Solution of Kuratowski's problem on function having the Baire property, I.

by

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Abstract. In this paper it is proved: ZFC + "there is measurable cardinal" is equiconsistent with ZFC + "there is a Baire metric space $X$, a metric space $Y$, and a function $f: X \to Y$ having the Baire property such that there is no meager set $F \subseteq X$ for which $f|X\setminus F$ is continuous".

In 1935 K. Kuratowski [11] posed the following problem: whether a function $f: X \to Y$ having the Baire property, where $X$ is completely metrizable and $Y$ is metrizable, is continuous apart from a meager set (cf. P. 6 [12]).

In this paper it will be proved:

**Theorem.** The following theories are equiconsistent:

1. ZFC + $\exists$ measurable cardinal;
2. ZFC + there is a complete metric space $X$, a metric space $Y$, and a function $f: X \to Y$ having the Baire property such that there is no meager set $F \subseteq X$ for which $f|X\setminus F$ is continuous;
3. ZFC + there is a Baire metric space $X$, a metric space $Y$, and a function $f: X \to Y$ having the Baire property such that there is no meager set $F \subseteq X$ for which $f|X\setminus F$ is continuous.

1. Definitions and the basic facts. Let $X$ be a topological space, and $A \subseteq X$. The set $A$ is said to have the Baire property if

$$A = (\overline{G} \setminus P_1) \cup P_2,$$

where $G$ is open and $P_1, P_2$ are meager sets (for basic facts see Kuratowski [10]). A map $f: X \to Y$ has the Baire property iff for each open set $V \subseteq Y$, $f^{-1}(V)$ has the Baire property.

In [4] the equivalence of the following statements has been proved: Let $X, Y$ be metric

(i) for each subspace $X^* = G \setminus F$ of $X$, where $G$ is a nonempty open set and $F$ is a meager set and for each partition $\mathcal{F}$ of $X^*$ into meager sets, there is a family $\mathcal{G} \subseteq \mathcal{F}$ such that $\mathcal{G}$ does not have the Baire property.

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(ii) for each map \( f : X \to Y \) having the Baire property, there exists a meager set \( F \subseteq X \) such that \( f(X \setminus F) \) is continuous.

1.2. In [3] the following has been proved: Let \( X \) be a complete metric space with weight \( \leq 2^{\omega} \) and let \( \mathcal{F} \) be a partition of \( X \) into meager sets. Then there exists a family \( \mathcal{F} \subseteq \mathcal{F} \) such that \( \mathcal{F} \) does not have the Baire property. As an immediate corollary from 1.2 and 1.1 we obtain a positive answer to Kuratowski’s problem in the case of a space \( X \) of weight \( \leq 2^{\omega} \).

1.3. Let \( \lambda \) be a regular cardinal, and assume that \( I \) is a \( \lambda \)-complete ideal that contains all singletons. The ideal \( I \) is precipitous if whenever \( S \) is a set of positive measure \( (S \in \mathcal{P}(\lambda) \setminus 1^+ \) and \( \{W_n : n < \omega\} \) are maximal \( I \)-partitions \( (W_n \cap b = a) \) for each \( b \in W_n \) and \( a \neq b \) then \( a \cap b \in I \) and \( a \subseteq S \) then \( S_{\text{add}} \in (b \cap a \in I^+) \) of \( S \) such that

\[
W_0 \supseteq W_1 \supseteq \ldots \supseteq W_n \supseteq \ldots
\]

\( (W_n \supseteq W_{n+1} \) denotes that \( W_{n+1} \) is refinement of \( W_n \) then there is a sequence of sets \( X_0 \supseteq X_1 \supseteq \ldots \supseteq X_n \supseteq \ldots \) such that \( X_n \in W_n \) for each \( n \), and \( \{X_n : n \in \omega\} = \emptyset \).

An ideal \( I \) over \( \lambda \) is \( \mu \)-saturated iff \( I(I) \) has the \( \mu \)-c.c. property.

1.4. If \( \kappa \) is a regular uncountable cardinal that carries a precipitous ideal, then \( \kappa \) is measurable in some transitive model of ZFC.

1.5. Assume \( V = L \). Then there is no precipitous ideal over any cardinal.

1.6. Con(ZFC + there is a precipitous ideal) iff Con(ZFC + there is a measurable cardinal).

1.7. If \( \kappa \) is a cardinal then let \( B(\kappa) \) denote a metric space \( (D(\kappa)^\omega) \) where \( D(\kappa) \) is a discrete space of cardinality \( \kappa \). For each natural \( n \) and function \( x : n \to \kappa \), let \( U(n) = \{f \in D(\kappa) : |n = x|\} \) (we will identify \( U(n) \) with \( x \)). The set \( U(\kappa) \) is a function from the natural number into \( \kappa \) is a canonical base for \( B(\kappa) \).

If \( X \) is a topological space in which the Baire theorem holds we call \( X \) a Baire space.

1.8. \( \square^* \) is the following statement: there is a set \( E \subseteq \kappa^+ \) and a sequence \( \langle C_n : n \in \kappa^+ \rangle \) such that

1. \( E \) is stationary in \( \kappa^+ \);
2. \( C_n \) is closed and unbounded in \( \kappa \);
3. if \( x \in C_n \) then \( C_n \subseteq C_{n+1} \);
4. if \( \gamma \) is a limit point of \( C_n \), then \( \gamma \notin E \) and \( \gamma \cap C_\gamma = C_n \).

If \( E \) is a stationary subset of \( \kappa \). Let \( \square^*(E) \) is the following assertion. There is a sequence \( \langle S_n : n \in E \rangle \) such that \( S_n \subseteq \kappa \) and for every \( X \subseteq \kappa \), the set \( \{x \in E : X \cap \kappa = S_n \} \) is stationary in \( \kappa \).

1.9. Let \( NS_\kappa \) denote the set of all nonstationary subsets of \( \kappa \).

§ 2. The case of a complete metric space.

2.1. Theorem. Assume that \( I \) is an \( \omega_1 \)-complete ultrafilter over the cardinal \( \kappa \). Then \( B(\omega_1) \) can be split into \( \kappa \) meager sets \( \{F_\alpha : \alpha < \kappa\} \) in such a way that for each \( \alpha \in \kappa \), the set \( \{F_\alpha : \alpha < \kappa\} \) has the Baire property.

Proof. Let \( \{F_\alpha : \alpha < \omega_1\} = I \).

Define \( F_0 = \{x \in B(\omega_1) : \alpha = \min \langle F_\alpha : \alpha \in \omega_1\rangle \} \).

It is easy to see that each \( F_\alpha \) is a meager subset of \( B(\omega_1) \).

Let \( \mathcal{F} \subseteq \mathcal{F} \) be \( \omega \)-closed, such that \( \mathcal{F} \) is \( \omega \)-complete.

We claim that \( \bigcup \{F_\alpha : \alpha \in A\} \) contains a set \( \{x \in B(\omega_1) : \alpha = \beta \} \) for each \( \alpha \in A \).

Indeed, if \( x \in \mathcal{F} \), then \( \bigcap \{F_\alpha : \alpha \in \omega_1 \} \subseteq F_\alpha = A \) and \( \mathcal{F} \subseteq \mathcal{F} \) has the Baire property.

It means that \( \bigcup \{F_\alpha : \alpha \in A\} \) has the Baire property.

2.1. Remark. It has been proved in [6] that if ZFC + there exists a measurable cardinal is consistent then, ZFC + there is a partition \( \mathcal{F} \subseteq \mathcal{F} \) into meager sets such that for each \( \mathcal{F} \subseteq \mathcal{F} \), the set \( \mathcal{F} \) has the Baire property, is consistent, too.

2.2. The following theorem does not involve advanced model theory. This is the reason we decided to present it here (other proofs involve the forcing method).

Theorem. Assume \( V = L + \) "there are no weakly compact cardinals." Then for any partition \( \mathcal{F} \) of a complete metric space \( X \) into meager sets there is \( \mathcal{F} \subseteq \mathcal{F} \) such that \( \mathcal{F} \) does not have the Baire property.

Proof. Assume not. Let \( \kappa = \omega \), and \( \mathcal{F} = \{F_\alpha : \alpha < \kappa\} \). It can be assumed that an ideal \( \mathcal{I} = \{A \in \mathcal{P}(\kappa) : \bigcup \{F_\alpha : \alpha \in A\} \) is meager \} \) is uniform and \( \omega \)-complete.

We now modify a standard construction of Suslin trees in \( L \) (for details see [2]).

There are two cases

Case 1. \( \kappa = \omega \) for some \( \mu \). Let \( E \subseteq \kappa^+ \), \( \langle C_\gamma : \gamma < \mu^+ \rangle \) satisfy \( \Gamma_\gamma(E) \). The tree will be constructed by induction on levels. The elements of \( T \) will be members of \( \kappa \) and \( \gamma < \beta \) to \( \gamma < \beta \). Let \( T_0 = \emptyset \).

If \( T_\alpha \) is defined then \( T_{\alpha+1} \) is obtained by adding two new ordinals as extensions of each member of \( T_\alpha \). In the limit stage, associate with each point \( x \in T_\alpha \), the \( \alpha \)-branch \( b^\alpha \) with \( x \in b^\alpha \).

Let \( \langle \gamma_n \rangle_\infty \) be a monotone enumeration of \( C_\alpha \) and \( \mathcal{V}(x) \) be the least \( v \) such that \( x \in T_{\gamma_n} \).

Define a sequence \( \langle p^\alpha_0 \rangle_{\infty} \langle p^\alpha_{n+1} \rangle_{\infty} \) of elements of \( T_\alpha \) as follows:

\[
\begin{align*}
& \langle p^\alpha_0 \rangle_{\infty} = \text{the least } y \in T_{\gamma_0} \text{ such that } x \leq y \\
& \langle p^\alpha_{n+1} \rangle_{\infty} = \text{the least } y \in T_{\gamma_n} \text{ such that } p^\alpha_n \leq y
\end{align*}
\]

The \( p^\alpha_n \) is unique \( y \in T_{\gamma_n} \) such that for all \( v < y \)

\[
\langle p^\alpha_n \rangle_{\infty} \leq y
\]

if it exists.

Let \( T_\alpha = \{y \in T_\alpha : (x \leq y) \} \). \( T_\alpha \) is defined as follows: If \( \alpha \notin E \), \( T_{\alpha+1} \) consists of the one point extensions of each \( b^\alpha \), \( x \in T_\alpha \). If \( \alpha \in E \) and \( S_\alpha \) is not a maximal antichain, do likewise.
If $x \in E$ and $S_x$ is a maximal antichain of $T_x$, then $S_x$ is a maximal antichain of $E$. 

Case 2. Because $\mu$ is not weakly compact, by Theorem VII.1.3 from [2], there is $E = \mu$ such that for each $x, E \cap x$ is not stationary in $x$. The construction in this case is similar to that of case 1. Let $T_0 = \emptyset$, and if $T_0$ is defined, $T_{n+1}$ has definition the same as case 1. Assume that $x$ is the limit. Let $\langle S_x, x < \mu \rangle$ satisfy $Q_0(E)$. If $x \in E$ then $T_x$ consists of the one-point extension of each $b_x$ for $x \in T_x$. If $x \in E$ and $S_x$ is not a maximal antichain of $T_x$ then the definition of $T_x$ is the same as in case $x \notin E$. 

If $x \in E$ and $S_x$ is a maximal antichain of $T_x$ then for any $x \in T_x$ is it possible to pick an $x$-branch $b_x$ such that $b_x \cap S_x \neq \emptyset$. Let $b_x^*$ denote such an $L$-minimal branch and let $T_x$ consist of a one-point extension of each $b_x^*$. 

Let $T_\omega$ denote a tree of height $\omega + 1$ such that $T_\omega[\omega]$ is a Cantor tree and $T_\omega$ is obtained by one-point extensions of each branch of $T_\omega[\omega]$. 

Claim. There does exist an embedding $\varphi$ of $T_\omega$ as a tree, into $T$, in such a way that for each $x < \omega + 1$, there is a $x^*_\omega$ for which $\varphi(T_\omega[\omega] \models \varphi(x^*_\omega)$. 

Assume that such an embedding exists. Then $T_\omega$ can be embedded via $\varphi$ into $T_\omega[\omega] \models \varphi(x^*_0).$ Let $\delta = \lim b_x$. In the tree $T_\delta$, with any point $\varphi(x)$ is associated branch $b\varphi(x)$ which is extended in the $\delta$-step. The number of such branches is countable. So there is a branch $b\varphi(x)$ in $T_\omega[\omega]$ such that $\varphi(x) \models \varphi(x^*_\omega)$. 

By the construction of $T$ there must exist a $y \in T$ such that $b_x^*$ coincides with the branch defined by $\varphi(x)$. But by minimality of elements in the branch, (Case 1, and part of Case 2), or $L$-minimality of the branch, there is an element $x \in T \models \varphi(x^*_\omega)$ such that $b_x^* = b\varphi(x^*_\omega)$. 

A contradiction. 

Now, for the proof of the theorem, it can be assumed that $\langle F_x, x \in A \rangle$ has the Baire property. Also, it can be assumed that, for each $x < \omega$ is an element of $T$. 

For $x \in T$ let $V_x = \bigcup \{ F_x \mid y \in \mathfrak{B} \}. $ 

There are $x_0, x_1, x_1$ such that $x_0, x_1 \in T_\omega[\omega]$. There, $V_x, V_x \neq \emptyset$ are nonmeager. Indeed, because $T$ is Suslin and by the assumption on $P$, in each level there is an $x$ such that $P_x$ is nonmeager, but since $F_x$ is $\omega$-complete and in $T$ there are no $\omega$-branches there must exist two noncompatible elements of $T_x$, $y \in T_x$ such that $V_y, V_y \neq \emptyset$. 

Now, let $G_0$ and $E_x$ for $x \in \omega$ be as follows 

(i) $G_0$ is an open nonvoid set; 

(ii) $V_x \subseteq G_x \cup \{ E_x^* \mid x \in \omega \};$ 

(iii) $E_x^*$ is a closed meager set; 

(iv) diam $G_1 < 1/2$. 

By induction on the length of $l$, where $x$ is a function from the natural numbers into $2$, define $\sigma_{\text{length} l}(x, l, G_0, \{ E^*_n \mid n \in \omega \})$ such that 

1. $\sigma_{\text{length} l}(x, l, G_0, \{ E^*_n \mid n \in \omega \})$ such that 

2. $x_0, x_1 \in \mathfrak{B}$ and if $\exists n \neq x_0$ then $x_0$ is $T$-incompatible with $x_0'$ and if $x_0 \geq x$ then $x_0 \geq x_0$; 

3. $G_0$ is an open set; 

4. $\text{diam } G_1 < \frac{1}{2^{\text{length} l}}$; 

5. $E^*_0$ is a closed meager set; 

6. $V_x = G_x \cup \{ E^*_x \mid n \in \omega \}$; 

7. $\{ G_x \subseteq G_x \cup \{ E^*_n \mid x \geq n \in m \} \} \text{ for } x \leq x_0$. 

Assume that for $m < \omega, x_0, x_0'$, $G_0$ and $E^*_0$ are defined. 

Using exactly the same argument as in step 0, it is possible to find $x_0' > x_0$ and invariable $x_0' \subseteq x_0$, $x_0' \subseteq x_0$ such that $V_x, x_0', G_0, E^*_0$ is nonmeager for $x = 0, 1$. 

Let $x = \sup\{ x_0': \text{length} x \leq m \}$. 

Now let $x_0, x_1 \in \mathfrak{B}$ and $x_0, x_1 \in T_x$. The set $G_0$ and $E^*_0$ can be easily found, since $V_x$ and $G_0$ has the Baire property and is nonmeager. Since $x$ is a complete metric space $\{ G_x \mid x \in \omega \}$ is a one-point set $x_0$. By the construction if $x \neq x_0$ and $F_x \models \mathfrak{B} \models E^*_0$ then $x_0 \models x_0$. 

Since $y \in V_x$ for all $n$ then $x_0' > x_0$ holds. It is possible to find $x_0'$ such that $E^*_0 > x_0'$ for all $x_0$ and $n$, and such that $E^*_0 > E^*_0$ in $x_0'$. This means that the tree of type $T_x$ can be embedded into $T$, a contradiction. 

2.4. Remark. It might be observed that this argument works only under the assumption that the space is complete. 

2.5. Remark. Assume $V = L$. Let $x$ be a regular cardinal and $I$ a $\omega$-complete uniform ideal (i.e. $x^* \subseteq I$) over $x$. 

Then a completion of the algebra $\text{Pr}(x)/I$ is isomorphic to the algebra of regular open subsets of $B(x^*)$. This means that two arbitrary $\omega$-complete uniform ideals over $x$ are similar. Indeed, since $V = L$ is assumed, then $I$ is not precipitous. Let $W_x = W_y \supseteq W_y$ be a sequence which contradicts to the precipitous. 

Let $x \in I^*$. We claim that there is an $n \in \omega$ such that 

$$[\{ y \in x \in I^* \land y \in W_y \}] = \omega^* $$ 

If it is not the case, then we can find a disjoint family such that $x \in \{ x \in I^* \land y \in W_y \}$ for some $y \in I$. 

Because $2^\omega = \omega^*$, then by induction using the fact pointed out above we can construct a dense subset of size $\omega^*$ in $P_0/I$ which looks like a canonical base for $B(\omega^*)$. 

Kuratowski’s problem:
§ 3. The case of incomplete metric spaces. Let \( I \) be an ideal over cardinal \( \kappa \). Consider the set \( X(I) = \{ x \in I^* \} \cap \{ x(n) = 0 \} \neq \emptyset \) and

\[ \forall_{\kappa \in I} \cap \{ x(m) m < n \} \in I^* \} \]

The set \( X(I) \) is considered as a subset of a complete metric space \( (I^*, \rho) \), where the set \( I^* \) is equipped with the discrete topology.

3.1. Proposition. \( X(I) \) is a Baire space iff \( I \) is a precipitous ideal.

Proof. Assume that \( I \) is not precipitous. Then by 1.3 there are families \( (X_n) \) of open sets \( X_n \supseteq X_{n+1} \) for any sequence \( X_0 \supseteq X_1 \supseteq \ldots \) such that \( X_n \neq X_{n+1} \). By 3.2 the set \( \{ x \in X(I) \} \cap \{ x(n) \neq 0 \} \) is dense in \( I^* \).

Now let \( X_n \) be a function, dom \( x = n \geq 1 \) and \( \forall_{\kappa \in I} \cap \{ x(k) k < n \} \in I^* \) and \( x(n-1) \in X_n \). Since \( X_n \) is maximal \( I \)-partition \( \bigcup U_n \) is dense and open and \( \bigcap \{ U_n n \in \kappa \} = \emptyset \).

Let us assume that \( X(I) \) is not the Baire space; that is, there are open dense \( U_n \) such that \( \bigcap \{ U_n n \in \kappa \} = \emptyset \). Let

\[ \Psi_n = \{ x \in X(I) \} \cap \{ x(n) = 0 \} \neq \emptyset \}

\[ \bigcup \Psi_n \] is dense and open and \( \bigcup \Psi_n \neq X(I) \). The sequence \( W_0 \supseteq W_1 \supseteq \ldots \) of \( I \)-partitions of \( \kappa \) can be defined by induction in such a way that

\[ \text{iff } x \in W_n \text{ then there is } x \in \Psi_n, \]

\[ \text{iff } x \in \{ x(k) k \in \text{dom}(x) \} \]

Of course for any sequence \( \gamma_0 \equiv \gamma_1 \equiv \ldots \) such that \( \gamma_n \in W_n \) the set \( \{ \gamma_n n \in \kappa \} = \emptyset \).

Remark. R. Pol observed that Proposition 3.1 can be proved using the game theoretic construction of precipitous ideals.

3.2. Theorem. Let \( I \) be a precipitous ideal over some regular cardinal. Then there is a decomposition \( (I^*, \rho) \) of the space \( I \) into meager sets such that \( (I^*, \rho) \) has the Baire property. In this case \( X(I) \) is a Baire space.

Proof. \( X(I) \) is a Baire space by Proposition 3.2.

Now, for each \( a \in I \) let

\[ F_a = \{ x \in X(I) \} \cap \{ x(a) n \neq a \} \}

It is not difficult to verify that \( F_a \) is a meager set in \( X(I) \) and \( \{ F_a a \in \kappa \} \) form a partition of \( X(I) \).

Let \( A \in I^* \) then

\[ \bigcup \{ F_a a \in A \} \equiv \{ x \in X(I) \} \exists_{a \in A} \exists_{s \in \kappa} \exists_{b \in \kappa} \{ x(s) = b \} \}

The set \( A \) is open. It is sufficient to have that \( P = \{ \{ F_a a \in A \} \equiv \{ x \in X(I) \} \exists_{a \in A} \exists_{s \in \kappa} \exists_{b \in \kappa} \{ x(s) = b \} \}

The set \( K \) is open. It is sufficient to have that \( P = \{ \{ F_a a \in A \} \equiv \{ x \in X(I) \} \exists_{a \in A} \exists_{s \in \kappa} \exists_{b \in \kappa} \{ x(s) = b \} \}

Theorem 3.3. If ZFC + there is a Baire space, and a decomposition \( \mathcal{F} \) of \( X \) into meager sets such that for any \( \mathcal{F} \equiv \mathcal{G} \), the set \( \bigcup \mathcal{F} \) has the Baire property, then \( \mathcal{F} \) is consistent with ZFC + there is a measurable cardinal and is consistent with ZFC.

Proof. Let \( \mathcal{F} = \{ F_a a \in \kappa \} \) be a decomposition of \( X \) such that

\[ \forall_{\kappa \in \kappa} \bigcup \{ F_a a \in \kappa \} \equiv \{ x \in X(I) \} \exists_{a \in A} \exists_{s \in \kappa} \exists_{b \in \kappa} \{ x(s) = b \} \}

have the Baire property. To see that, define \( U \supseteq U \supseteq \{ x \in X(I) \} \neq \emptyset \). If \( W \in \mathcal{F} \) then \( x \in \{ W \} \equiv \{ x \in X(I) \} \neq \emptyset \). If \( x \in X(I) \) then \( x \in \{ F_a a \in \kappa \} \equiv \{ x \in X(I) \} \neq \emptyset \).

Let \( B \) be the Boolean algebra of regular open subsets of \( X \) and \( G \)-generic ultrafilter over \( B \). We consider the "generic ultrapower" \( X(\mathcal{V})/G \) (this is actually a limit ultrapower in Keisler sense (c.f. sec. 6.4 in [1])).

The generic ultrapower is a model of ZFC. We have a fundamental theorem in this form

\[ \mathcal{V}(\mathcal{V})/G \equiv \mathcal{V}(\mathcal{F}_1 \ldots \mathcal{F}_n) \text{ iff } \{ x \in X(I) \} \equiv \mathcal{F}(\mathcal{F}_1 \ldots \mathcal{F}_n) \in G \]

where \( \mathcal{F}_1 \ldots \mathcal{F}_n \in \mathcal{F}(\mathcal{V}) \). We have a natural embedding \( j_\mathcal{V} : \mathcal{V}(\mathcal{V})/G \rightarrow X(\mathcal{V})/G \) where \( e : X \rightarrow V \) such that \( \forall_{\kappa \in \kappa}(\mathcal{V}_x) = \mathcal{V}(\mathcal{V})/G \) is well defined: if \( \mathcal{F}_1 \ldots \mathcal{F}_n \in \mathcal{F}(\mathcal{V}) \) then \( \mathcal{F}_1 \ldots \mathcal{F}_n \in \mathcal{F}(\mathcal{V}) \) is measurable in \( X \) and by Baer category theorem \( \exists_{\kappa \in \kappa} \exists_{\kappa \in \kappa} \mathcal{F}_1 \ldots \mathcal{F}_n \in \mathcal{F}(\mathcal{V}) \)

Of course \( j_\mathcal{V}(\mathcal{V}) \equiv \mathcal{V} \).

Lemma. \( X \) is a measurable cardinal in some transitive model of ZFC.

From this lemma the theorem follows immediately.

The proof of it is almost the same as the proof of Theorem 86 a in [8] so we give only a sketch of it (this technique is due to Solovay).

Proof of the lemma. Let \( K \) be a class of strong limit cardinals \( \kappa \) such that \( \kappa \) is an \( \kappa \)-supercompact cardinals. Then \( \kappa \) is an \( \kappa \)-supercompact cardinals. Let \( \kappa = \gamma_n n \in \omega \) and \( \lambda = \sup \lambda \).

"X" is a transitive model of ZFC. From this model the theorem follows immediately.
Sublemma (compare with Lemma 35.12 in [8]). There exists an $L[A]$-ultrafilter $W$ over $x$ such that $W$ is nonprincipal $L[A]$-normal and $L[A]$-$\mathfrak{c}$-complete, iterable and every iterated ultrapower $\text{FN}^{[\mathfrak{c}]}(L[A])/W$ is well-founded.

Proof. There is a $\gamma$ and $S \in G$ such that

$$S \uparrow d \rightarrow \gamma \in \text{Fin}(V) \uparrow G$$

where $d: X \rightarrow \text{Ordinals}$, such that $d(x) = \gamma$ if $x \in F_x$. Let

$$U = \{Y \in P(\alpha) \cap L[A]; S \cap \bigcup \{F_{\alpha} \mid \alpha \in Y\} \text{ is nonmeager} \}.$$  

Of course $U$ is $L[A]$-ultrafilter. Kunen’s argument shows that for any $Y \in L[A] \cap P(\alpha)$ there is a finite set $E \subseteq x \cup K$ and a formula $\varphi$ such that

$$Y = \{\xi \in x[L[A] \uparrow \varphi(\xi, E, A)\}.$$  

By the definition of $\gamma$ we have $\bigcup \{F_{\alpha} \mid \alpha \in Y\} \cap S$ is nonmeager set then

$$\bigcup \{F_{\alpha} \mid \alpha \in Y\} \cap S \uparrow \gamma \in \text{J}d(Y)$$

and by the fact that $j_{\text{D}}(A) = A$ and $j_{\text{D}}(E) = E$ there is

$$\bigcup \{F_{\alpha} \mid \alpha \in Y\} \cap S \uparrow L[A] \uparrow \varphi(\gamma, E, A)$$

where the forced formula is about $V$, and thus, hence

$$S \uparrow \gamma \in \text{J}d(Y)$$

which means that $S \uparrow \bigcup \{F_{\alpha} \mid \alpha \in Y\} \in G$. Let $f: u \rightarrow x$ which represents $(f$ is constant on $F\rangle)\uparrow \alpha$. In fact we can assume that $f$ is defined on $\alpha$. Let $W = f_\alpha(F)$. $(\forall \alpha$ denotes the dual filter to $\alpha$.) Since $f$ is $\mathfrak{c}$-complete $W$ is $L[A] \uparrow \mathfrak{c}$-complete. From this point the proof is exactly the same as the proof of Lemma 35.11 and Theorem 86 in [8]. Now similar techniques we will prove:

**Theorem 3.4.** If $\text{ZFC} + \exists$ measurable cardinal is consistent then the following is consistent with $\text{ZFC}$: There is a metric Baire noncomplete space $X$ and a partition $\mathcal{F}$ of $X$ into meager sets such that for any $\mathcal{F}' \subseteq \mathcal{F}$, the set $\bigcup \mathcal{F}'$ has the Baire property. But there does not exist a complete metric space with the same decomposition property.

Proof. Let us work in $L[D] = V$ where $D$ is a normal $\mathfrak{c}$-complete ultrapower over $x$. By Kunen’s theorem $x$ is the only measurable cardinal in $V$ (see [6]). Now let $N = V[G_1][G_2]$ where $N$ is obtained by Levy collapsing forcing which collapses $x$ to $\mathfrak{c}$ and later adding $\mathfrak{c}$ Cohen reals. By [8] and [9] $\mathfrak{c}$ carries the precipitous ideal $I$ over $\mathfrak{c}$. Of course by Theorem 3.2 the space $X(I)$ has the required decomposition. Let us assume that in $N$ there is a complete metric space $X$ with the decomposition $\mathcal{F}$. Then by Theorem 1.2, $x = |\mathcal{F}| \geq \mathfrak{c}$, (we assume that $x$ is the smallest cardinality of family with this property) we can repeat twice the arguments of Theorem 3.3. In fact we can get a transitive model such that $x$ is a measurable cardinal and later extend it to $M$ such that $x$ and $\mathfrak{c}$ are measurable, of course $x > \mathfrak{c}$ and such that there is an elementary embedding $i: M \rightarrow L[D]$, a contradiction, because $L[D]$ has only one measurable cardinal.

§ 4. Final conclusions and remarks.

4.1. Proof of the theorem. Theorem is an immediately consequence of Theorems 1.1, 2.1, 3.2 and 3.3.

4.2. Let $A \ni x$, where $x$ is a regular cardinal and let

$$A^* = \{x \in B(\alpha) \mid \sup \{\alpha(\eta) \mid \eta \ni \alpha \in A\}.$$

It is known [6] that if $A$ is a stationary subset of $C_m(\alpha) = \{\alpha \mid \eta \ni \alpha \in A\}$, then $A^*$ is Baire. P. Pol has strengthened this result in the following way: If $B \subseteq B(\alpha)$ and if $A$ is stationary such that for each $\alpha \in A$ the set $(A)^* \cap B$ is nonmeager in $(A)^*$ then $B$ is a Baire space. He proved also the similar theorem for Baire metric spaces (not necessary complete). Now we are ready to prove the Proposition 4.3.

4.3. PROPOSITION. If the density of $X$ is inaccessible and $X$ is a Baire metric space then for each decomposition $\mathcal{F}$ of $X$ into meager sets there is $\mathcal{F} \subseteq \mathcal{F}'$ such that $\mathcal{F}'$ does not have the Baire property.

It is sufficient to consider a decomposition of $B(\alpha)$ into $\mathfrak{c}$ meager sets. In this case let

$$i(\beta) = \min \{\delta \mid \beta \in \mathcal{F}, \mathcal{F} \uparrow \beta \subseteq \mathcal{F} \uparrow \delta \}.$$  

If $i$ is regressive on some stationary set $\delta \subset C_m(\alpha)$ then there is $\eta \ni \alpha$ and stationary $\delta \subseteq A$ such that $i(\delta) = \eta$. Because $\eta < \alpha$, there is $\mathcal{F} \subseteq \mathcal{F}'$ such that $|\mathcal{F}| < \alpha$ and $\mathcal{F}'$ is a nonmeager subset of $B(\alpha)$.

There is $\mathcal{F} \subseteq \mathcal{F}'$ such that $\mathcal{F} \uparrow \mathcal{F}'$ does not have the Baire property. Assume, that $i$ is NS, -almost (nonstationary) identity on $C_m(\alpha)$. Let $A_0, A_1 \subseteq C_m(\alpha)$ be the disjoint stationary subsets such that $i_{\alpha} = i_A$, for $i = 0, 1$ and $A_0 \cup A_1 = A$. If $\alpha \subseteq A$ there is $\mathcal{F} \subseteq \mathcal{F}'$ such that $\mathcal{F} \cap \mathcal{F}'$ is nonmeager in $(A)^*$. By assumption on $\beta$ (and on $A$) the set $\mathcal{F} \cap \mathcal{F}'$ covers a nonmeager subset of $(A)^*$. Let

$$\mathcal{F} = \bigcup \{\mathcal{F} \mid \beta \in A_0\} \quad \text{and} \quad \mathcal{F}' = \bigcup \{\mathcal{F} \mid \beta \in A_1\}$$

then $\mathcal{F} \cap \mathcal{F}'$ are every where nonmeager subsets of $B(\alpha)$ (be $P$. Theorem 3.3). This means that $\mathcal{F} \cap \mathcal{F}'$ fails to have the Baire property.

4.4. COROLLARY. Assume $V = L$.

Let $F: X \rightarrow K(Y)$ be a lower Baire-measurable function, where $X, Y$ are metric space, $X$ a complete space and $K(Y)$ a compact nonvoid subset of $Y$. Then there is a Baire-measurable selector of $F$.

P.\lambda Partition relations

by

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Abstract. We study the partition relations $X \rightarrow (I^\alpha)^*\mu$, $X \rightarrow (\text{nsf})^\alpha$, and $X \rightarrow (\text{nsf})^\alpha, (I^\alpha)^*\mu$ where $X \subseteq P_\lambda$, $\lambda > \kappa$, $I$ is a proper, nonprincipal $\kappa$-complete ideal on $P_\lambda$, and $\text{nsf}$ is an unbounded homogeneous function (see 1.3, 2.1 below).

Theorem. If $\kappa^+ = \lambda$, then $\kappa$ is $\lambda$-inaccessible if $X \rightarrow (\text{nsf})^\alpha$ holds for some $X \subseteq P_\lambda$. (1.2, 4.3).

Theorem. If $X \rightarrow (\text{nsf}^\alpha)^\kappa$ holds for some $X \subseteq P_\lambda$, then $\kappa$ is almost $\lambda$-inaccessible. (1.7).

Theorem. If $\kappa^+ = \lambda$ and $\kappa$ is almost $\lambda$-inaccessible, then $X \rightarrow (I^\alpha)^*\mu$ holds for every $X \in NA_\alpha^\kappa$. (4.2).

Theorem. If $\kappa^+ = \lambda$, then $\kappa$ is mildly $\lambda$-inaccessible if $X \rightarrow (\text{nsf})^\alpha$ holds for every $X \subseteq I^\alpha_\kappa$ and $n \geq 2$. (2.4)

Theorem. If $\kappa^+ = \lambda$ and $\kappa$ has the $\lambda$-Shelah property, then $X \rightarrow (\text{nsf})$, $\text{nsf}^\alpha$ holds for every $X \subseteq NS_\lambda^\kappa$. (5.4)

All of the ideal-theoretic notation is explained in 0.0 and 0.4.

0. Introduction

0.0. Notation and basic facts. Unless we specify otherwise, $\kappa$ denotes an uncountable regular cardinal and $\lambda$ a cardinal $\geq \kappa$. For any such pair, $P_\lambda$, denotes the set $\{x \subseteq \lambda : |x| < \kappa\}$.\n
The basic combinatorial notions are defined here for $P_\lambda$ as in Jech [12]. For any $X \subseteq P_\lambda$, $\mathcal{I}$ denotes the set $\{y \subseteq P_\lambda : x \subseteq y\}$. $X \subseteq P_\lambda$ is said to be unbounded iff $(\forall x \in P_\lambda)(X \cap x \neq \emptyset)$, and $\mathcal{I}_x$ denotes the ideal of not unbounded subsets of $P_\lambda$.

In the sequel, an "ideal on $P_\lambda$" is always a "proper, nonprincipal, $\kappa$-complete ideal on $P_\lambda$" extending $\mathcal{I}_x$ unless we specify otherwise. Further, for any ideal $I$ on $P_\lambda$, $I^*$ denotes the set $\{X \subseteq P_\lambda : x \notin X\}$, and $I^*$ the filter dual to $I$. $\text{E}^{\mathcal{F}}_x$ denotes $\mathcal{I}_x^*$.\n
* AMS(MOS) subject classification (1980) primary 03E55, secondary 03E05.

Some of the results of this paper were presented at the 1983 Annual Meeting of the A.S.L. in Denver, Colorado on 8 January, 1983.

The author wishes to thank J. E. Baumgartner for a copy of his very interesting notes [2], and C. A. DiPrisco, D. H. Pelletier, D. J. Velleman, and W. S. Zwicker for their helpful remarks.