THEOREM. *If* $\lambda^{<\varkappa} = \lambda$ *and* $\varkappa$ *has the* λ*-Shelah property, then for any stationary subset* S *of* $P_\varkappa\lambda$, $\{x \in P_\varkappa\lambda\colon S \cap P_{\varkappa_x} x \in NS^+_{\varkappa_x x}\} \in NSh^*_{\varkappa\lambda}$.

Proof. Suppose not; let $S \in NS^+_{\varkappa\lambda}$ be such that

$$X = \{x \in P_\varkappa\lambda\colon S \cap P_{\varkappa_x} x \in NS_{\varkappa_x x}\} \in NSh^+_{\varkappa\lambda}.$$

In view of 3.4 (2) in [2] we may assume w.l.o.g. that $(\forall x \in X)(|[x]^{<\varkappa_x}| = |x|)$.

For each $x \in X$, let $c_x\colon x^2 \to P_{\varkappa_x} x$ be such that

$$C_x = \{z \in P_{\varkappa_x} x\colon (\forall \alpha, \beta \in z)(c_x(\alpha, \beta) \subseteq z)\} \subseteq P_{\varkappa_x} x - S.$$

Now let $c\colon \lambda^2 \to P_\varkappa\lambda$ be such that

$$(\forall x \in P_\varkappa\lambda)(H_x = \{y \in X \cap \hat{x}\colon c_y \restriction x^2 = c \restriction x^2\} \in NS^+_{\varkappa\lambda}),$$

and set $C = \{x \in P_\varkappa\lambda\colon (\forall \alpha, \beta \in x)(c(\alpha, \beta) \subseteq x)\}$.

Pick $x \in C \cap S$ and then pick $y \in H_x$ such that $x \in P_{\varkappa_y} y$. Then

$$(\forall \alpha, \beta \in x)(c_y(\alpha, \beta) = c(\alpha, \beta) \subseteq x),$$

thus $x \in S \cap C_y$. This is the required contradiction. ■

References

[1] D. M. Carr, *$P_\varkappa\lambda$ generalizations of weak compactness*, Z. Math. Logic Grundlag. Math. 31 (1985), 393–401.

[2] — *$P_\varkappa\lambda$ partition relations*, this journal, this issue.

[3] T. K. Menas, *On strong compactness and supercompactness*, Ann. Math. Logic, 7 (1974), 327–359.

DEPARTMENT OF MATHEMATICS
YORK UNIVERSITY
North York, Ontario
M3J 1P3
Canada

Received 2 December 1985

A general theory of superinfinitesimals

by

Benjamin Benninghofen (Iowa) and **Michael M. Richter** (Kaiserlautern)

Abstract. In this paper the concept of a "superinfinitesimal number" is defined in terms of a generalized notion of monads. This allows to extend the concept to very general situations. A transfer theorem relates properties of generalized monads with those of ordinary monads. Some applications are given, mostly to the theory of monads.

Introduction. The idea of infinitesimals and monads in nonstandard analysis has been applied successfully to general topology and functional analysis (cf. [Lu] and [Str–Lu] Ch. 8–10). We extend this theory in a new way.

The main result of this article is a transfer theorem that allows us to compare "orders of infinity" by extending certain formal properties of monads indexed by standard points to "π-monads" indexed at nonstandard points. For example, L'Hospital's rule from calculus involves a limit of derivatives. If $f(x)$ tends to infinity as x tends to zero, ζ is a small positive standard number, and ξ is a positive infinitesimal, then $f(\zeta)/f(\xi)$ is infinitesimal. In the proof of L'Hospital's rule (Proposition 4.1) given below, we choose ζ infinitesimal and ξ *superinfinitesimal* so that we may transfer the statement, "$f(\zeta)/f(\xi)$ is infinitesimal" to the infinitesimal index ζ. The notion of "superinfinitesimal" is relative.

We will use the framework of Internal Set Theory (IST) (see [Ne] or [Ri]). The full strength of internal set theory (namely that it axiomatizes the whole universe of sets) is not used, however. We work with bounded formulas and these can be interpreted in a suitable universe. Refering to internal set theory means for those readers who prefer to think in terms of superstructures that all one has to know about the superstructure is that the axioms of IST are valid.

The identification of the particular class of properties subject to the transfer is the main content of the transfer theorem. This relies on a extension of Nelson's reduction algorithm applied to a class of formulas very much like these encountered in the topological languages of [Fl–Zi].

Unfortunately, the class of formulas which we can transfer is not a simple one. However, it is a useful one.

We give a number of new applications to the theory of monads in section 4. Further applications to topology are in [Ben–Ri–Str]; Benninghofen and Stroyan [Ben–Str] give applications to a kind of "bounded inductive limits" of locally convex spaces and Benninghofen [Ben 3] applies the theory to the generalized Riemann (or "Kurzweil") integral.

Acknowledgment. The authors are very grateful to Keith Stroyan for his help in preparing this manuscript. Numerous discussions with him and many suggestions he made have led to a clarification of the content as well as to an improvement in the presentation.

§ 1. Preliminaries. Let **x** be the set superstructure based on a set of individuals $\mathbf{x}_0 \supseteq R$ as in [Str–Lu, (3.2)]. With **x** we associate the first order language $L(\mathbf{x}, \in, =)$ which contains besides the basic predicates "$\in$" (for membership) and "$=$" (for equality) a constant for each individual and entity of **x**. $L(\mathbf{x}, \in, =)$ is interpreted in **x** in the usual way. The interpretation of $L(\mathbf{x}, \in, =)$ in any nonstandard superstructure model $^*\mathbf{x}$ of **x** assigns to each constant the nonstandard extension of the corresponding interpretation in **x**.

Nelson proposed Internal Set Theory (IST) as an axiomatic theory of all of set theory. This theory has an addition to the axioms of ZFC, three new axioms (I), (S) and (T) (see below). The predicate language in which the axioms are formulated contains besides "$\in$" and "$=$" a new predicate "standard (x)". If the standard predicate is interpreted in $^*\mathbf{x}$ by the domain of standard individuals and entities of $^*\mathbf{x}$ the axioms (I), (S) and (T) have a meaning in $^*\mathbf{x}$ but are not necessarily true there. It should be remarked that $^*\mathbf{x}$ is not uniquely determined by the validity of IST. Working in IST in this way simply means that we are only interested in consequences of these axioms and do not need other specific properties of the model.

Of course, not all axioms from ZFC remain true when interpreted in a superstructure. But the only property one really loses is the full strength of the axioms of replacement which we do not need in the intended applications. On the other hand, it provides no difficulty to have the constants from **x** (with their usual interpretation in $^*\mathbf{x}$) in the language.

In order to remind the reader that we are working in internal set theory we sometimes attach a "*" to classical concepts (which still have their usual definitions), e.g. $^*\boldsymbol{N}$ or $^*\boldsymbol{R}$.

In [Ne] bounded formulae play an important role. In a bounded formula the quantifiers appear only in the form

$$(\forall x)[x \in A \Rightarrow \ldots] \text{ or } (\exists y)[y \in B \wedge \ldots].$$

We sometimes abbreviate these as $(\forall x \in A)$ or $(\exists y \in B)$ in semi-formal shorthand. Here A and B are constants in the language. When we restrict ourselves to the fixed superstructure each formula is of course equivalent to a bounded formula. This means that only results in [Ne] depending on the use of bounded formulae remain valid in our context.

We will now precisely formulate those parts of Internal Set Theory on which the paper relies.

Our predicate language will be

$$L = L(\mathbf{x}, \in, =, \text{standard})$$

where standard is new unary predicate symbol.

We need some notation:

$\varphi(y_1, \ldots, y_n)$ means that all free variables of φ are among the $y_1, \ldots, y_n$.
$\forall^{st} x \varphi$ means $\forall x$ (standard $(x) \Rightarrow \varphi$).
$\exists^{st} x \varphi$ means $\exists x$ (standard $(x) \wedge \varphi$).
$\forall^{stfin} x \varphi$ means $\forall^{st} x$ (x finite $\Rightarrow \varphi$).
$\exists^{stfin} x \varphi$ means $\exists^{st} x$ (x finite $\wedge\ \varphi$).

Formulae containing the standard predicate are called *external*; internal formulae are those from $L(\mathbf{x}, \in, =)$.

The three axioms of Internal Set Theory are:

Transferaxiom (T).

$$\forall^{st} t_1, \ldots, t_n\ [\forall^{st} x \varphi \Leftrightarrow \forall x \varphi].$$

for each internal φ with the only free variables $t_1, \ldots, t_n$, x.

Axiom (I) *for the ideal point.*

$$[(\forall^{stfin} Z)\ (\exists t)\ (\forall x \in Z)\ \varphi] \Leftrightarrow [\exists t\ \forall^{st} x\ \varphi]$$

for each internal φ.

Axiom (S) *for standard set formation.*

$$\forall^{st} Z\ \exists^{st} Y\ \forall^{st} x\ (x \in Y \Leftrightarrow x \in Z \wedge \varphi).$$

for each (external or internal) φ.

In addition we assume all formulae from $L(\mathbf{x}, \in, =)$ true in **x** as axioms. This does not allow the formation of subset using external formulae. So, if A is an internal set and φ an external formula then there is in general no internal set B such that

$$a \in B \Leftrightarrow a \in A \wedge \varphi(a).$$

This set B exists of course as an external set and we denote it by

$$B = {}^{E}\{x \in A \mid \varphi\}$$

In the special case where φ is the standard predicate we also write

$$B^{st} = \{x \in B \mid \text{standard}(x)\}.$$

A kind of internal substitute for B is provided by axiom (S). The (standard) set Y in (S) is denoted by

$$Y = {}^{S}\{x \in Z \mid \varphi\}.$$

1.1. Definition. A subformula ψ of a formula φ in L occurs *positively* (resp. *negatively*) in φ if ψ is inside the scope of an even (resp. odd) number of negation symbols when "$\to$" and "$\leftrightarrow$" are replaced by equivalent statements using $\neg$, $\wedge$ and $\vee$.

1.2. Definition. A formula φ in L is in *reduced form* (RF) if it is of the form $(Q_1^{st}x_1 \in U_1) \ldots (Q_m^{st}\, x_m \in U_m)\psi$ where $Q_i \in \{\forall, \exists\}$ and ψ is internal. If $m = 1$ then such a formula is called a Q_1^{st}*-formula*.

1.3. Definition. A formula φ in L in which the standard predicate occurs at most negatively is called a *monadic formula*. If in addition, all the bounds on the external quantifiers, $U_1, \ldots, U_n$ satisfy $\mathrm{card}(U_i) \leqslant \alpha$, then we say φ is α*-monadic*.

Nelson refers to monadic formulae as "universal semi-internal formulae". In [Ne] the following theorems are proved: *Extended axiom of choice*:

(SF) If $\varphi(u, v)$ is any formula and U, V are standard:

$$[(\forall^{st}u \in U)\ (\exists^{st}v \in V)\varphi(u, v)] \Leftrightarrow [(\exists^{st}\tilde{v} \in V^U)(\forall^{st}u \in U)\varphi(u, \tilde{v}(u))].$$

Extended transfer principles:

(T$\forall$) If $\varphi = \varphi(x, t_1, \ldots, t_n)$ is an $\forall^{st}$-formula, then also (T) holds.

(T$\exists$) If $\varphi = \varphi(x, t_1, \ldots, t_n)$ is an $\exists^{st}$-formula, then

$$\forall^{st}t_1, \ldots, t_n[\exists^{st}x\varphi \Leftrightarrow \exists x\varphi].$$

The proof of the following two theorems is similar to the proof of (T$\forall$), (T$\exists$).

Extended principles of the ideal point:

(I$\forall$) If φ is an $\forall^{st}$-formula, then (I) also holds.

(I$\exists$) If φ is an $\exists^{st}$-formula, then

$$((\exists^{stfin}x')(\forall y)(\exists x \in x')\varphi) \Leftrightarrow ((\forall y)(\exists^{st}x)\varphi).$$

In [Ne] it is shown that each formula can be transformed into an equivalent formula in reduced form using (I), (SF) and some conventional rules. The transformation into a reduced form can actually be given by an algorithm which is called the *reduction algorithm* (RA for short). Monadic formulae can be transformed by the (RA) without using (SF) into the form

$$(\forall^{st}x_1 \in U_1) \ldots (\forall^{st}x_m \in U_m)\psi$$

with ψ internal. In other words, monadic formulae are equivalent to reduced formulae with only $\forall^{st}$ as external quantifier and, moreover, the reduction to this form does not need (SF). Furthermore, if the initial formula is α-monadic, then $\mathrm{card}(U_i) \in \alpha$ for $1 \leqslant i \leqslant m$.

§ 2. Monads, α-monads and monadic formula. If A is standard and $\mathscr{F}$ is a standard filter on A then the external set $M := \mu(\mathscr{F}) := {}^E\{x \in A | (\forall^{st}F \in \mathscr{F})\ (x \in F)\}$ is called a *monad* on A (namely the monad of $\mathscr{F}$).

If $\mathscr{F}$ has a basis $\mathscr{B}$ with card $\mathscr{B} \leqslant \alpha$, α a cardinal number, then M is called an α*-monad*.

For the properties of monads we refer to [Lu] or [Str–Lu] where they are discussed in detail. To some extent this paper can be regarded as an extension of Luxemburg's article.

Next we will describe the structure of those sets which are monads of some filter. The theorem is the converse of the observation that $\mu(\mathscr{F})$ is defined by a monadic formula. The proof can be found in [Ben 1] or [Ri]; for completeness we include it, using some simplifications which are due to K. Potthoff.

2.1. The Syntactical Theorem on Monads. *Let A be a standard set, and let $\varphi = \varphi(x)$ be a monadic formula with only x free. Then*:

(a) *$M := {}^E\{x \in A | \varphi(x)\}$ is a monad.*

(b) *If φ is an α-monadic formula, then M is an α-monad.*

Proof. We may assume that $\varphi(x)$ is of the form

$$(\forall^{st}\ y \in E)\ \psi(x, y)$$

where ψ is an internal formula. The obvious candidate for the filter whose monad should be M is

$$\mathscr{F} := {}^S\{F \subseteq A | M \subseteq F\}.$$

Clearly $M \subseteq \mu(\mathscr{F})$ holds. We obtain a base for $\mathscr{F}$ by taking for finite $U \subseteq E$ the sets

$$B_U := \{x \in A|\ (\forall y \in U)\ \psi(x, y)\}.$$

Now we have:

$$x \in M \Leftrightarrow (\forall^{stfin}U \subseteq E)\ (\forall y \in U)\ \psi(x, y) \Leftrightarrow (\forall^{stfin}\ U \subseteq E)\ (x \in B_U) \Leftrightarrow x \in \mu(\mathscr{F}),$$

which shows (a).

Now $\mathscr{F} = \{F \subseteq A|\ (\exists^{fin}U \subseteq E)\ (\{x \in A|\ (\forall y \in U)\ \psi(x, y)\})\}$ holds and therefore $\mu(\mathscr{F})$ is a $\mathrm{card}(E)$-monad. Then (b) follows from the fact that $\mathrm{card}(E) \leqslant \alpha$ if φ is an α-monad.

The theorem shows that filter monads correspond to monadic formulae. It is well known that monads are either standard or external (cf. [Lu]). In particular α-monads for finite α are standard and proper α-monads for infinite α are external.

§ 3. Families of monads. In applications one is often interested in infinitesimal or infinite numbers of different size. This is related to the question of what a "nonstandard monad" or a "monad of nonstandard filter" might be. One expects such nonstandard monads, e.g. when one takes the union of infinitely many monads (cf. § 4).

We will define "superinfinitesimals" in such a way that they form the "monad of some nonstandard filter". To carry out this approach systematically we need not only to generalize the concept of a monad but to do it in such a way that we obtain a transfer theorem in order to prove properties about these new monads. Transfer amounts to interchange of quantifiers. To make it possible we need to impose some syntactical restrictions in the defining formulae of the generalized monads. We start, however, with the set-theoretical definition.

Suppose $\mathscr{F} = (\mathscr{F}_i | i \in I)$ is a standard family of filters $\mathscr{F}_i$ on sets X_i. This implies that $(X_i | i \in I)$ is again a standard family and the product filter $\mathscr{F} = \prod(\mathscr{F}_i | i \in I)$ is a standard filter on the standard set $X = \prod(X_i | i \in I)$. Therefore the monad $\mu(\mathscr{F})$ on X is well defined. The central concept now is:

3.1. Definition. The π-*monads* of the family $\mathscr{F}$ are

$$^{\pi}\mu_j(\mathscr{F}) := {}^{E}\{x \in X_j | \ (\forall^{\mathrm{st}} F \in \prod(\mathscr{F}_i | i \in I))(x \in F(j))\} \text{ for } j \in I .$$

Hence the π-monads are the projections of the ordinary monad of the product filter on the coordinate sets which may be standard or nonstandard.
As expected for standard $j \in I$ we get nothing new:

$$^{\pi}\mu_j(\mathscr{F}) = \mu(\mathscr{F}_j) = \{x | \ (\forall^{\mathrm{st}} F \in \mathscr{F}) \ (x \in F(j))\} .$$

Example. Put $I = {}^*N$ and consider the constant family $(\mathscr{F}_n | n \in {}^*N)$ where each $\mathscr{F}_n = \mathscr{U}_R(0)$, the neighborhood filter of 0 in the *reals.

Taking the standard function

$$F(n) = \{x | \ |x| < 1/n\}$$

we observe that $1/m \notin {}^{\pi}\mu_m(\mathscr{F})$ and therefore ${}^{\pi}\mu_m(\mathscr{F}) \neq \mu$ for nonstandard m where μ is the usual monad of 0. More precisely, for $x \neq 0$ we have

$$x \in {}^{\pi}\mu_m(\mathscr{F}) \Leftrightarrow \left|\frac{1}{x}\right| > f(m) \text{ for all standard } f .$$

This example also shows that π-monads do not only depend on a special internal filter but on the whole family of filters.

Because we are more interested in monads rather than filters we replace a standard family $\{\mathscr{F}_i | i \in I\}$ equivalently by the external family $(M_i)_{i \in I^{\mathrm{st}}}$ of monads which determines the family of filters by axiom (S).

If no confusion arises we write

$$^{\pi}M_j \text{ for } {}^{\pi}\mu_j(\mathscr{F})$$

or ${}^{\pi}M[j]$ if $M_i = M = \mu(\mathscr{F}_i)$ for all standard $i \in I$.

For standard $j \in I$ we then have

$$^{\pi}M_j = M_j .$$

Next we extend the syntactical description of monads by formulae to the case of π-monads.

If $M_i = {}^{E}\{x \in X_i | \ \varphi_i(x)\}$ we want to construct a formula ${}^{\pi}\varphi(i, x)$, such that

$$(\forall j \in I) \ ({}^{\pi}M_j = {}^{E}\{x \in X_j | \ {}^{\pi}\varphi(j, x)\}) .$$

The construction of ${}^{\pi}\varphi(i, x)$ from the $\varphi_i(x)$ is quite natural: $i \in I$ is (with some provisions) considered as an argument and then allowed to be nonstandard. The next definition will make this precise.

3.2. Definition. (a) A formula $\varphi(x_0, \dots, x_{l-1}, V_0, \dots, V_{n-1}, X_0, \dots, X_{m-1})$ with free variables $x_0, \dots, x_{l-1}$ is called *admissible* if

(i) the V_k occur only as $((\forall^{\mathrm{st}} u \in V_k) \ \psi)$ or $((\exists^{\mathrm{st}} u \in V_k) \ \psi)$ and all external quantifications are of this form;

(ii) bounded variables are different from free variables and different quantifiers bind different variables.

(b) If φ is admissible, I is standard, U_k and A_j are constants denoting standard functions with domain I, then the family

$$(\varphi_i(x_0, \dots, x_{l-1}) | \ i \in I) ,$$

$$\varphi_i(x_0, \dots, x_{l-1}) = \varphi(x_0, \dots, x_{l-1}, U_0(i), \dots, U_{n-1}(i), A_0(i), \dots, A_{m-1}(i)) ,$$

is called an *admissible family* with *base formula* φ.

(c) If $(\varphi_i(x_0, \dots, x_{l-1}) | \ i \in I)$ is an admissible family with base formula φ then we perform the following syntactic transformations:

$$\psi_1(x_0, \dots, x_{l-1}) = \varphi(x_0, \dots, x_{l-1}, \prod_{i \in I} U_0(i), \dots, \prod_{i \in I} U_{n-1}(i), A_0, \dots, A_{m-1});$$

ψ_1 in general will not be a very meaningful formula.

$$\psi_2(x, x_0, \dots, x_{l-1}) = \varphi(x_0, \dots, x_{l-1}, \prod_{i \in I} U_0(i), \dots, \prod_{i \in I} U_{n-1}(i), A_0 x, \dots, A_{m-1}(x)) .$$

Note that the A_j are constants denoting functions. $\psi_3(x, x_0, \dots, x_{l-1})$ is obtained from ψ_2 by replacing each subformula of the form

$$(Q^{\mathrm{st}} u_k \in \prod_{i \in I} U_k(i))\chi(\dots, u_k, \dots)$$

by

$$(Q^{\mathrm{st}} u_k \in \prod_{i \in I} U_k(i))\chi(\dots, u_k(x), \dots) .$$

If the family $(\varphi_i | i \in I)$ has contained meaningful formulae then ψ_3 also will make sense.

${}^{\pi}\varphi(x, x_0, \dots, x_{l-1}) = \psi_3(x, x_0, \dots, x_{l-1})$ is called *product form* of the admissible family $(\varphi_i | i \in I)$ or the π-*transform* of φ with respect to

$$(U_k(i), A_l(i), i \in I, 0 \leqslant k < n, 0 \leqslant l < m) .$$

It is clear that ${}^{\pi}\varphi$ is of the form

$$\psi(x, x_0, \dots, x_{l-1}, \prod_{i \in I} U_0(i), \dots, \prod_{i \in I} U_{n-1}(i), A_0, \dots, A_{m-1})$$

for some admissible

$$\psi(x, x_0, \dots, x_{l-1}, W_0, \dots, W_{n-1}, Y_0, \dots, Y_{m-1}) .$$

In the intended applications the variable x will range over I. Therefore we will in general use the variable i instead of x, i.e., our π-transforms are of the form ${}^{\pi}\varphi(i, x_0, \ldots, x_{l-1})$.

As a first example we consider a family of monads $(M_i)_{i \in I^{st}}$ with the corresponding standard family of filters $\{\mathcal{F}_i | \in I\}_i$ for standard $i \in I$ we have

$$x \in M_i \Leftrightarrow (\forall^{st} u \in \mathcal{F}_i)(x \in u).$$

The formulae on the right hand side form an admissible family with $\varphi(x, V) = (\forall^{st} u \in V)\ (x \in u)$ as a base formula. We see that the π-transform ${}^{\pi}\varphi(i, x)$ of φ with respect to this family of filters defines the π-monads ${}^{\pi}M_i$ for all $i \in I$.

The idea now is that the formula ${}^{\pi}\varphi(i, x_0, \ldots, x_{l-1})$ might allow metatheorems which are not available for the corresponding filters (or galaxies, skies, etc.). The formula ${}^{\pi}\varphi$ extends the logical description of the usual monads to the π-monads and can be regarded as a transfer. Because of the additional parameter i the situation is more involved than in the theorem of monads. There the reduction algorithm RA was used; here we have to define a more refined reduction procedure.

3.3. Definition. Let $Q \in \{\forall, \exists\}$, $\bar{\forall} = \exists$, $\bar{\exists} = \forall$; $\underline{\mathrm{fr}}(\varphi)$ denotes the set of free variables of φ.

The rules of the (nondeterministic) topological reduction algorithm (TRA) are:

($\neg$) $\neg(Q^{st} u \in U)\varphi \mid\!\!\rightsquigarrow (\bar{Q}^{st} u \in U)\neg\varphi$;

($\vee$) if $u \notin \underline{\mathrm{fr}}(\psi)$, $v \notin \underline{\mathrm{fr}}(\varphi)$:

$$((Q^{st} u \in U)\varphi) \vee \psi \mid\!\!\rightsquigarrow (Q^{st} u \in U)(\varphi \vee \psi);$$

$$\varphi \vee ((Q^{st} v \in V)\psi) \mid\!\!\rightsquigarrow (Q^{st} v \in V)(\varphi \vee \psi).$$

($\wedge$) analogous to ($\vee$)

($\rightarrow$) if $u \notin \underline{\mathrm{fr}}(\psi)$, $v \notin \underline{\mathrm{fr}}(\varphi)$:

$$((Q^{st} u \in U)\varphi) \rightarrow \psi \mid\!\!\rightsquigarrow (\bar{Q}^{st} u \in U)(\varphi \rightarrow \psi);$$

$$\varphi \rightarrow ((Q^{st} v \in V)\psi) \mid\!\!\rightsquigarrow (Q^{st} v \in V)(\varphi \rightarrow \psi).$$

($\forall$) $(\forall x)(\forall^{st} u \in U)\varphi \mid\!\!\rightsquigarrow (\forall^{st} u \in U)(\forall x\varphi)$.

($\exists$) $(\exists x)(\exists^{st} u \in U)\varphi \mid\!\!\rightsquigarrow (\exists^{st} u \in U)(\exists x\varphi)$.

($\forall\wedge$) if $\varphi \wedge \psi$ is *not* internal:

$$(\forall x)(\varphi \wedge \psi) \mid\!\!\rightsquigarrow (\forall x\psi) \wedge (\forall x\psi).$$

($\exists\vee$) if $\varphi \vee \psi$ is not internal:

$$(\exists x)(\varphi \vee \psi) \mid\!\!\rightsquigarrow (\exists x\varphi) \vee (\exists x\psi).$$

(TI$\forall$) $(\exists x)(\forall^{st} u \in U)\varphi \mid\!\!\rightsquigarrow (\forall^{st} u \in U)(\exists x\varphi)$

if φ is an $\forall^{st}$-formula.

(TI$\exists$) $(\forall x)(\exists^{st} u \in U)\varphi \mid\!\!\rightsquigarrow (\exists^{st} u \in U)\ (\forall x\varphi)$

if φ is an $\exists^{st}$-formula.

The (TRA) does not reduce all formulae to a reduced form. The rules (TI$\forall$), (TI$\exists$) do not lead in general to equivalent formulae; TI$\exists$ is not even correct in the sense that it always leads from valid formulae to valid formulae.

We will therefore define a special class of formulae for which the (TRA) is correct. This will also determine the class of properties for which we can show a transfer theorem. To make it as big as possible we must concede a somewhat lengthy definition.

First we have to single out a class of formulae which behave nicely with respect to positive and negative occurrences of certain subformulae.

Let $\varphi = \varphi(x_0, \ldots, x_{l-1}; U_0, \ldots, U_{n-1}; A_0, \ldots, A_{m-1})$. If Φ is a subformula of φ we write $\Phi = \Phi(\bar{v}, \bar{y})$, if $\bar{v} = (v_0, \ldots, v_{r-1})$ contains the free variables of Φ which are externally quantified in φ and $\bar{y} = (y_0, \ldots, y_{p-1})$ contains the remaining free variables of Φ.

From now on we assume that each U_k is nonempty and equipped with a partial order $\underline{[}_k$ such that $(U_k, \underline{[}_k)$ and $(U_k, \underline{]}_k)$ are directed. In our examples it is either an obvious existing natural order or set inclusion.

3.4. Definition. (1) An internal subformula $\Phi = \Phi(\bar{v}, \bar{y})$ of φ *occurs monotonically in* φ (relative to "$\underline{[}$") if we have

$$(\forall y_0, \ldots, y_{p-1})(\forall v_0, \ldots, v_{r-1})(\forall v'_0, \ldots, v'_{r-1})$$
$$[(\Phi(\bar{v}, \bar{y}) \wedge \bar{v} \leqslant \bar{v}') \Rightarrow (\Phi(\bar{v}', \bar{y}))]$$

where $\leqslant$ is defined as follows:

$$\bar{v} \leqslant \bar{v}': \begin{cases} v_e \,\underline{[}\, v'_e \text{ for all } e \in r, \text{ if } \Phi \text{ occurs positively in } \varphi; \\ v_e \,\underline{]}\, v'_e \text{ for all } e \in r, \text{ if } \Phi \text{ occurs negatively in } \varphi. \end{cases}$$

(2) φ is called *completely monotonic* if it is admissible and if every maximal internal formula Φ of φ occurs monotonically (relative to "$\underline{[}$").

(This is the case if the maximal internal subformulae Φ are "built up by monotonic subformulae" of φ).

For the notion of "monotonic" one should compare the language L_t to topological model theory [Fl–Zi].

3.5. Definition. φ is *simply reducible* (s. r.) if φ is completely monotonic and if it can be transformed by the rules of the (TRA) into a formula in RF.

We remark that any admissible φ can be turned into an equivalent completely monotonic formula. After such a transformation the TRA might no longer be applicable even if it was in the first place. Later on an example shed more light on these conditions. It should also be emphasized that the concept of a simply reducible formula is — despite its very technical character — easily handled in application (see § 4).

3.6. Lemma. *If φ is completely monotonic and is transformed into ψ by the* (TRA) *then ψ is also completely monotonic and $\varphi \Leftrightarrow \psi$ holds.*

Proof. We have to show that the application of each TRA rule to a completely monotonic formula φ produces an equivalent completely monotonic ψ. This will be carried out only for the rule (TI∃) and the case where φ is of the form $(\forall x)(\exists^{st}u \in U)\Phi(x, U)$, Φ being the maximal internal subformula of φ. We need to prove

$$(\forall x)(\exists^{st}u \in U)\ \Phi(x, u) \Leftrightarrow (\exists^{st}u \in U)(\forall x)\ \Phi(x, u).$$

"$\Leftarrow$" is trivial.

"$\Rightarrow$: By (I∃) we have

$$(\exists^{st\,fin}U' \subseteq U)(\forall x)(\exists u \in U')(\Phi(x, u)).$$

Now choose a $u_0 \in U$ such that $u \sqsubseteq u_0$ holds for all $u \in U'$; then $\Phi(x, u_0)$ follows for all x because Φ occurs monotonically. In the next lemma φ is the product form of some set of admissible φ_i's but not necessarily monotonic.

3.7. LEMMA. *If* $^{\pi}\varphi(i) = {}^{\pi}\varphi(i, U_0, \ldots, U_{n-1}, A_0(i), \ldots, A_{m-1}(i))$ *has no free variables and is in* RF, *then*

$$(\forall^{st}i \in I)\ {}^{\pi}\varphi(i) \Leftrightarrow ((\forall i \in I)\ {}^{\pi}\varphi(i)).$$

Proof. Using (SF) from (RA) we can transform the formula $(\forall^{st}i \in I)\ {}^{\pi}\varphi(i)$ into the form

$$(Q_0^{st}\tilde{u}_0 \in U_0) \ldots (Q_{n-1}^{st}\tilde{u}_{n-1} \in U_{n-1})(\forall^{st}i \in I)$$
$$\Phi(\tilde{u}_0(i), \ldots, \tilde{u}_{n-1}(i), A_0(i), \ldots, A_{m-1}(i)).$$

Then we apply (T) and bring the quantifier $\forall i \in I$ to the left. This proves "$\Rightarrow$"; the other direction is trivial. Next we come to the main result of this section.

3.8. TRANSFER THEOREM. *Suppose* $\varphi_i = \varphi(U_0(i), \ldots, U_{m-1}(i); A_0(i), \ldots, A_k(i))$ *is s.r. and has no free variables for standard* $i \in I$; *assume furthermore that* $^{\pi}\varphi(i)$ *is the product form of the* φ_i's.

Then we have

$$(\forall^{st}i \in I)\varphi_i \Leftrightarrow (\forall^{st}i \in I)\ {}^{\pi}\varphi(i) \Leftrightarrow (\forall i \in I)^{\pi}\ \varphi(i).$$

Proof. We have:

(1) $(\forall^{st}i \in I)(\varphi_i \Leftrightarrow {}^{\pi}\varphi(i))$, which gives the first equivalence. By assumption (TRA) transforms each φ_i into some φ_i^0 in reduced form; the same rules reduce $^{\pi}\varphi(i)$ to some $({}^{\pi}\varphi)^0(i)$ in RF and we get

(2) ${}^{\pi}(\varphi^0)(i) = ({}^{\pi}\varphi)^0(i)$ for all $i \in I$.

Furthermore we have by the correctness of the (TRA) for s.r. formulae

(3) $(\forall^{st}i \in I)(\varphi_i \Leftrightarrow \varphi_i^0)$ and

(4) $(\forall i \in I)({}^{\pi}\varphi(i) \Leftrightarrow ({}^{\pi}\varphi)^0(i))$.

The previous lemma yields

(5) $(\forall^{st}i \in I)({}^{\pi}(\varphi)(i)) \Leftrightarrow (\forall i \in I)({}^{\pi}(\varphi)(i))$.

The rest follows from applying (1), (4), (2), (5), (2) and (4) in this order to $(\forall^{st}i \in I)\varphi_i$.

If $(\forall^{st}F \in \mathscr{F}_i)((t_1, \ldots, t_n) \in F)$ is a subformula of a completely monotonic Formula φ_i we use for this subformula the abbreviation $(t_1, \ldots, t_n) \in M_i$, where $M_i = \mu(\mathscr{F}_i)$. The corresponding subformula of $^{\pi}\varphi$ is denoted by $(t_1, \ldots, t_n) \in {}^{\pi}M_i$.

3.9. COROLLARY. *If the monad family* $(M_i)_{i\in I^{st}}$ *is defined by* $M_i = {}^{E}\{x \in X_i | \varphi_i(x)\}$, *where* φ_i *is a completely monotonic monadic formula for standard* $i \in I$, *then*:

$$^{\pi}M_j = {}^{E}\{x \in X_j | {}^{\pi}\varphi(j, x)\}\ \textit{for all}\ j \in I.$$

Proof. We first write $\varphi_i(x)$ as $^{\pi}\varphi(i, x)$ for standard $i \in I$. We have:

$$(\forall^{st}i \in I)(\forall x \in X_i)[x \in M_i \Leftrightarrow {}^{\pi}\varphi(i, x)].$$

Obviously a completely monotonic monadic formula is s.r. which implies that: "$(\forall^{st}i \in I)(\forall x \in X_i)(x \in M_i \Leftrightarrow {}^{\pi}\varphi(i, x))$" is s.r. too. Now the transfer theorem implies the desired result:

$$(\forall i \in I)(\forall x \in X_i)[x \in {}^{\pi}M_i \Leftrightarrow {}^{\pi}\varphi(i, x)].$$

In the next example we discuss the necessity of the assumptions in the transfer theorem; we see in particular that "s.r." cannot be replaced by "completely monotonic".

EXAMPLE. If (X, d) is a metric space, we define for $A \subseteq X$, $a \in X$, $r \in [0, \infty[$

$$d(a, A) := \inf_{x \in A} d(a, x);$$

$$K(a, r) := \{x \in X |\ d(x, a) < r\};$$

$$I := {}^*N;$$

$$\psi_i := (\forall x \in X)(\exists^{st}a \in X)(\forall^{st}\varepsilon \in]0, \infty[)(d(x, a) < \varepsilon).$$

This formula which says that "X is compact" is not completely monotonic. If we convert it to a completely monotonic one we obtain another description of compactness:

$$\varphi_i := (\forall x \in X)(\exists^{st}u \in \mathscr{P}_\omega(X)(\forall^{st}\varepsilon \in]0, \infty[)(d(x, u) < \varepsilon).$$

Now we specialize X to be the compact unit interval $[0, 1] \subseteq {}^*\boldsymbol{R}$ and we get $(\forall^{st}i \in {}^*N)\varphi_i$.

Now assume $(\forall i \in {}^*N)^{\pi}\varphi(i)$. This is equivalent to

$$(\forall i \in {}^*N)(\forall x \in [0, 1])(\exists^{st}\tilde{u} \in \mathscr{P}_\omega[0, 1])^{*N})(\forall^{st}\tilde{u} \in]0, \infty[^{*N})(d(x, \tilde{u}(i)) < \tilde{\varepsilon}(i)).$$

Using (SF), (I) and (T) this is transformed further into

$$(\forall^{st}\tilde{\tilde{\varepsilon}}: \mathscr{P}_\omega([0, 1])^{*N} \to]0, \infty[^{*N})(\exists^{fin}\tilde{u}' \subseteq \mathscr{P}_\omega([0, 1])^{*N})$$
$$(\forall i \in {}^*N)(\forall x \in [0, 1])(\exists \tilde{u} \in \tilde{u}')(d(x, \tilde{u}(i)) < \tilde{\tilde{\varepsilon}}(\tilde{u})(i))$$

and finally into

$$(\forall \tilde{\tilde{\varepsilon}}\colon \mathscr{P}_\omega([0,1])^{*N} \to (]0,\infty[)^{*N})(\forall^{\text{fin}}\tilde{u}' \subseteq \mathscr{P}_\omega([0,1]^{*N}))$$
$$(\forall i \in {}^*N)\Big([0,1] \subseteq \bigcup_{\tilde{u}\in\tilde{u}'} \bigcup_{y\in\tilde{u}(i)} K(y; \tilde{\tilde{\varepsilon}}(\tilde{u})(i))\Big).$$

We will show that this last formula is false.

We put $\tilde{\tilde{\varepsilon}}(\tilde{u})(n) := \dfrac{1}{2nd\cdot \underline{\text{card}}(\tilde{u}(n))}$.

Let Λ be the *Lebesgue-measure on ${}^*\boldsymbol{R}$.
We take a *finite $\tilde{u}' \subseteq (\mathscr{P}_\omega[0,1])^{*N}$ such that

$$[0,1] \subseteq \bigcup_{\tilde{u}\in\tilde{u}'} \bigcup_{y\in\tilde{u}(i)} K(y; \tilde{\tilde{\varepsilon}}(\tilde{u})(i)) \text{ for all } i \in {}^*N.$$

Then:

$$1 = \Lambda([0,1]) \leqslant \sum_{\tilde{u}\in\tilde{u}'} \sum_{y\in\tilde{u}(i)} \big(K(y; \tilde{\tilde{\varepsilon}}(\tilde{u})(i))\big)$$
$$= \sum_{\tilde{u}\in\tilde{u}'} \sum_{y\in\tilde{u}(i)} 2\cdot\tilde{\tilde{\varepsilon}}(\tilde{u})(i) = \sum_{\tilde{u}\in\tilde{u}'} \sum_{y\in\tilde{u}(i)} \frac{1}{i\,\underline{\text{card}}\,\tilde{u}(i)} = \sum_{\tilde{u}\in\tilde{u}'} \frac{1}{i} = \frac{\text{card}\,\tilde{u}'}{i}.$$

Hence $\underline{\text{card}}\,\tilde{u}' \geqslant i$ for all $i \in {}^*N$, a contradiction to the *finiteness of $\tilde{u}'$; therefore $\neg^*\varphi(i)$ holds for some i.

In the final section we will give some applications of the theory obtained so far.

§ 4. Some applications. First we consider the nontrivial part of de l'Hospital's rule mentioned in the introduction. For standard $a \in {}^*\boldsymbol{R}$ we have the monads

$$\mu_+(a) = {}^E\{x \in {}^*\boldsymbol{R} \mid x \geqslant a, x \approx a\},\ \mu_-(a) = {}^E\{x \in {}^*\boldsymbol{R} \mid x \leqslant a, x \approx a\};$$

furthermore let $\mu_{\boldsymbol{R}}(\infty)$ resp. $\mu_{\boldsymbol{R}}(-\infty)$ denote the monads of the positive resp. negative infinite real numbers.

4.1. PROPOSITION (de l'Hospital). *Suppose M is one of the monads $\mu_-(a)$, $\mu_+(a)$, $\mu_{\boldsymbol{R}}(\infty)$, $\mu_{\boldsymbol{R}}(-\infty)$, g and f are standard real-valued differentiable functions with domain D, $M \subseteq D \subseteq {}^*\boldsymbol{R}$.*

Assume for all $x \in M$: $f(x), g(x) \in \mu_{\boldsymbol{R}}(+\infty)$, $g'(x) \neq 0$ and $\dfrac{f'(x)}{g'(x)} \approx d$, d standard.

Then $(\forall x \in M)\left(\dfrac{f(x)}{g(x)} \approx d\right)$ holds.

Proof. We fix some arbitrary $z \in M$. If $x \in {}^\pi M[z]$ (say $x \leqslant z$), then by the mean value theorem for some η, $x \leqslant \eta \leqslant z$,

$$d \approx \frac{f'(\eta)}{g'(\eta)} = \frac{f(x)-f(z)}{g(x)-g(z)} = \frac{f(x)}{g(x)}\cdot\frac{1-\dfrac{f(z)}{f(x)}}{1-\dfrac{g(z)}{g(x)}}$$

is true.

The relation $x \in {}^\pi M[z]$ implies furthermore (by 3.8)

$$\frac{f(z)}{f(x)} \approx 0 \approx \frac{g(z)}{g(x)}.$$

Hence we get $d \approx \dfrac{f(x)}{g(x)}$ for all $x \in {}^\pi M[z]$. Using Proposition 4.4 (which we will prove below) at this place we obtain our conclusion

$$(\forall x \in M)\left(\frac{f(x)}{g(x)} \approx d\right).$$

Now we turn to applications in the general theory of monads.
For a standard set X the set

$$\boldsymbol{F} := \boldsymbol{F}(X)$$

of all filters on X (this time including the improper filter of all subsets of X): the ordering

$$F \subseteq G \text{ iff } \mu(F) \subseteq \mu(G),\ F, G \text{ standard}$$

makes $\boldsymbol{F}$ a complete standard lattice. This lattice is studied in [Lu], II, 3. It has simple arbitrary infima but the infinite suprema provide some difficulties for which the families of monads are useful.

First we observe that for a family of monads on X

$$\prod (M_i \mid i \in I^{\text{st}}) := \bigcap (M_i \mid i \in I^{\text{st}}) = {}^E\{x \in X \mid (\forall^{\text{st}} i \in I)(\forall x \in M_i)\}$$
$$(\text{in short: } \bigcap_{i\in I} M_i)$$

is again a monad by the theorem of monads. In fact, it is the greatest monad N s.t.

$$(\forall^{\text{st}} i \in I)(N \subseteq M_i).$$

From this follows that

$$\mu(\prod_{i\in I} F_i) = \bigcap_{i\in I} \mu(F_i),$$

where Π is the lattice operation.

The corresponding method to describe the lattice supremum $\amalg$ fails, however, because

$${}^E\{x \in X \mid (\exists^{\text{st}} i \in I)(x \in M_i)\}$$

needs not be a monad if I is infinite. Therefore we define the sup as:

$$\coprod (M_i \mid i \in I^{\text{st}}) := \bigcup ({}^\pi M_i \mid i \in I) = {}^E\{x \in X \mid (\exists i \in I)(x \in {}^\pi M_i)\},$$

which in short will be denoted by $\bigcup_{i\in I} M_i$.

This is, again by the theorem on monads, a monad too. Of course, for each standard $i \in I$ we have

$$M_i \subseteq \bigcup_{i \in I} M_i .$$

Now suppose that for some monad N

$$(\forall^{st} i \in I)(M_i \subseteq N)$$

is true. By the transfer Theorem 3.8 of § 3 we get

$$(\forall i \in I)({}^{\pi}M_i \subseteq {}^{\pi}N[i] \subseteq N)$$

and

$$\bigcup_{i \in I} M_i \subseteq N .$$

This means that $\bigcup_{i \in I} M_i$ is the smallest monad containing all M_i, $i \in I$ standard and shows

$$\mu(\coprod_{i \in I} F_i) = \bigcup_{i \in I} \mu(F_i)$$

is in fact true. Therefore we now have a description of the complete lattice $(F, [)$ in terms of monads. This allows us also to describe the S-topology (cf., e.g., [Ro]).

For an external set $A \subseteq X$ the hull in the S-topology is the filter monad

$$\bar{A} = \mu({}^{S}\{Y \subseteq X | A \subseteq Y\}) .$$

Observe that the S-topology is an external topology on external sets.

From above, we now obtain:

4.2. Proposition. *Suppose that $\varphi_i(x)$ are completely monotonic monadic formulae for standard $i \in I$ and*

$$A = {}^{E}\{x \in X| \ (\exists^{st} i \in I)\varphi_i(x)\} .$$

Then

$$\bar{A} = {}^{E}\{x \in X| \ (\exists i \in I)^{\pi}\varphi(i, x)\} ,$$

where ${}^{\pi}\varphi$ is the product form of the φ_i's.

The next lemma says that the elements in the monads of an ultrafilter are in some sense indistinguishable.

4.3. Lemma. *Let N be an ultramonad (i.e., the monad of an ultrafilter) and $\varphi(x)$ an external formula with no other free variables besides x.*

Then (i) *and* (ii) *are equivalent*:

(i) $(\exists x \in N)\varphi(x)$

(ii) $(\forall x \in N)\varphi(x)$

Proof. We may assume that $\varphi(x)$ is of the form

$$(\exists^{st} w \in W)(\forall^{st} v \in V)\Phi(x, w, v) \text{ where } \Phi(x, w, v) \text{ is internal} .$$

The implication (ii) ⇒ (i) is trivial.

For (i) ⇒ (ii): First choose some $a \in N$; then we can find a standard $w \in W$ s.t. $(\forall^{st} v \in V)\Phi(a, w, v)$ holds.

We put

$$M := \{x \in N | \forall^{st} \ v \in V \Phi(x, w, v)\} ;$$

by the theorem on monads M is a monad; we also have $a \in M \subseteq N$. Thus $N = M$ because N is a ultramonad;

$$\Rightarrow (\forall x \in N)(\forall^{st} v \in V)\Phi(x, w, v)$$
$$\Rightarrow (\forall x \in N)(\exists^{st} w \in W)(\forall^{st} v \in V)\Phi(x, w, v)$$
$$\Rightarrow (\forall x \in N)\varphi(x) .$$

Next we will see how far a π-monad determines the monad itself.

4.4. Proposition. *Let $M \subseteq X$ be a monad of some filter $\mathscr{F}$ and $\varphi(x)$ be an external formula with no other free variables besides x. Let I be standard and $i \in I$. Then*

$$[(\forall x \in {}^{\pi}M[i])\varphi(x)] \Rightarrow [(\forall x \in M)\varphi(x)]$$

holds.

Proof. $N := \mu({}^{S}\{Y \subseteq I| \ i \in Y\})$ is an ultramonad s.t. $i \in N$ holds; this N is called the ultramonad of I.

We may assume that $\varphi(x)$ is completely monotonic and of the form $(\forall^{st} u \in U)(\exists^{st} v \in V)\Phi(x, u, v)$ where $\Phi(x, u, v)$ is internal. Then we have by the previous lemma and the axioms (I) and (T) the implications:

$$(\forall i \in N)(\forall x \in {}^{\pi}M[i])(\forall^{st} u \in U)(\exists^{st} v \in V)\Phi(x, u, v)$$
$$\Rightarrow (\forall^{st} u \in U)(\forall i \in N)(\forall x \in {}^{\pi}M[i])(\exists^{st} v \in V)\Phi(x, u, v)$$
$$\Rightarrow (\forall^{st} u \in U)(\exists i \in I)(\forall x \in X)[(\forall^{st} \tilde{F} \in \mathscr{F}^I)(x \in F(i)) \Rightarrow (\exists^{st} v \in V\Phi(x, u, v))]$$
$$\Rightarrow (\forall^{st} u \in U)(\exists v \in V)(\exists^{st} \tilde{F} \in \mathscr{F}^I)(\exists^{st} i \in I)(\forall x \in \tilde{F}(i)\Phi(x, u, v))$$
$$\Rightarrow (\forall^{st} u \in U)[(\forall x \in X)(\forall^{st} i \in I)(\forall^{st} \tilde{F} \in \mathscr{F}^I)(x \in \tilde{F}(i)) \Rightarrow ((\exists^{st} v \in V)\Phi(x, u, v))]$$
$$\Rightarrow (\forall^{st} u \in U)(\forall x \in M)(\exists^{st} v \in V)\Phi(x, u, v)$$
$$\Rightarrow (\forall x \in M)(\forall^{st} u \in U)(\exists^{st} v \in V)\Phi(x, u, v)$$
$$\Rightarrow (\forall x \in M)\varphi(x) .$$

Returning to the S-topology we observe that standard functions are continuous in the S-topology.

4.5. Proposition. *Let $f\colon A \to B$ be a standard function. Then for any external $E \subseteq A$ which is defined by an external formula*

$$f(\bar{E}) = \overline{f(E)} .$$

Proof. We may assume $E = \{x \in A| \ \varphi(x)\}$, $\varphi(x) = (\exists^{st} i \in I)\psi(i, x)$ where $\psi(i, x)$ is a completely monotonic monadic formula.

Then we obtain

$$\bar{E} = {}^{E}\{x \in A \mid (\exists i \in I)\psi(i, x)\};$$

$$f(\bar{E}) = {}^{E}\{y \in B \mid (\exists x \in A)[(f(x) = y) \wedge ((\exists i \in I)\psi(i, x))]\};$$

$$f(E) = {}^{E}\{y \in B \mid (\exists x \in A)[(f(x) = y) \wedge (\exists^{st} i \in I\psi(i, x))]\}$$

$$= {}^{E}\{y \in B \mid (\exists^{st} i \in I)(\exists x \in A)[f(x) = y \wedge \psi(i, x)]\};$$

$$\overline{f(E)} = {}^{E}\{y \in B \mid (\exists i \in I)(\exists x \in A)[f(x) = y \wedge \psi(i, x)]\}$$

$$= {}^{E}\{y \in B \mid (\exists x \in A)[f(x) = y \wedge \exists i \in I\psi(i, x)]\}.$$

This shows that $f(\bar{E}) = \overline{f(E)}$.

Robinson's lemma for real sequences says that a sequence (a_n) which is infinitesimal for all standard n is infinitesimal up to some nonstandard $\omega \in {}^*N$.

If we express this in terms of monads we are led to the following slightly more general observation.

4.6. Proposition. *Suppose I is standard and $M_i = M$ is a monad for all standard $i \in I$. If $N \subseteq I$ is also a monad then for all $x \in M$ there is some $i \in N$ s.t. $x \in {}^{\pi}M[i]$.*

Proof. Assume $M = \mu(\mathscr{F})$, $N = \mu(\mathscr{G})$. We have to show:

$$(\forall x \in M)(\exists i \in N)(x \in {}^{\pi}M[i])$$

By RA we see that this is equivalent to

$$(\forall G \in \mathscr{G})(\forall \tilde{F} \in \mathscr{F}^I)(\exists F \in \mathscr{F})(\forall x \in F)(\exists i \in G)(x \in \tilde{F}(i))$$

which follows from

$$(\forall G \in \mathscr{G})(\exists i \in G)(\forall \tilde{F} \in \mathscr{F}^I)(\exists F \in \mathscr{F})(F \subseteq F(i)).$$

To show the last statement we take for $G \in \mathscr{G}$ any $i \in G$ and put $F := \tilde{F}(i)$.

Finally we obtain a characterization of uncountable sets in terms of π-monads.

For a standard set X the external set $X \setminus X^{st}$ is a monad; the corresponding filter contains the cofinite subsets of X. We define a family of monads indexed by ${}^*N$:

$$M = \neg st(X) = X \setminus X^{st},\ M_n = M \text{ for } n \in {}^*N^{st};$$

finally we put

$$W := {}^{E}\{x \in X \mid (\forall n \in {}^*N)(x \in M[n])\}.$$

4.7. Proposition. $\text{card}(X) > \aleph_0$ *iff* $W \neq \emptyset$.

Proof. We have

$${}^{\pi}M[n] = {}^{E}\{x \in X \mid (\forall^{st} f\colon {}^*N \to \mathscr{P}_{\omega}(X))(x \notin f(n))\}$$

and

$$W = \{x \in X \mid (\forall^{st} f\colon {}^*N \to \mathscr{P}_{\omega}(X))(x \notin \text{im}(f))\}.$$

Therefore $W = \emptyset$ for countable X. If X is uncountable then we have some $x \notin \text{im}(f)$ for finitely many standard f and use the axiom (I) of the ideal point.

W is again a monad and it is easy to see that ${}^{\pi}W[n] = W$ holds for all $n \in {}^*N$ from which follows that the filter belonging to W is δ-complete.

References

[Ben1] B. Benninghofen, *Über Nichtstandard — Methoden und Monaden*, Dipl. Arbeit Aachen, 1982.

[Ben2] — *Infinitesimalien und Superinfinitesimalien*, Dissertation Aachen, 1982.

[Ben3] — *A nonstandard approach to the generalized Riemann integral.*

[Ben–Ri–Str] B. Benninghofen, M. M. Richter and K. D. Stroyan, *Superinfinitesimals in Topology*, preprint.

[Ben–Str] B. Benninghofen and K. D. Stroyan, *Bounded Weak — Star Continuity*, preprint submitted for publication.

[Fl–Zi] J. Flum and M. Ziegler, *Topological model Theory*, SLN in Mathematics 769 (1980).

[Lu] W. A. J. Luxemburg, *A General Theory of Monads*. In: *Applications of Model Theory of Algebra, Analysis and Probability* (ed. W. A. J. Luxemburg), 18–86.

[Ne] E. Nelson. *Internal Set Theory*, Bull. Amer. Math. Soc. 83 (1977), 1165—1198.

[Ne1] — *The Syntex of Nonstandard Analysis.*

[Pu] Ch. Puritz, *Skies, constellations and monads*, in *Contributions to Nonstandard Analysis* (eds. W. A. J. Luxemburg, A. Robinson), 1972.

[Ri] M. M. Richter, *Ideale Punkte, Monaden und Nichtstandard-Methoden*, Wiesbaden 1982.

[Ro] A. Robinson, *Non-standard Analysis*, Amsterdam 1966.

[Str–Lu] K. Stroyan and W. A. J. Luxemburg, *Introduction to the Theory of Infinitesimals*, New York 1976.

Received 16 December 1985;
in revised form 17 December 1986