# A note on the $\lambda$-Shelah property

by

**Donna M. Carr** (East Lansing, Mich.)

In this brief note, we use some results from section 3 of [2] to prove that *if* $\lambda^{<\varkappa} = \lambda$ *and* $\varkappa$ *has the* $\lambda$*-Shelah property, then for any stationary subset* $S$ *of* $P_\varkappa\lambda$, $\{x \in P_\varkappa\lambda\colon S \cap P_{\varkappa_x}x \in NS^+_{\varkappa_x x}\} \in NSh^*_{\varkappa\lambda}$. Notice that this is a natural $P_\varkappa\lambda$ analogue of a well known property of weakly compact cardinals.

Our notation here is the same as in [1], [2].

The $\Pi^1_1$-indescribability characterization of the $\lambda$-Shelah property given in [1] can also be used to prove the above result. This will be presented elsewhere.

Our proof here requires the following proposition which is interesting in its own right.

PROPOSITION. *If* $\lambda^{<\varkappa} = \lambda$ *and* $\varkappa$ *has the* $\lambda$*-Shelah property, then for any* $(c_x\colon x \in P_\varkappa\lambda)$ *such that* $(\forall x \in P_\varkappa\lambda)(c_x\colon x^2 \to P_{\varkappa_x}x)$,

$$(\exists c\colon \lambda^2 \to P_\varkappa\lambda)(\forall x \in P_\varkappa\lambda)(\{y \in \hat{x}\colon c_y \restriction x^2 = c \restriction x^2\} \in NS^+_{\varkappa\lambda}).$$

Proof. Set $X = \{x \in P_\varkappa\lambda\colon |[x]^{<\varkappa_x}| = |x|\}$ and notice that by 3.4 (2) in [2] this is in $NSh^*_{\varkappa\lambda}$.

Let $p\colon \lambda^2 \to \lambda$ be a bijection, and recall that $\{x \in P_\varkappa\lambda\colon p''(x^2) = x\}$ is cub in $P_\varkappa\lambda$ and hence is in $NSh^*_{\varkappa\lambda}$. Thus $X_1 = \{x \in X\colon p''(x^2) = x\} \in NSh^*_{\varkappa\lambda}$.

For each $x \in X_1$, let $\varphi_x\colon P_{\varkappa_x}x \to x$ be a bijection, and define $f_x\colon x \to x$ by $f_x(\alpha) = \varphi_x c_x p^{-1}(\alpha)$. Now let $f\colon \lambda \to \lambda$ be such that

$$(\forall x \in P_\varkappa\lambda)(H_x = \{y \in X_1 \cap \hat{x}\colon f_y \restriction x = f \restriction x\} \in NS^+_{\varkappa\lambda}).$$

Pick $(\alpha, \beta) \in \lambda^2$ and set $p(\alpha, \beta) = \gamma$. Notice that for any $y \in H_{\{\alpha,\beta,\gamma\}}$, $f(\gamma) = f_y(\gamma) = \varphi_y c_y p^{-1}(\gamma) = \varphi_y c_y(\alpha, \beta)$. Thus define $c\colon \lambda^2 \to P_\varkappa\lambda$ by $c(\alpha, \beta) = \varphi_y^{-1} fp(\alpha, \beta)$ where $y$ is any element of $H_{\{\alpha,\beta,p(\alpha,\beta)\}}$.

To see that this works, pick $x \in P_\varkappa\lambda$ and let $y$ be any element of $H_x$. Then $x \cup p''(x^2) \subseteq y \cup p''(y^2)$, so for any $(\alpha, \beta) \in x^2$, $c(\alpha, \beta) = \varphi_y^{-1} fp(\alpha, \beta) = \varphi_y^{-1} f_y p(\alpha, \beta) = \varphi_y^{-1}\varphi_y c_y p^{-1} p(\alpha, \beta) = c_y(\alpha, \beta)$. ■

Remark. Recall that for any $c\colon \lambda^2 \to P_\varkappa\lambda$, $\{x \in P_\varkappa\lambda\colon (\forall \alpha, \beta \in x)(c(\alpha, \beta) \subseteq x)\}$ is cub in $P_\varkappa\lambda$, and for any cub subset $C$ of $P_\varkappa\lambda$ there is a $c\colon \lambda^2 \to P_\varkappa\lambda$ such that $\{x \in P_\varkappa\lambda\colon (\forall \alpha, \beta \in x)(c(\alpha, \beta) \subseteq x)\} \subseteq C$ (Menas [3]).

THEOREM. *If $\lambda^{<\varkappa} = \lambda$ and $\varkappa$ has the $\lambda$-Shelah property, then for any stationary subset $S$ of $P_\varkappa\lambda$, $\{x \in P_\varkappa\lambda \colon S \cap P_{\varkappa_x} x \in NS^+_{\varkappa_x x}\} \in NSh^*_{\varkappa\lambda}$.*

Proof. Suppose not; let $S \in NS^+_{\varkappa\lambda}$ be such that

$$X = \{x \in P_\varkappa\lambda \colon S \cap P_{\varkappa_x} x \in NS_{\varkappa_x x}\} \in NSh^+_{\varkappa\lambda}.$$

In view of 3.4 (2) in [2] we may assume w.l.o.g. that $(\forall x \in X)(|[x]^{<\varkappa_x}| = |x|)$.

For each $x \in X$, let $c_x \colon x^2 \to P_{\varkappa_x} x$ be such that

$$C_x = \{z \in P_{\varkappa_x} x \colon (\forall \alpha, \beta \in z)(c_x(\alpha, \beta) \subseteq z)\} \subseteq P_{\varkappa_x} x - S.$$

Now let $c \colon \lambda^2 \to P_\varkappa\lambda$ be such that

$$(\forall x \in P_\varkappa\lambda)(H_x = \{y \in X \cap \hat{x} \colon c_y \restriction x^2 = c \restriction x^2\} \in NS^+_{\varkappa\lambda}),$$

and set $C = \{x \in P_\varkappa\lambda \colon (\forall \alpha, \beta \in x)(c(\alpha, \beta) \subseteq x)\}$.

Pick $x \in C \cap S$ and then pick $y \in H_x$ such that $x \in P_{\varkappa_y} y$. Then

$$(\forall \alpha, \beta \in x)(c_y(\alpha, \beta) = c(\alpha, \beta) \subseteq x),$$

thus $x \in S \cap C_y$. This is the required contradiction. ■

### References


[1] D. M. Carr, *$P_\varkappa\lambda$ generalizations of weak compactness*, Z. Math. Logic Grundlag. Math. 31 (1985), 393–401.

[2] — *$P_\varkappa\lambda$ partition relations*, this journal, this issue.

[3] T. K. Menas, *On strong compactness and supercompactness*, Ann. Math. Logic, 7 (1974), 327–359.



DEPARTMENT OF MATHEMATICS
YORK UNIVERSITY
North York, Ontario
M3J 1P3
Canada


*Received 2 December 1985*

# A general theory of superinfinitesimals

by

**Benjamin Benninghofen** (Iowa) and **Michael M. Richter** (Kaiserlautern)

**Abstract.** In this paper the concept of a "superinfinitesimal number" is defined in terms of a generalized notion of monads. This allows to extend the concept to very general situations. A transfer theorem relates properties of generalized monads with those of ordinary monads. Some applications are given, mostly to the theory of monads.

**Introduction.** The idea of infinitesimals and monads in nonstandard analysis has been applied successfully to general topology and functional analysis (cf. [Lu] and [Str–Lu] Ch. 8–10). We extend this theory in a new way.

The main result of this article is a transfer theorem that allows us to compare "orders of infinity" by extending certain formal properties of monads indexed by standard points to "$\pi$-monads" indexed at nonstandard points. For example, L'Hospital's rule from calculus involves a limit of derivatives. If $f(x)$ tends to infinity as $x$ tends to zero, $\zeta$ is a small positive standard number, and $\xi$ is a positive infinitesimal, then $f(\zeta)/f(\xi)$ is infinitesimal. In the proof of L'Hospital's rule (Proposition 4.1) given below, we choose $\zeta$ infinitesimal and $\xi$ *superinfinitesimal* so that we may transfer the statement, "$f(\zeta)/f(\xi)$ is infinitesimal" to the infinitesimal index $\zeta$. The notion of "superinfinitesimal" is relative.

We will use the framework of Internal Set Theory (IST) (see [Ne] or [Ri]). The full strength of internal set theory (namely that it axiomatizes the whole universe of sets) is not used, however. We work with bounded formulas and these can be interpreted in a suitable universe. Refering to internal set theory means for those readers who prefer to think in terms of superstructures that all one has to know about the superstructure is that the axioms of IST are valid.

The identification of the particular class of properties subject to the transfer is the main content of the transfer theorem. This relies on a extension of Nelson's reduction algorithm applied to a class of formulas very much like these encountered in the topological languages of [Fl–Zi].

Unfortunately, the class of formulas which we can transfer is not a simple one. However, it is a useful one.