

References


Automorphisms of the Loeb algebra

by

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Abstract. Let $(Ω, L(Ω), L(μ))$ be a uniform hyperfinite probability space in a sufficiently saturated nonstandard model of analysis. We prove: 1. Every automorphism of the measure algebra over $Ω$ is induced by an invertible point transformation. 2. Some automorphisms are not given by internal transformations. 3. The restriction of every automorphism to a small subalgebra is given by an internal transformation.

We discuss applications to ergodic theory and hyperfinite measure theory.

1. Introduction. Suppose $T$ is an invertible transformation, measurable in both senses, of a probability space $(X, ℬ, m)$. $T$ induces a Boolean (σ-) automorphism $Φ = Φ_T$ of the measure algebra $[ℬ]$ associated with $(X, ℬ, m)$. Considerations from Ergodic Theory motivate the converse question: When is a given automorphism $Φ$ induced by a transformation $T$?

The answer in “always” for most common spaces (von Neumann [14], Choksi [4]). For those spaces $(X, ℬ, m)$ and automorphisms $Φ$ of $[ℬ]$ not induced by a transformation, some authors have asked weaker questions, for example (Panzone and Segovia [15]), whether $Φ$ is induced by a transformation $T$ of a thick subset of $X$.

We consider here the question when $(X, ℬ, m)$ is the uniform hyperfinite probability space $(Ω, L(Ω), L(μ))$ deeply investigated by Loeb [13], Anderson [1] and others. This space has a variety of “universality” properties (Anderson [1], Hoover [9], Keisler [12]) which allow questions about more general or common spaces to be reduced to questions about $Ω$. (For a further discussion, see Section 5.)

Our main result, Theorem 4.1, is that in the presence of sufficient saturation, every measure algebra automorphism is indeed given by a permutation of $Ω$.

Since in application the most useful transformations of $Ω$ are the internal ones, we consider whether the transformation in Theorem 4.1 can always be taken to be internal. Theorem 4.3 gives a negative answer. However, the restriction of $Φ$ to any sufficiently small subset of $[L(Ω)]$ is induced by an internal permutation; this is Theorem 4.4. (Another proof of Theorem 4.4, using Hall’s “Marriage Lemma”, appears in Ross [16].)

We give some applications of these results in Section 5. Proposition 5.1 shows