References

[1] A. Bruckner, R. O'Malley and B. Thomson, *Path derivatives: a unified view of certain generalized derivatives*, Trans. Amer. Math. Soc. 283 (1984), 97–125.

[2] R. O'Malley, *Decomposition of approximate derivatives*, Trans. Amer. Math. Soc. 69 (1978), 243–247.

[3] R. O'Malley and C. E. Weil, *Selective, bi-selective and composite differentiation*, Acta Math. Hung. 43 (1984), 31–36.

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Fix-finite and fixed point free approximations of symmetric product maps

by

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Abstract. Let X be a locally finite simplicial complex with the weak topology. It is shown that every symmetric product map $f\colon X\to X^n/G$ is homotopic to a symmetric product map $f'\colon X\to X^n/G$ so that all fixed points of f' are isolated and so that the fixed point set of f' is finite (countable) if X is a finite (countable) complex. In the former case f' can be chosen so that, for *every* $m\geqslant 1$, its set of periodic points of period $m\leqslant 1$ is finite as well. If X is a noncompact manifold, then f can be homotoped to a fixed point free symmetric product map.

1. Introduction. Let X be a topological space and X^n be the n-fold Cartesian product of X with the product topology. Any (proper or improper) subgroup G of the symmetric group S_n of all permutations of $\{1, 2, \ldots, n\}$ acts on X^n as a group of homeomorphisms by permuting its factors. Let X^n/G be the orbit space with the quotient topology induced by the quotient map $q\colon X^n\to X^n/G$. Then a map (i.e. a single-valued continuous function) $f\colon X\to X^n/G$ is called a *symmetric product map*, and a point $x\in X$ is called a *fixed point* of the symmetric product map f if $f(x) = q(z)$, where $z\in X^n$, implies that x is a coordinate of z. Fixed points of symmetric product maps have been studied e.g. by S. Kwasik [5], C. N. Maxwell [8], [9], S. Masih [6], [7], Nancy Rallis [11] and C. Vora [17], [18]. Periodic points of symmetric product maps (see the definition in § 3) have been considered by Nancy Rallis [12].

In this paper we extend to symmetric product maps the Hopf approximation theorem which states that every selfmap of a compact polyhedron is homotopic to a fix-finite one (see e.g. [2], Ch. VIII A, Theorem 2, p. 118), and also prove related results for noncompact polyhedra. In Theorem 1 we show that every symmetric product map $f\colon X\to X^n/G$ is homotopic to a symmetric product map $f'\colon X\to X^n/G$, which has a finite (countable) fixed point set, if X is a finite (countable) polyhedron. Theorem 2 states that if X is finite, then f' can be chosen so that it has, for *every* $m\geqslant 1$, at most finitely many points of period $\leqslant m$. Finally, Theorem 1 is used to show, in Theorem 3, that if X is a noncompact manifold, then f is homotopic to a fixed point free symmetric product map.

As $X^n/G = X$ if $n = 1$, these theorems extend the Hopf approximation theorem, an approximation theorem for periodic points by Boju Jiang [4], p. 62, and, for noncompact manifolds, a result by G.-H. Shi [15], Theorem 1. A standard technique for the proof of the Hopf approximation theorem is the so-called "Hopf construction" [2], pp. 117–118, which yields a stronger result, as it leads to a fix-finite map which has all fixed points situated in maximal simplexes. But the Hopf construction is lengthy, and we replace it here by an idea from [15] in order to obtain a fairly simple proof of Theorem 1.

We also use the fact that a symmetric product map can be interpreted as a multifunction. For if $\pi\colon X^n \to X$ is the multifunction given by $\pi(x_1, x_2, \ldots, x_n) = \{x_1, x_2, \ldots, x_n\}$, then a symmetric product map $f\colon X \to X^n/G$ defines a multifunction $\varphi\colon X \to X$ by $\varphi = \pi \circ q^{-1} \circ f$. We say that φ is *induced* by f. Clearly φ is finite-valued, as each point image consists of at most n points, and continuous (i.e. both upper and lower semi-continuous, see e.g. [1], p. 109, for definitions). If the fixed point set $\operatorname{Fix}\varphi$ of φ is defined, as usual, by $\operatorname{Fix}\varphi = \{x \in X \mid x \in \varphi(x)\}$, then $\operatorname{Fix}\varphi = \operatorname{Fix} f$. For $n = 2$ each $\varphi(x)$ consists of one or two points, so in this case φ belongs to the class of continuous multifunctions which have the property that each point image consists of either one or n points. A Lefschetz number for such multifunctions has been defined by B. O'Neill [10]. Our results are also related to those in [13] concerning fix-finite approximations of multifunctions whose point images consist of exactly n points.

This research was conducted while I was a visiting faculty member at the University of California, Los Angeles. I wish to thank Robert F. Brown for many stimulating and helpful discussions.

2. Simplicial multifunctions. In order to prove Lemma 1, which is the main tool for all our results, we need two definitions concerning finite-valued multifunctions from [13]. We say that a multifunction $\varphi\colon X \to Y$ from a topological space X to a topological space Y *splits into maps* if there exist finitely many maps $f_i\colon X \to Y$, where $i = 1, 2, \ldots, n$, so that $\varphi(x) = \{f_1(x), f_2(x), \ldots, f_n(x)\}$ for all $x \in X$. Now let $|K|$ denote a polyhedron, i.e. a simplicial complex K with the weak topology. Then we say that a multifunction $\varphi\colon |K| \to |L|$ from a polyhedron $|K|$ into a polyhedron $|L|$ is a *simplicial multifunction from* $|K|$ to $|L|$ if, for every closed simplex $\bar{\sigma} \in |K|$, the restriction $\varphi|\bar{\sigma}$ splits into maps f_i so that each f_i maps $\bar{\sigma}$ affinely onto a simplex $\bar{\tau}_i \in |L|$.

LEMMA 1. *Let $X = |K|$ be a polyhedron and $f\colon X \to X^n/G$ a symmetric product map. Then there exist a subdivision $|K'|$ of $|K|$ and a symmetric product map $f'\colon X \to X^n/G$ homotopic to f so that the induced multifunction of f' is a simplicial multifunction $\varphi'\colon |K'| \to |K|$.*

Proof. Let K^n denote the usual product triangulation of X^n, let $\mathrm{Sd}(K^n)$ be the first barycentric subdivision of K^n, and let $K(n, G)$ be the triangulation of X^n/G constructed from the set of equivalence classes of vertices of $\mathrm{Sd}(K^n)$ under the action of G by C. N. Maxwell [8], p. 809. (See also [7], p. 144.) Choose a subdivision $|K'|$ of $|K|$ so that $f\colon X \to X^n/G$ has a simplicial approximation $f''\colon |K'| \to |K(n, G)|$. As $\pi\colon |\mathrm{Sd}(K^n)| \to |K|$ is not simplicial, the induced multifunction of f'' need not be simplicial either. We remedy this situation by using the simplicial map $s\colon |\mathrm{Sd}(K^n)| \to |K^n|$ which associates with each vertex of $\mathrm{Sd}(K^n)$, obtained as the barycenter of a simplex τ of $|K^n|$, the last vertex of τ. (This map is denoted by Φ in [8], p. 809, and called a standard map in [3], p. 35.) We now define a multifunction $\varphi'\colon |K'| \to |K|$ by

$$\varphi' = \pi \circ s \circ q^{-1} \circ f''$$

(see Diagram 1).

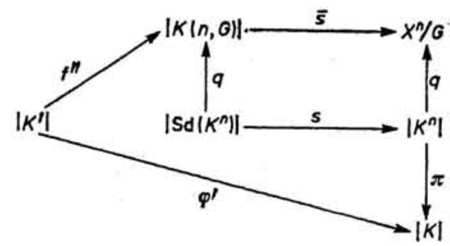

Diagram 1

As the map s is equivariant, it induces a map $\bar{s}\colon |K(n, G)| \to X^n/G$ so that $q \circ s = \bar{s} \circ q$ (see Diagram 1), and we can define a map $f'\colon |K'| \to X^n/G$ by

$$f' = \bar{s} \circ f''.$$

Then

$$\pi \circ q^{-1} \circ f' = \pi \circ q^{-1} \circ \bar{s} \circ f'' = \pi \circ s \circ q^{-1} \circ f'' = \varphi',$$

so φ' is the multifunction induced by f'. We will show that f' and φ' satisfy Lemma 1.

To see that φ' is a simplicial multifunction, let $\bar{\sigma}$ be a closed simplex of $|K'|$ and $\bar{\varrho}$ be a closed simplex of $|\mathrm{Sd}(K^n)|$ with $q(\bar{\varrho}) = f''(\bar{\sigma})$. If $\bar{\varrho}_1$ is any other simplex of $|\mathrm{Sd}(K^n)|$ with $q(\bar{\varrho}_1) = f''(\bar{\sigma})$, then $\bar{\varrho}_1 = g\bar{\varrho}$ for some $g \in G$. Now we have for any point $z_1 \in \bar{\varrho}_1$

$$\pi \circ s(z_1) = \sum_{i=1}^{n} \pi_i \circ s(z_1),$$

where $\pi_i\colon |K^n| \to |K|$ is the projection onto the ith factor. Hence, if $z_1 = gz$ for $z \in \bar{\varrho}_1$, then

$$\pi \circ s(z_1) = \sum_{i=1}^{n} \pi_i \circ s \circ gz = \sum_{i=1}^{n} \pi_i \circ gs(z) = \sum_{i=1}^{n} \pi_{g(i)} \circ s(z) = \pi \circ s(z).$$

Thus, if $r\colon f''(\bar{\sigma}) \to \bar{\varrho}$ is the affine map obtained from the multifunction $q^{-1}\colon f''(\bar{\sigma}) \to |\mathrm{Sd}(K^n)|$ by restriction of the range to $\bar{\varrho}$, then

$$\varphi'|\bar{\sigma} = \sum_{i=1}^{n} \pi_i \circ s \circ r \circ f''|\bar{\sigma}$$

splits into the n maps $h_i = \pi_i \circ s \circ r \circ f''|\bar{\sigma}$, where $i = 1, 2, \ldots, n$. As the $\pi_i\colon |K^n| \to |K|$ are simplicial, the h_i are affine maps onto simplexes of $|K|$. So φ' is a simplicial multifunction.

It remains to show that f' is homotopic to f. As z and $s(z)$ lie in the same closed simplex of $|K^n|$, we can define a homotopy $H\colon X^n \times I \to X^n$ from the identity map on X^n to s by the condition that $H(z, t)$ is the point on the line segment from z to $s(z)$ for which

$$d(z, H(z, t)) = t \cdot d(z, s(z)),$$

where d is the barycentric metric of $|K^n|$. It follows from the facts that s is equivariant and d is invariant under the action of g that

$$H(gz, t) = gH(z, t) \quad \text{for all } z \in X^n, t \in I \text{ and } g \in G.$$

Thus H induces a homotopy $\bar{H}\colon (X^n/G) \times I \to X^n/G$ from the identity on X^n/G to $\bar{s}$, and Diagram 2, in which 1 denotes the identity on I, is commutative.

$$\begin{array}{ccccc} & & X^n \times I & \xrightarrow{\;H\;} & X^n \\ & & \big\downarrow{\scriptstyle q \times 1} & & \big\downarrow{\scriptstyle q} \\ X \times I & \xrightarrow{\;f'' \times 1\;} & (X^n/G) \times I & \xrightarrow{\;\bar{H}\;} & X^n/G \end{array}$$

Diagram 2

If we define the homotopy $F\colon X \times I \to X^n/G$ by $F = \bar{H} \circ (f'' \times 1)$, then F is a homotopy from f'' to f'. As f'' is a simplicial approximation of f, we see that f' is homotopic to f.

6. Symmetric product maps with isolated fixed points and periodic points. The proof of the next lemma uses an idea from the proof of Lemma 3 in [15].

LEMMA 2. *Let $X = |K|$ be a polyhedron, $|K'|$ a subdivision of $|K|$ and $f\colon X \to X^n/G$ a symmetric product map which induces a simplicial multifunction $\varphi\colon |K'| \to |K|$. Then each open simplex of $|K'|$ contains at most n fixed points of f.*

Proof. Let σ be an open simplex of $|K'|$, let $\bar{\sigma}$ be the corresponding closed simplex and let $\varphi|\bar{\sigma}$ split into affine maps $h_i\colon \bar{\sigma} \to \bar{\tau}_i$ from $\bar{\sigma}$ onto closed simplexes $\bar{\tau}_i$ of $|K|$. We assume by way of contradiction that σ contains $\geqslant n+1$ points of $\mathrm{Fix} f = \mathrm{Fix}\varphi$, which implies that there exists an index j so that $h_j\colon \bar{\sigma} \to \bar{\tau}_j$ has at least two fixed points on $\bar{\sigma}$. Let x_1 be one of them. Then it follows from $h_j(x_1) = x_1$ that τ_j is the carrier of x_1 in $|K|$, and as $|K'|$ subdivides $|K|$ and as σ is the carrier of x_1 in $|K'|$, the dimension $\dim\sigma \leqslant \dim\tau_j$. But h_j is affine, so $\dim\sigma \geqslant \dim\tau_j$, and we see that in fact $\dim\sigma = \dim\tau_j$.

h_j has at least one other fixed point x_2 on σ, so the line segment in $\bar{\sigma}$ through x_1 and x_2 is pointwise fixed under the affine map h_j. Let this line segment intersect the boundary of σ in y_1 and y_2. Repeating the above argument for y_1 and y_2 instead of x_1, we see that the carriers of y_1 and y_2 in $|K'|$ and $|K|$ must have the same dimension. But this implies that y_1 and y_2 lie on the same face of σ, which is impossible. Hence σ can contain at most one fixed point of h_j, and therefore at most n fixed points of φ.

Recall that a fixed point x of f is isolated if there exists a neighbourhood U of x so that f has no other fixed point on U. We now obtain Theorem 1 immediately from Lemmas 1 and 2.

THEOREM 1. *Let $X = |K|$ be a polyhedron and $f\colon X \to X^n/G$ a symmetric product map. Then f is homotopic to a symmetric product map $f'\colon X \to X^n/G$ so that*

(i) *all fixed points of f' are isolated if K is locally finite,*
(ii) *the fixed point set of f' is countable if K is countable,*
(iii) *the fixed point set of f' is finite if K is finite.*

Remark. A simplicial complex with the weak topology is metrizable if and only if it is locally finite [16], p. 119. An inspection of the proofs of Lemmas 1 and 2 shows that in this case the symmetric product map f' of Theorem 1 can be chosen, as in the Hopf approximation theorem for maps (see e.g. [2], Ch. VIII A, Theorem 2, p. 118), so that the distance between f and f' in the sup metric is arbitrarily small.

Nancy Rallis [12], Definition 2.1, defined that a point $x_1 \in X$ is a *periodic point of the symmetric product map* $f\colon X \to X^n/G$ *with period* $\leqslant m$ if there exist points $x_2, x_3, \ldots, x_m, x_{m+1} = x_1$ such that, for $2 \leqslant j \leqslant m+1$, $f(x_{j-1}) = q(z_j)$ implies that x_j is a coordinate of z_j. Hence x_1 is a periodic point of period $\leqslant m$ if and only if x_1 is a fixed point of the multifunction $\varphi^m = \varphi \circ \varphi \circ \ldots \circ \varphi\colon X \to X$. If $X = |K|$ is a compact polyhedron, then it is possible to approximate $f\colon X \to X^n/G$ by a symmetric product map $f'\colon X \to X^n/G$ which has finitely many periodic points of period $\leqslant m$ for *all* periods $m \geqslant 1$. To do so, take for each simplex of $|K|$ of positive dimension the distance between its barycenter and its boundary, and let ε be the shortest of these distances. If $|K|$ is compact, then $\varepsilon > 0$, and the subdivision K' of K in Lemma 1 can be chosen so that its mesh $\mu(K') < \varepsilon$. It is straightforward to check, along the lines of the proof of the Approximation Theorem in [4], pp. 62–63, that then the simplicial multifunction φ' induced by f' has the property that $\mathrm{Fix}\varphi'^m$ is finite for all $m \geqslant 1$. We leave this task to the reader, but state the result.

THEOREM 2. *Let $X = |K|$ be a compact polyhedron and $f\colon X \to X^n/G$ a symmetric product map. Then f is homotopic to a symmetric product map $f'\colon X \to X^n/G$ so that f' has, for every $m \geqslant 1$, at most finitely many points of period $\leqslant m$.*

4. Fixed point free symmetric product maps on noncompact manifolds. We will now show, again using ideas from [15], that if $X = |K|$ is a noncompact manifold, then $f\colon X \to X^n/G$ can be homotoped to a fixed point free map. Lemma 3 will replace [15], Lemma 1 in the final argument.

LEMMA 3. (Moving of fixed points). *Let M be a topological manifold, let $f_a\colon M \to M^n/G$ be a symmetric product map and A an arc in M from a to b with $\mathrm{Fix} f_a \cap A = \{a\}$. Then f_a is homotopic to $f_b\colon M \to M^n/G$ so that $\mathrm{Fix} f_b = (\mathrm{Fix} f_a - \{a\}) \cup \{b\}$.*

Proof. As $\operatorname{Fix} f_a$ is closed, there exists a tubular neighbourhood N of A in M such that $\operatorname{Fix} f_a$ intersects the closure of N in $\{a\}$ only. Let $h\colon M, \{a\} \to M, \{b\}$ be a homeomorphism which is homotopic to the identity and is equal to the identity on $M-N$, and let $h^n\colon M^n \to M^n$ be given by

$$h^n(x_1, x_2, \ldots, x_n) = \big(h(x_1), h(x_2), \ldots, h(x_n)\big).$$

Then $gh^n = h^n g$ for all $g \in G$, and so h^n induces a homeomorphism $\bar{h}\colon M^n/G \to M^n/G$. We define $f_b\colon M \to M^n/G$ by

$$f_b = \bar{h} \circ f_a \circ h^{-1}$$

(see Diagram 3).

$$\begin{array}{ccccccc} M & \xleftarrow{\ h\ } & M & \xrightarrow{\ f_a\ } & M^n/G & \xrightarrow{\ \bar{h}\ } & M^n/G \\ & & & & \uparrow q & & \uparrow q \\ & & & & M^n & \xrightarrow{\ h^n\ } & M^n \end{array}$$

Diagram 3

It is straightforward to check that f_b has the properties of Lemma 3.

THEOREM 3. *Let M be a countable locally finite simplicial complex with the weak topology, which is a connected noncompact manifold. Then every symmetric product map $f\colon M \to M^n/G$ is homotopic to a symmetric product map $g\colon M^n/G \to M^n/G$, which is fixed point free.*

Proof. Using Theorem 1 and Lemma 3 we can homotope f to $f'\colon M \to M^n/G$ so that $\operatorname{Fix} f'$ is countable and all fixed points of f' are isolated and lie in maximal simplexes. If $\dim M \geqslant 2$, then M is 2-dimensionally connected (see [15] for the definition), and the proof of Theorem 2 can be completed by a construction of g from f' which is quite analogous to the construction of F and f, case $|K| = |L|$, in the proof of Theorem 1 in [15], if we use our Lemma 3 instead of [15], Lemma 1. If $\dim M = 1$, then M is the real line, and the proof of Theorem 1 in [15] still works.

Remark. The hypothesis in Theorem 3 that M is a manifold can almost certainly be changed to include all 2-dimensionally connected polyhedra. But the proof of such a result would be considerably longer. Theorem 1 would have to be strengthened to yield a symmetric product map f' which has all fixed points contained in maximal simplexes, and the proof of Lemma 3 would change from a very simple to a very lengthy one. (Compare the proof of [15], Lemma 1 in [14] or in [2], Ch. VIII C, proof of Lemma 6, p. 135 ff.)

References

[1] C. Berge, *Topological Spaces*, Oliver & Boyd, Edinburgh and London, 1963.

[2] R. F. Brown, *The Lefschetz Fixed Point Theorem*, Scott, Foresman and Co., Glenview, Ill., 1971.

[3] P. J. Hilton and S. Wylie, *Homology Theory*, Cambridge University Press, 1960.

[4] Boju Jiang, *Lectures on Nielsen Fixed Point Theory*, Contemporary Mathematics v. 14, Amer. Math. Society, Providence, R. I., 1983.

[5] S. Kwasik, *Fixed points of symmetric product mappings of some nonmetrizable spaces*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. 25 (1977), 1271–1277.

[6] S. Masih, *Fixed points of symmetric product mappings of polyhedra and metric absolute neighborhood retracts*, Fund. Math 80 (1973), 149–156.

[7] —, *On the fixed point index and Nielsen fixed point theorem for symmetric product mappings*, Fund. Math. 102 (1979), 143–158.

[8] C. N. Maxwell, *Fixed points of symmetric product mappings*, Proc. Amer. Math. Soc. 8 (1957), 808–815.

[9] —, *The Degree of Multiple-Valued Maps of Spheres*, Springer Lecture Notes 666 (1977), 123–141.

[10] B. O'Neill, *Induced homology homomorphism for set-valued multifunctions*, Pacific J. Math. 7 (1957), 1179–1184.

[11] N. Rallis, *A fixed point index theory for symmetric product maps*, Manuscripta Math. 44 (1983), 279–308.

[12] —, *Periodic points of symmetric product mappings*, Fund. Math. 117 (1983), 61–66.

[13] H. Schirmer, *Fix-finite approximation of n-valued multifunctions*, Fund. Math. 121 (1983), 83–91.

[14] G.-H. Shi, *On the fewest numbers of fixed points and Nielsen numbers*, Acta Math. Sinica 16 (1966), 223–232.

[15] —, *On the fewest number of fixed points for infinite complexes*, Pacific J. Math. 103 (1982), 377–387.

[16] E. H. Spanier, *Algebraic Topology*, McGraw Hill, New York 1966.

[17] C. Vora, *Fixed points of certain symmetric product mappings of metric manifolds*, Fund. Math. 85 (1974), 19–24.

[18] —, *Fixed points of certain symmetric product mappings of a compact A*-ANR, Math. Student 42 (1974), 379–396.

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