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Solution of Kuratowski's problem on function having the Baire property, I.

by

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Abstract. In this paper it is proved: ZFC + "there is measurable cardinal" is equiconsistent with ZFC + "there is a Baire metric space X, a metric space Y, and a function $f\colon X\to Y$ having the Baire property such that there is no meager set $F\subseteq X$ for which $f|X\setminus F$ is continuous".

In 1935 K. Kuratowski [11] posed the following problem: whether a function $f\colon X\to Y$ having the Baire property, where X is completely metrizable and Y is metrizable, is continuous apart from a meager set (cf. P. 6 [12]).

In this paper it will be proved:

THEOREM. *The following theories are equiconsistent*:

(1) ZFC + $\exists$ *measurable cardinal*;

(2) ZFC + *there is a complete metric space X, a metric space Y, and a function $f\colon X\to Y$ having the Baire property such that there is no meager set $F\subseteq X$ for which $f|X\setminus F$ is continuous*;

(3) ZFC + *there is a Baire metric space X, a metric space Y, and a function $f\colon X\to Y$ having the Baire property such that there is no meager set $F\subseteq X$ for which $f|X\setminus F$ is continuous.*

1. Definitions and the basic facts. Let X be a topological space, and $A\subseteq X$. The set A is said to have the *Baire property* if

$$A = (G\setminus P_1)\cup P_2,$$

where G is open and P_1, P_2 are meager sets (for basic facts see Kuratowski [10]). A map $f\colon X\to Y$ has the *Baire property* iff for each open set $V\subseteq Y$, $f^{-1}(V)$ has the Baire property.

1.1 In [4] the equivalence of the following statements has been proved: Let X, Y be metric

(i) for each subspace $X^* = G\setminus F$ of X, where G is a nonempty open set and F is a meager set and for each partition $\mathscr{F}$ of X^* into meager sets, there is a family $\mathscr{F}'\subseteq\mathscr{F}$ such that $\mathscr{F}'$ does not have the Baire property.

(ii) for each map $f\colon X\to Y$ having the Baire property, there exists a meager set $F\subseteq X$ such that $f|X\setminus F$ is continuous.

1.2. In [3] the following has been proved: Let X be a complete metric space with weight $\leqslant 2^{\omega}$ and let $\mathscr{F}$ be a partition of X into meager sets. Then there exists a family $\mathscr{F}'\subseteq\mathscr{F}$ such that $\mathscr{F}'$ does not have the Baire property. As an immediate corollary from 1.2 and 1.1 we obtain a positive answer to Kuratowski's problem in the case of a space X of weight $\leqslant 2^{\omega}$.

1.3. Let λ be a regular cardinal, and assume that I is λ-complete ideal over λ containing all singletons. The ideal I is *precipitous* if whenever S is a set of positive measure ($S\in\mathscr{P}(\lambda)\setminus I = I^+$) and $\{W_n|\ n<\omega\}$ are maximal I-partitions (if $a, b\in W_n$ and $a\neq b$ then $a\cap b\in I$ and if $a\in I^+$ and $a\subseteq S$ then $\exists_{b\in W_n}(b\cap a\in I^+)$ of S such that

$$W_0\geqslant W_1\geqslant \ldots \geqslant W_n\geqslant \ldots$$

($W_n\geqslant W_{n+1}$ denotes that W_{n+1} is refinement of W_n) then there exists a sequence of sets $X_0\supseteq X_1\supseteq \ldots \supseteq X_n\supseteq \ldots$ such that $X_n\in W_n$ for each n, and $\bigcap\{X_n|\ n\in\omega\} = \emptyset$. An ideal I over λ is *μ-saturated* iff $P(\lambda)/I$ has the μ-c.c. property.

1.4. If $\varkappa$ is a regular uncountable cardinal that carries a precipitous ideal, then $\varkappa$ is measurable in some transitive model of ZFC.

1.5. Assume $V=L$. Then there is no precipitous ideal over any cardinal

1.6. Con(ZFC + there is a precipitous ideal) iff Con(ZFC + there is a measurable cardinal).

1.7. If $\varkappa$ is a cardinal then let $B(\varkappa)$ denote a metric space $(D(\varkappa))^{\omega}$ where $D(\varkappa)$ is a discrete space of cardinality $\varkappa$. For each natural n, and function $x\colon n\to\varkappa$ let $U(x) = \{f\in{}^{\omega}\varkappa|\ f|n = x\}$ (we will identify $U(x)$ with x). The set $\{U(x)|\ x$ is a function from the natural number into $\varkappa\}$ is a canonical base for $B(\varkappa)$.

If X is a topological space in which the Baire theorem holds we call X a *Baire space*.

1.8. $\square_{\varkappa}$ is the following statement: there is a set $E\subseteq\varkappa^+$ and a sequence $\langle C_\lambda|\lambda<\varkappa^+, \lambda$ is limit$\rangle$, such that:

(i) *E is stationary in $\varkappa^+$*,

(ii) C_λ is closed and unbounded in λ,

(iii) if $\mathrm{cf}\lambda<\varkappa$ then $|C_\lambda|<\varkappa$,

(iv) if γ is a limit point of C_λ, then $\gamma\notin E$ and $\gamma\cap C_\lambda = C_\gamma$.

If E is a stationary subset of $\varkappa$. Let $\diamondsuit_{\varkappa}(E)$ is the following assertion. There is a sequence $\langle S_\alpha|\alpha\in E\rangle$ such that $S_\alpha\subseteq\alpha$ and for every $X\subseteq\varkappa$, the set

$$\{\alpha\in E|\ X\cap\alpha = S_\alpha\}$$

is stationary in $\varkappa$.

1.9. Let $\mathrm{NS}_{\varkappa}$ denote the set of all nonstationary subsets of $\varkappa$.

§ 2. The case of a complete metric space.

2.1. THEOREM. *Assume that j is an ω_1-complete ultrafilter over the cardinal $\varkappa$. Then $B(2^{\varkappa})$ can be split into $\varkappa$ meager set $\{F_\alpha|\alpha<\varkappa\}$ in such a way that for each $A\subseteq\varkappa$, the set $\{F_\alpha|\alpha\in A\}$ has the Baire property.*

Proof. Let $\{P_\alpha|\ \alpha<2^{\varkappa}\} = j$.

Define $F_\alpha = \{x\in B(2^k)|\ \alpha = \min\bigcap\{P_{x(n)}|\ n\in\omega\}$.

It is easy to see that each F_α is a meager subset of $B(2^{\varkappa})$.

Let $A\in j$. Then there is a $\beta\in 2^{\varkappa}$, such that $P_\beta = A$.

We claim that $\bigcup\{F_\alpha|\ \alpha\in A\}$ contains a set $\{x\in B(2^{\varkappa})|$ there is an $n\in\omega$ such that $x(n)=\beta\} = V$. The set V is open and dense.

Indeed, if $x\in V$ then $\bigcap\{P_{x(n)}|\ n\in\omega\}\subseteq P_\beta = A$ and $\min\bigcap\{P_{x(n)}|\ n\in\omega\}\in A$.

It means that $\bigcup\{F_\alpha|\ \alpha\in A\}$ has the Baire property.

2.2. Remark. It has been proved in [6] that if ZFC + there exists a measurable cardinal is consistent then, ZFC + there is a partition $\mathscr{F}$ of $B(c^+)$ into meager sets such that for each $\mathscr{F}'\subseteq\mathscr{F}$ the set $\bigcup\mathscr{F}'$ has the Baire property, is consistent, too.

2.3. The following theorem does not involve advanced model theory. This is the reason we decided to present it here (other proofs involve the forcing method).

THEOREM. *Assume $V = L$ + "there are no weakly compact cardinals". Then for any partition $\mathscr{F}$ of a complete metric space X into meager sets there is $\mathscr{F}'\subseteq\mathscr{F}$ such that $\bigcup\mathscr{F}'$ does not have the Baire property.*

Proof. Assume not. Let $\varkappa = |\mathscr{F}|$, and $\mathscr{F} = \{F_\alpha|\ \alpha<\varkappa\}$. It can be assumed that an ideal $V = \{A\in P(\varkappa)|\ \bigcup\{F_\alpha|\ \alpha\in A\}$ is meager$\}$ is uniform and $\varkappa$-complete.

We now modify a standard construction of Suslin trees in L (for details see [2]). There are two cases

Case 1. $\varkappa = \mu^+$ for some μ. Let $E\subseteq\mu^+$, $\langle C_\lambda|\ \lambda<\mu^+\ \&\ \lambda$ is limit$\rangle$ be as in 1.8 and $\langle S_\alpha|\ \alpha<\mu^+\rangle$ satisfy $\diamondsuit_{\mu^+}(E)$. The tree will be constructed by induction on levels. The elements of T will be members of $\varkappa$, and $\alpha<_T\beta\to\alpha<\beta$. Let $T_0 = \{0\}$. If T_α is defined then $T_{\alpha+1}$ is obtained by adding two new ordinals as extensions of each member of T_α. In the limit stage, associate with each point $x\in T|\alpha$, the α-branch b_x^α with $x\in b_x^\alpha$.

Let $\langle\gamma_\nu|\ \nu<\lambda\rangle$ be a monotone enumeration of C_α and $\bar\nu(x)$ be the least ν such that $x\in T_{\gamma_\nu}$. Define a sequence $\langle p_\nu^x|\ \bar\nu(x)\leqslant\nu<\lambda\rangle$ of elements of T_α as follows:

$p_{\bar\nu(x)}^x$ — the least $y\in T_{\gamma_{\bar\nu(x)}}$ such that $x\leqslant_T y$

$p_{\nu+1}^x$ — the least $y\in T_{\gamma_{\nu+1}}$ such that $p_\gamma^x\leqslant_T y$

p_ν^x — the unique $y\in T_{\nu_\eta}$ such that for all $\nu<y$

$p_\nu^x\leqslant_T y$ if it exists.

Let $b_x^\alpha = \{y\in T|\alpha|(\exists_{\nu<\lambda})(y\leqslant_T p_\nu^x)\}$. T_α is defined as follows: If $\alpha\notin E$, T_α consists of the one point extensions of each b_x^α, $x\in T|\alpha$: If $\alpha\in E$ and S_α is not a maximal antichain, do likewise.

If $\alpha \in E$ and S_α is a maximal antichain T_α consists of one-point extensions of each b_α^x for x lying above an element of S_α.

Case 2. Because μ is not weakly compact, by Theorem VII. 1.3 from [2], there is $E \subseteq \mu$ such that for each α, $E \cap \alpha$ is not stationary in α. The construction in this case is similar to those of case 1. Let $T_0 = \{0\}$, and if T_α is defined, $T_{\alpha+1}$ has definition the same as case 1. Assume that α is the limit. Let $\langle S_\alpha | \ \alpha < \mu \rangle$ satisfy $\diamondsuit_\mu(E)$. If $\alpha \notin E$ than T_α consists of the one-point extension of each b_α^x for $x \in T|_\alpha$. If $\alpha \in E$ and S_α is not a maximal antichain of $T|\alpha$ then the definition of $T|\alpha$ is the same as in case $\alpha \notin E$.

If $\alpha \in E$ and S_α is a maximal antichain of $T|\alpha$ then for any $x \in T|\alpha$ it is possible to pick an α-branch b_x such that $b_x \cap S_\alpha \neq \emptyset$. Let b_x^α denote such an L-minimal branch and let T_α consist of a one-point extension of each b_α^x.

Let T_∞ denote a tree of height $\omega+1$ such that $T_\infty|\omega$ is a Cantor tree and T_∞ is obtained by one-point extensions of each branch of $T_\infty|\omega$.

Claim. *There does exist an embedding φ of T_∞, as a tree, into T, in such a way that for each $\alpha < \omega+1$, there is a β_α for which*

$$\varphi(T_\infty)_\alpha \subseteq T_{\beta_\alpha}.$$

Assume that such an embedding exists. Then T_∞ can be embedded via φ into $T|_{(\lim_n \beta_n)+1}$. Let $\delta = \lim_n \beta_n$. In the tree $T|\delta$, with any point $\varphi(x)$ is associated branch $b_{\varphi(x)}^\delta$ which is extended in the δ-step. The number of such branches is countable. So there is a branch b in $T_\infty|\omega$ such that $\varphi''(b) \neq b_{\varphi(x)}^\delta$ for all $x \in T_\infty|\omega$.

By the construction of T there must exist a $y \in T$ such that b_y^δ coincides with the branch defined by $\varphi''(b)$. But by minimality of elements in the branch, (Case 1, and part of Case 2), or L-minimality of the branch, there is an element $x \in T_\infty|\omega$ such that

$$b_y^\delta = b_{\varphi(x)}^\delta,$$

a contradiction.

Now, for the proof of the theorem, it can be assumed that $\{F_\alpha | \ \alpha \in A\}$ has the Baire property. Also, it can be assumed that, each $\alpha < \varkappa$ is an element of T.

For $x \in T$ let

$$V_x = \bigcup \{F_\beta | \beta_T > x\}.$$

There are α_0, x_0, x_1 such that $x_0, x_1 \in T_{\alpha_0}$ and V_{x_0}, V_{x_1} are nonmeager. Indeed, because T is Suslin and by the assumption on F, in each level there is an x such that V_x is nonmeager, but since F is $\varkappa$-complete and in T there are no $\varkappa$-branches there must exists two noncompatible elements of T, x, y such that V_x and V_y are nonmeager.

Now, let G_i and $\{E_n^i | \ n \in \omega\}$ for $i = 0, 1$ be as follows

(i) G_i is an open nonvoid set;

(ii) $V_{x_i} \supseteq G_i \setminus \bigcup \{E_n^i | \ n \in \omega\}$;

(iii) E_n^i is a closed meager set;

(iv) $\operatorname{diam} G_i < 1/2$.

By induction on the length of s, where s is a function from the natural numbers into 2, define $\alpha_{\text{length } s}$, x_s, G_s, $\{E_n^s | \ n \in \omega\}$ such that

(1) $\alpha_{\text{length } s} = \alpha > \alpha_m$ if $m < \text{length } s$;

(2) $x_s \in T_\alpha$, if $\exists_n s(n) \neq s'(n)$ then x_s is T-incomparable with x_s', and if $s' \supseteq s$ then $x_{s'T} > x_s$;

(3) G_s is an open set;

(4) $\operatorname{diam} G_s < \dfrac{1}{2^{\text{length } s}}$;

(5) E_n^s is a closed meager set;

(6) $V_{x_s} \supseteq G_s \setminus \bigcup \{E_n^s | \ n \in \omega\}$;

(7) $\operatorname{cl} G_{s'} \subseteq G_s \setminus \bigcup \{E_n^{s''} | \text{length } s'' \leqslant m \text{ and } n < m\}$ for $s' \supseteq s$.

Assume that for $m < \omega$, x_s's, G_s's and $\{E_n^s | \ n \in \omega\}$'s are defined.

Using exactly the same argument as in step 0, it is possible to find $\alpha^s > \alpha_m$ and incomparable x_{s0}^*, x_{s1}^* such that $V_{x^*_{s\frown i}} \cap G_s$ is nonmeager for $i = 0, 1$.

Let $\alpha = \sup\{\alpha^s | \text{ length } s \leqslant m\}$.

Now let $x_{s\frown i}|_T > x^*_{s\frown i}$ and $x_{s\frown i} \in T_\alpha$. The set G_{si} and $\{E_n^s | \ n \in \omega\}$ can be easily found, since $V_{si} \cap G_s$ has the Baire property and is nonmeager. Since X is a complete metric space $\bigcap \{G_s | \ s \in {}^\omega 2\}$ is a one-point set $\{y_s\}$. By the construction if $s \neq s'$ and $F_{\alpha_{s'}} \ni y_{s'}$, $y_s \in F_{\alpha_s}$ then $\alpha_s \neq \alpha_{s'}$.

Since $y_s \in V_{x_{s|n}}$ for all n then $\alpha_{sT} > x_{s|n}$ holds. It is possible to find α_s^* such that $a^*_{sT} > x_s|_n$ for all s and n, and such that $\alpha_s^* \in T_{\sup\{\alpha_m | m \in \omega\}}$. This means that the tree of type T_∞ can be embedded into T, a contradiction.

2.4. Remark. It can be observed that this argument works only under the assumption that the space is complete.

2.5. Remark. Assume $V = L$. Let $\varkappa$ be a regular cardinal and I a $\varkappa$-complete uniform ideal (i.e. $[\varkappa]^{<\varkappa} \subseteq I$) over $\varkappa$.

Then a completion of the algebra $P(\varkappa)/I$ is isomorphic to the algebra of regular open subsets of $B(\varkappa^+)$. This means that two arbitrary $\varkappa$-complete uniform ideals over $\varkappa$ are similar. Indeed, since $V = L$ is assumed, then I is not precipitous. Let $W_0 \geqslant W_1 \geqslant \ldots$ be a sequence which contradict to the precipitous.

Let $x \in I^+$. We claim that there is an $n \in \omega$ such that

$$|\{y | y \subseteq x \in I^+ \ \& \ y \in W_n\}| = \varkappa^+.$$

If it is not the case, then we can find a disjoint family

$$\mathscr{W}_k = \{y | y \cap x \in I^+ \ \& \ y \cup t \in W_n \text{ for some } t \in I\}$$

such that $x \setminus \bigcup \mathscr{W}_n \in I$ and if $n > m$ then $y \supseteq y_1$ or $y \cap y_1 = \emptyset$ for $y \in \mathscr{W}_m$, $y_1 \in \mathscr{W}_n$.

Of course $x \cap \bigcap \{\bigcup \mathscr{W}_n | \ n \in \omega\} \neq \emptyset$. Hence, there is a sequence $w_n \in \mathscr{W}_n$ such that $\bigcap \{w_n | \ n \in \omega\} \neq \emptyset$, a contradiction with the assumption on I.

Because $2^\varkappa = \varkappa^+$, then by induction using the fact pointed out above we can construct a dense subset of size $\varkappa^+$ in $P(\varkappa)/I$ which looks like a canonical base for $B(\varkappa^+)$.

§ 3. The case of incomplete metric spaces. Let I be an ideal over cardinal $\varkappa$. Consider the set $X(I) = \{x \in {}^{\omega}(I^+) | \bigcap \{x(n)| \ n \in \omega\} \neq \emptyset$ and

$$\forall_{n\in\omega} \bigcap \{x(m)| \ m<n\} \in I^+\}.$$

The set $X(I)$ is considered as a subset of a complete metric space $(I^+)^{\omega}$, where the set I^+ is equipped with the discrete topology.

3.1. Proposition. *$X(I)$ is a Baire space iff I is a precipitous ideal.*

Proof. Assume that I is not precipitous. Then by 1.3 there are families (maximal I-partitions) W_n for $n \in \omega$, such that $W_n \geqslant W_{n+1}$ for any sequence $X_0 \supseteq X_1 \supseteq \ldots$ and such that $X_n \in W_n$ for any $n \in \omega$, then $\bigcap \{X_n| \ n \in \omega\} = \emptyset$.

Let $U_m = \{x|x$ is a function, $\operatorname{dom} x = n \geqslant 1$ and $\forall_{k<n} x(k) \in I^+$ and $\bigcap \{x(k)|k<n\} \in I^+$ and $x(n-1) \in W_m\}$. Since W_m is a maximal I-partition $\bigcup U_n$ is dense and open and $\bigcap \{\bigcup U_n| \ n \in \omega\} = \emptyset$.

Let us assume that $X(I)$ is not the Baire space; that is, there are open dense U_n's such that $\bigcap \{U_n| \ n \in \omega\} = \emptyset$. Let

$$\mathscr{U}_m = \{x|x \text{ is function } \operatorname{dom} x = n \geqslant 1, \ \forall_{k<n} x(k) \in I^+ \text{ and } \bigcap \{x(k)| \ k<n\} \in I^+, \ x \subseteq U_n\}.$$

$\bigcup \mathscr{U}_n$ is dense and open and $\bigcup \mathscr{U}_n \subseteq U_n$. The sequence $W_0 \geqslant W_1 \geqslant \ldots$ of I-partitions of $\varkappa$ can be defined by induction in such way that

$$\text{iff } y \in W_m \text{ then there is } x \in \mathscr{U}_m, \quad y \subseteq \bigcap \{x(k)| \ k \in \operatorname{dom} x\}.$$

Of course for any sequence

$$y_0 \supseteq y_1 \supseteq \ldots$$

such that $y_n \in W_n$ the set $\bigcap \{y_n| \ n \in \omega\}$ is empty.

Remark. R. Pol observed that Proposition 3.1 can be proved using the game theoretic characterization of precipitous ideals.

3.2. Theorem. *Let I be a precipitous ideal over some regular cardinal. Then there is decomposition $\mathscr{F}$ of the space $X(I)$ into meager sets such that if $\mathscr{F}' \subset \mathscr{F}$ then $\bigcup \mathscr{F}'$ has the Baire property. In this case $X(I)$ is a Baire space.*

Proof. $X(I)$ is a Baire space by Proposition 3.2.

Now, for each $\alpha \in \varkappa$ let

$$F_\alpha = \{x \in X(I)| \ \min \cap \{x(n)| \ n \in \omega\} = \alpha\}.$$

It is not difficult to verify that F_α is a meager set in $X(I)$ and $\{F_\alpha| \ \alpha < \varkappa\}$ form a partition of $X(I)$.

Let $A \in I^+$ then

$$\bigcup \{F_\alpha| \ \alpha \in A\} \supseteq \{x \in X(I)| \ \exists_{n\in\omega} \exists_{B \subseteq A} (B \in I^+)((x(n) = B))\} = K.$$

The set K is open. It is sufficient to have that $P = \bigcup \{F_\alpha| \ \alpha \in A\}$, K is meager in $X(I)$. Assume, *a contrario*, that P is not meager. Then there is an $n \in \omega$ and a function $s\colon n \to I^+$ such that $C = \bigcap \{s(m)| \ m<n\} \in I^+$ and P is nonmeager in s. But $C \cap A \in I$. This means that P is not nonmeager in s. Hence P is meager and $\{F_\alpha| \ \alpha \in A\}$ has the Baire property.

Theorem 3.3. *If* ZFC + *there is a Baire space X, and a decomposition $\mathscr{F}$ of X into meager sets such that for any $\mathscr{F}' \subseteq \mathscr{F}$, the set $\bigcup \mathscr{F}'$ has the Baire property, is consistent, then* ZFC + *there is a measurable cardinal is consistent as well.*

Proof. Let $\mathscr{F} = \{F_\alpha| \ a<\varkappa\}$ be a decomposition of x such that

$$\forall_{A \subseteq \varkappa} \bigcup \{F_\alpha| \ \alpha \in A\} \text{ has the Baire property}$$

and such that an ideal $\nabla = \{A| \bigcup \{F_\alpha| \ \alpha \in A\}$ is meager$\}$ is $\varkappa$-complete and uniform. Then $\varkappa$ is regular cardinal.

Now let $\mathrm{FN}(V)$ denote the class of all functions $\mathrm{FN}(V) = \{f \in {}^X V| \exists_{U_f}$ family of open disjoint sets, $\bigcup U_f$ is dense in X, and $\forall_{\alpha\in\varkappa} \forall_{U \in U_f} f$ is constant on $U \cap F_\alpha\}$ where V denotes the universe. It is very easy to check that if $f, g \in \mathrm{FN}(V)$, the sets

$$\{x|f(x) = g(x)\} \quad \text{and} \quad \{x|f(x) \in g(x)\}$$

have the Baire property. To see that, define $U_f \wedge U_g = \{U \cap V|U \in U_f, V \in U_g\}$. If $W \in U_f \wedge U_g$, then $\{x \in W|f(x) \overset{\in}{=} g(x)\} = \bigcup \{F_\alpha \cap W| \ \alpha \in A\}$ for some A, hence $\{x \in W|f(x) \overset{\in}{=} g(x)\}$ has the Baire property. By the Banach localization theorem [2], the set $\{x|f(x) \overset{\in}{=} g(x)\}$ has the Baire property.

Let B denote the Boolean algebra of regular open subsets of X and G-generic ultrafilter over B. Let us consider the "generic ultrapower" $\mathrm{FN}(V)/G$ (this is actually a limit ultrapower in Keisler sense (cf. sec. 6.4 in [1])).

The generic ultrapower is a model of ZFC. (We have a fundamental theorem in this form

$$\mathrm{FN}(V)/G \models \varphi([f_1] \ldots [f_n]) \quad \text{iff} \quad \{x \in X| \models \varphi(f_1(x) \ldots f_n(x))\} \in G$$

where $f_1 \ldots f_n \in \mathrm{FN}(V)$). We have a natural embedding $j_G\colon V \to \mathrm{FN}(V)/G$ $j_G(x) = [c_x]$ where $c_x\colon X \to V$ such that $\forall_{t\in X} c_x(t) = x$ $\mathrm{FN}(V)/G$ is well founded: if $S \Vdash f_1 \ni f_2 \ni \ldots \ni f_n \ni \ldots$ then $\{x|f_n(x) \ni f_{n+1}(x)\}$ is comeager in S and by Baire category theorem $\exists_{x_0 \in S} f_0(x_0) \ni \ldots \ni f_n(x_0) \ni \ldots$

Of course $j_G(\varkappa) > \varkappa$.

Lemma. *$\varkappa$ is a measurable cardinal in some transitive model of* ZFC.

From this lemma the theorem follows immediately.

The proof of it is almost the same as the proof of Theorem 86 a in [8] so we give only a sketch of it (this technique is due to Solovay).

Proof of the lemma. Let K be a class of strong limit cardinals $\nu > 2^{\varkappa}$ such that $\operatorname{cf} \nu > \varkappa$ let $\gamma_0 < \gamma_1 < \ldots < \gamma_n < \ldots$ $n \in \omega$ be elements of K such that $|\gamma_n \cap K| = \gamma_n$ for all $n \in \omega$. Let $A = \{\gamma_n| \ n \in \omega\}$ and $\lambda = \sup A$.

Sublemma (compare with Lemma 35.12 in [8]). There exists an $L[A]$-ultrafilter W over $\varkappa$ such that W is nonprincipial $L[A]$-normal and $L[A]$-$\varkappa$-complete, iterable and every iterated ultrapower $\mathrm{FN}^{(\alpha)}(L[A])/W$ is well founded.

Proof. There is a γ and $S \in G$ such that

$$S \Vdash d = \check{\gamma} \quad \text{in } \mathrm{FN}(V)/G$$

where d: $X \to$ Ordinals, such that $d(x) = \alpha$ iff $x \in F_\alpha$. Let

$$U = \{Y \in P(n) \cap L[A] | S \cap \bigcup \{F_\alpha | \ \alpha \in Y\} \text{ is nonmeager}\}.$$

Of course U is $L[A]$-ultrafilter. Kunen's argument shows that for any $Y \in L[A] \cap P(\varkappa)$ there is a finite set $E \subseteq \varkappa \cup K$ and a formula φ such that

$$Y = \{\xi \in \varkappa | L[A] \vDash \varphi(\xi, E, A)\}.$$

By the definition of γ we have $\bigcup \{F_\alpha | \ \alpha \in Y\} \cap S$ is nonmeager set then

$$\bigcup \{F_\alpha | \ \alpha \in Y\} \cap S \Vdash \check{\gamma} \in j_G(\check{Y})$$

and by the fact that $j_G(A) = A$ and $j_G(E) = E$ there is

$$\bigcup \{F_\alpha | \ \alpha \in Y\} \cap S \Vdash L[A] \vDash \varphi(\gamma, E, A)$$

where the forced formula is about V, and thus true, hence

$$S \Vdash \gamma \in j_G(Y)$$

which means that $S \Vdash \bigcup \{F_\alpha | \ \alpha \in Y\} \in G$ Let f: $u \to \varkappa$ which represents $\varkappa$ (f is constant on F_α's) in fact we can assume that f is defined on $\varkappa$. Let $W = f_*(\nabla)$. (∇ denotes the dual filter to Δ.) Since ∇ is $\varkappa$-complete W is $L[A]$-$\varkappa$-complete. From this point the proof is exactly the same as the proof of Lemma 35.11 and Theorem 86 in [8]. Now similar techniques we will prove:

THEOREM 3.4. *If* ZFC + $\exists$ *measurable cardinal is consistent then the following is consistent with* ZFC: *There is a metric Baire noncomplete space X and a partition $\mathcal{F}$ of X into meager sets such that for any $\mathcal{F}' \subseteq \mathcal{F}$, the set $\bigcup \mathcal{F}'$ has the Baire property. But there does not exist a complete metric space with the same decomposition property.*

Proof. Let us work in $L[D] = V$ where D is a normal $\varkappa$-complete ultrafilter over $\varkappa$. By Kunen's theorem $\varkappa$ is the only measure cardinal in V (see [6]). Now let $N = V[G_1][G_2]$ where N is obtained by Levy collapsing forcing which collapses $\varkappa$ to ω_1 and later adding ω_2 Cohen reals. By [8] and [9] ω_1 carries the precipitous ideal I over ω_1.

Of course by Theorem 3.2 the space $X(I)$ has the required decomposition. Let us assume that in N there is a complete metric space X with similar decomposition $\mathcal{F}$. Then by Theorem 1.2, $\varkappa = |\mathcal{F}| \geqslant \omega_2^N$, (we assume that $\varkappa$ is the smallest cardinality of family with this property) we can repeat twice the arguments of Theorem 3.3. In fact we can get a transitive model such that $\varkappa$ is a measurable cardinal and later extend it to M such that $\varkappa$ and ω_1^N are measurable, of course $\varkappa > \omega_1^N$ and such that there is an elementary embedding i: $M \to L[D]$, a contradiction, because $L[D]$ has only one measurable cardinal.

§ 4. Final conclusions and remarks.

4.1. Proof of the theorem. Theorem is an immediately consequence of Theorems 1.1, 2.1, 3.2 and 3.3.

4.2. Let $A \subseteq \varkappa$, where $\varkappa$ is a regular cardinal and let

$$A^* = \{x \in B(\varkappa) | \ \sup\{x(n) | \ n \in \omega\} \in A\}.$$

It is known [5] that if A is a stationary subset of $C_\omega(\varkappa) = \{\alpha \in \varkappa | \text{ cf. } \alpha = \omega\}$, then A^* is Baire. R. Pol has strengthened this result in the following way: If $B \subseteq B(\varkappa)$ and if A is stationary such that for each $\alpha \in A$ the set $\{\alpha\}^* \cap B$ is nonmeager in $\{\alpha\}^*$ then B is a Baire space. He proved also the similar theorem for Baire metric spaces (not necessary complete). Now we are ready to prove the Proposition 4.3.

4.3. PROPOSITION. *If the density of X is inaccessible and X is a Baire metric space then for each decomposition $\mathcal{F}$ of X into meager sets there is $\mathcal{F}' \subset \mathcal{F}$ such that $\mathcal{F}'$ does not have the Baire property.*

It is sufficient to consider a decomposition of $B(\varkappa)$ into $\varkappa$ meager sets.

In this case let

$$i\colon C_\omega(\varkappa) \to \varkappa \text{ be a function defined as follows}$$

$i(\beta) = \min\{\alpha | (\exists_{\mathcal{F}' \subseteq \mathcal{F}} \forall_{F \in \mathcal{F}'} (F \cap \alpha^* \neq \emptyset) \text{ and } \bigcup \mathcal{F}' \text{ is nonmeager in } \{\beta\}^*\}$.

If i is regressive on some stationary set $A \subseteq C_\omega(\varkappa)$ then there is $\eta \in \varkappa$ and stationary $B \subseteq A$ such that $i(B) = \{\eta\}$. Because $|\eta| < \varkappa$, there is $\mathcal{F}' \subseteq \mathcal{F}$ such that $|\mathcal{F}'| < \varkappa$ and $\bigcup \mathcal{F}'$ is a nonmeager subset of $B(\varkappa)$.

There is $\mathcal{F}'' \subseteq \mathcal{F}'$ such that $\bigcup \mathcal{F}''$ does not have the Baire property.

Assume, that i is $\mathrm{NS}_\varkappa$-almost (nonstationary) identity on $C_\omega(\varkappa)$.

Let $A_0, A_1 \subseteq C_\omega(\varkappa)$ be the disjoint stationary subsets such that $i|_{A_i} = \mathrm{id}_{A_i}$ for $i = 0, 1$ and $A_0 \cup A_1 = A$.

If $\alpha \in A$ there is $\mathcal{F}_\alpha \subset \mathcal{F}$ such that $\bigcup \mathcal{F}_\alpha$ is nonmeager in $\{\alpha\}^*$. By assumption on i (and on A) the set $\mathcal{F}_\alpha^* = \mathcal{F}_\alpha \setminus \bigcup \{\mathcal{F}_\beta^* | \ \beta < \alpha\}$ covers a nonmeager subset of $\{\alpha\}^*$.

Let

$$\mathcal{F}^0 = \bigcup \{\mathcal{F}_\alpha^* | \ \alpha \in A_0\} \quad \text{and} \quad \mathcal{F}' = \bigcup \{\mathcal{F}_\beta^* | \ \beta \in A_1\}$$

then $\mathcal{F}^0$ and $\mathcal{F}'$ are every where nonmeager subsets of $B(\varkappa)$ (be R. Pol theorem). This means that $\bigcup \mathcal{F}'$ fails to have the Baire property.

4.4. COROLLARY. *Assume $V = L$.*

Let F: $X \to K^+(Y)$ be a lower Baire-measurable function, where X, Y are metric space, X a complete space and $K^+(Y)$ a compact nonvoid subset of Y. Then there is a Baire-measurable selector of F.

Proof. By [7] and the theorem.

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$P_\varkappa\lambda$ Partition relations

by

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Abstract. We study the partition relations $X \to (I^+)^n$, $X \to (uhf)^n$, and $X \to (uhf, I^+)^n$ where $X \subseteq P_\varkappa\lambda$, $n \geqslant 1$, I is a proper, nonprincipal $\varkappa$-complete ideal on $P_\varkappa\lambda$, and a *uhf* is an unbounded homogeneous function (see 1.3, 2.1 below).

THEOREM. *If $\lambda^{<\varkappa} = \lambda$, then $\varkappa$ is λ-ineffable iff $X \to (NS^+_{\varkappa\lambda})^2$ holds for some $X \subseteq P_\varkappa\lambda$.* (4.2, 4.3).

THEOREM. *If $X \to (SNS^+_{\varkappa\lambda})^2$ holds for some $X \subseteq P_\varkappa\lambda$, then $\varkappa$ is almost λ-ineffable.* (1.7).

THEOREM. *If $\lambda^{<\varkappa} = \lambda$ and $\varkappa$ is almost λ-ineffable, then $X \to (I^+_{\varkappa\lambda})^2$ holds for every $X \in NAIn^+_{\varkappa\lambda}$.* (4.2).

THEOREM. *If $\lambda^{<\varkappa} = \lambda$, then $\varkappa$ is mildly λ-ineffable iff $X \to (uhf)^n$ holds for every $X \in I^+_{\varkappa\lambda}$ and $n \geqslant 2$.* (2.4)

THEOREM. *If $\lambda^{<\varkappa} = \lambda$ and $\varkappa$ has the λ-Shelah property, then $X \to (uhf, NSh^+_{\varkappa\lambda})^2$ holds for every $X \in NSh^+_{\varkappa\lambda}$.* (5.4).

All of the ideal-theoretic notation is explained in 0.0 and 0.4.

0. Introduction

0.0. Notation and basic facts. Unless we specify otherwise, $\varkappa$ denotes an uncountable regular cardinal and λ a cardinal $\geqslant \varkappa$. For any such pair, $P_\varkappa\lambda$ denotes the set $\{x \subseteq \lambda:\ |x| < \varkappa\}$.

The basic combinatorial notions are defined here for $P_\varkappa\lambda$ as in Jech [12]. For any $x \in P_\varkappa\lambda$, $\hat{x}$ denotes the set $\{y \in P_\varkappa\lambda:\ x \subseteq y\}$. $X \subseteq P_\varkappa\lambda$ is said to be *unbounded* iff $(\forall x \in P_\varkappa\lambda)(X \cap \hat{x} \neq 0)$, and $I_{\varkappa\lambda}$ denotes the *ideal of not unbounded subsets of* $P_\varkappa\lambda$. In the sequel, an "ideal on $P_\varkappa\lambda$" is always a "proper, nonprincipal, $\varkappa$-complete ideal on $P_\varkappa\lambda$ extending $I_{\varkappa\lambda}$" unless we specify otherwise. Further, for any ideal I on $P_\varkappa\lambda$, I^+ denotes the set $\{X \subseteq P_\varkappa\lambda:\ X \notin I\}$, and I^* the filter dual to I; $FSF_{\varkappa\lambda}$ denotes $I_{\varkappa\lambda}{}^*$

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