Contents of Volume 128, Number 1

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ISBN 83-01-07602-X ISSN 0016-2736

DRUKARNIA UNIWERSYTETU JAGIELLOŃSKIEGO W KRAKOWIE

FUNDAMENTA
MATHEMATICAE
128 (1987)

The multiple intersection property for path derivatives

by

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Abstract. A collection, $E = \{E_x\colon x \in R\}$, is a system of paths, if each set E_x has x as a point of accumulation. For such a system E the derivative, $F'_E(x)$, of a function F at a point x is just the usual derivative at x relative to the set E_x. The goal of this paper is the investigation of properties that F and its derivative F'_E must have under certain natural assumptions about the collection E. In particular, it is shown that most of the familiar properties of approximate derivatives and approximately differentiable functions follow in this setting from three conditions on the collection E relating to the "thickness" of the sets E_x and the way in which the sets intersect.

The purpose of this paper is to extend to path derivatives a result previously established for approximate derivatives [2]. That result is the decomposition property of approximate derivatives. More precisely, we deal with two ideas (see [1], [2], and [3]):

(1) A function $f\colon R \to R$ is said to be *compositely differentiable* to another function $g\colon R \to R$ if there is a sequence of closed sets X_n such that $\bigcup_{n=1}^{\infty} X_n = R$ and for each n, the restriction of f to X_n differentiates to the restriction of g to X_n. (The function $g|X_n$ are called a *decomposition* of g.)

(2) A collection of sets $E = \{E_x\colon x \in R\}$ is called a *path system* if, for each x, E_x contains x and has x as a limit point. Then, relative to this system, $f\colon R \to R$ is said to have $g\colon R \to R$ as path derivative, if, for each x, the restriction of f to E_x, differentiates, at x, to $g(x)$.

In general, (1) is a more restrictive property than (2). For each x, merely pick a X_n containing and let $E_x = X_n$. Yet such a technique may fail to satisfy (2). This is because (1) does not require that x is a limit point of X_n. However, the minimal way to avoid this difficulty is to demand that, for each x, $g(x)$ is a derived number of f at x. Then (1) implies (2). This will be shown later.

Alternately, (2) will not imply (1) unless some additional condition is placed on the paths in the system. Similar situations were encountered in [1] and solved by demanding that E_x and E_y intersect each other in some prescribed fashion when x

and y are in a sense close. Further, it is known from [2] that each approximately differentiable function is compositely differentiable to its approximate derivative. Moreover, clearly each such pair satisfies (1) via a path system E when for each x E_x has density 1 at x. A study of the proof used in [2] with the viewpoint of finding a suitable intersection condition leads to the following definition.

A system of paths, $E = \{E_x\colon x \in R\}$, has the *multiple intersection property* if there is a positive function δ, defined on R, such that for each triple of numbers, $x_1 \leqslant x_2 \leqslant x_3$, not all equal, if

$$[x_1, x_3] \subset \bigcap_{i=1}^{3} \big(x_i - \delta(x_i), x_i + \delta(x_i)\big),$$

then

$$\bigcap_{i=1}^{3} E_{x_i} \cap [x_j, x_{j+1}] \neq \emptyset \quad \text{for } j = 1, 2.$$

In the case where $x_j = x_{j+1}$, $j = 1$ or 2, this represents the basic intersection property of [1]

THEOREM 1. *If $f\colon R \to R$ has $g\colon R \to R$ as a path derivative relative to a system $E = \{E_x\colon x \in R\}$ where E has the multiple intersection property, then f is compositely differentiable to g.*

Proof. Let $A_n = \{x\colon \delta(x) > n^{-1}$ and $\left|\dfrac{f(y)-f(x)}{y-x}\right| \leqslant n$ when $y \in E_x$ and $|x-y| \leqslant n^{-1}\}$. Then $\bigcup_{n=1}^{\infty} A_n = R$; and using the basic intersection property as in Theorem 4.5 of [1] it can be shown that if $x_1, x_2 \in A_n$ and $|x_1 - x_2| \leqslant n^{-1}$, then $|f(x_1) - f(x_2)| \leqslant n|x_1 - x_2|$. Further, the function f is continuous on the closure of A_n, which will be denoted A_n^*. Again, if $x_1, x_2 \in A_n^*$ and $|x_1 - x_2| \leqslant n^{-1}$, then $|f(x_1) - f(x_2)| \leqslant n|x_1 - x_2|$.

Remark 1. Up to here the proof parallels that of [2]. If the parallel was to continue, we would next show that f restricted to A_n^* differentiates to the restriction of g to A_n^*. However, the difficulty lies in the fact that A_n^* can have isolated points. It will be indicated later why isolated points pose a problem.

Let $H_n = \{x\colon x$ is an isolated point of $A_n^*\} = \{x\colon x$ is an isolated point of $A_n\}$. For each n, $A_n^* \setminus H_n$ is a closed set and H_n is countable. Let $C = \bigcup_{n=1}^{\infty} H_n$. Then C is countable. If C is nonempty, we arrange its terms in to a sequence, possibly finite, which we denote by q_n.

Let $X_n = (A_n^* \setminus H_n) \cup \{q_n\}$. We claim f will satisfy (1) with these X_n. Let n be fixed. We note that differentiability relative to X_n need only be proved at points of $A_n^* \setminus H_n$. This is because if q_n does not belong to $A_n^* \setminus H_n$, it will be an isolated point of X_n and differentiability is taken to hold vacuously. Let $x_k \in A_n^* \setminus H_n$, $k = 0, 1, 2, \ldots$, converge to x_0. We may assume without loss of generality that

(i) $x_0 = 0 = f(0)$,
(ii) $x_k > x_{k+1} > 0$,
(iii) $0 < x_k < \min(\delta(0), n^{-1})$, $k = 1, 2, 3, \ldots$

Let k be fixed. Since $x_k \in A_n^*$ but not H_n, we can select two points $y_k < z_k$ such that

(a) $0 < y_k < z_k < n^{-1}$,
(b) $x_k \notin [y_k, z_k]$,
(c) $y_k, z_k \in A_n$,
(d) $\max(|x_k - y_k|, |x_k - z_k|, |y_k - z_k|) < \min(\delta(x_k), \delta(0), (k+1)^{-1} \cdot x_k)$.

With these conditions the multiple intersection property can be applied to the triple $0, y_k, z_k$. Therefore

$$E_0 \cap E_{y_k} \cap E_{z_k} \cap [y_k, z_k] \neq \emptyset.$$

Let w_k belong to this intersection. We will assume $x_k < y_k < w_k < z_k$; alternate cases are handled similarly. Then it follows that

$$\frac{w_k}{x_k} \to 1, \quad \frac{y_k}{x_k} \to 1, \quad w_k \to 0, \quad \frac{f(w_k)}{w_k} \to g(0), \tag{1}$$

and

$$\left|\frac{f(x_k) - f(y_k)}{x_k - y_k}\right| \leqslant n, \quad \left|\frac{f(w_k) - f(y_k)}{w_k - y_k}\right| \leqslant n \tag{2}$$

hold.

Then by (b) and (2) $|x_k - y_k| + |y_k - w_k| = |x_k - w_k|$, and hence

$$\left|\frac{f(x_k) - f(w_k)}{x_k - w_k}\right| \leqslant n. \tag{3}$$

Now, (1) and (3) together yield

$$\frac{f(x_k)}{x_k} = \left[\frac{f(x_k) - f(w_k)}{x_k - w_k} \cdot \left[1 - \frac{w_k}{x_k}\right] + \frac{w_k}{x_k}\frac{f(w_k)}{w_k}\right] \to g(0),$$

which is the desired result.

Remark 2. For points in H_n it would have been impossible to guarantee the existence of points y_k and z_k satisfying (a) through (d). This is the difficulty mentioned in Remark 1. To complete the circle between the multiple intersection property and composite differentiability we will need the following proposition. (In the course of the proof we will need to introduce a lemma.)

PROPOSITION 1. *If $f\colon R \to R$ is compositely differentiable to $g\colon R \to R$ and, for each x, $g(x)$ is a derived number of f at x, then there is a sequence of perfect sets P_n such that $\bigcup_{n=1}^{\infty} P_n = R$ and f is compositely differentiable to g relative to the sequence P_n. That is, then (1) implies (2).*

Proof. Let X_n be the original sequence of closed sets such that $\bigcup_{n=1}^{\infty} X_n = R$ and for which the restriction of f to X_n differentiates to the restriction of g to X_n. As in Theorem 1, it is the isolated points of X_n which cause difficulty, and they are handled one at a time. Express each X_n as $Q_n \cup C_n$ where Q_n is perfect and C_n is countable. Then let $C = \bigcup_{n=1}^{\infty} C_n$. Let the elements of C be arranged in to a sequence x_n, possibly finite, listing each element only once. Let n be fixed and consider $S_n = Q_n \cup \{x_n\}$. If x_n belongs to Q_n, we have that $S_n \subset X_n$ is a perfect set and f differentiates relative to S_n to g. Then we take $P_n = S_n$. If x_n does not belong to Q_n, we must find a path E_{x_n} which is a perfect set containing x_n and such that the restriction of f to E_{x_n} is differentiable to g on E_{x_n}, not only at x_n. Then we will take $P_n = Q_n \cup E_{x_n}$. We formalize this as a lemma.

LEMMA. *If $f\colon R \to R$ and $g\colon R \to R$ satisfy the hypotheses of the proposition then for each x there is a perfect path E_x such that f is differentiable relative to E_x to g, not only at x.*

Proof. Let V be the union of open intervals I such that each point x in I satisfies the conclusion of the lemma. Any interval which is in the interior of one of the original X_n would be in V. Therefore by the Baire category theorem we are assured that V is a dense open subset of R. If we show $V = R$ we are finished. Assume $W = R \setminus V \neq \varnothing$. We first show that W is a perfect set. Assume instead that W has an isolated point y.

Since $g(y)$ is a derived number of f at y, we can find a monotone sequence of points y_k converging to y such that $\dfrac{f(y_k)-f(y)}{y_k-y} \to g(y)$. Assume without loss of generality that y_k is a decreasing sequence, and that y_k belong to V for all k. For each k, choose $\delta_k > 0$ such that the sequence of intervals $[y_k-\delta_k, y_k+\delta]$ are pairwise disjoint and contained in V. Since y_k belongs to V, we can find a perfect set E_k such that

(i) $y_k \in E_k$,

(ii) f restricted to E_k differentiates to g in E_k,

(iii) $E_k \subset [y_k-\delta_k, y_3+\delta_k]$,

(iv) $\left|\dfrac{f(z)-f(y)}{z-y} - \dfrac{f(y_k)-f(y)}{y_k-y}\right| < |y_k-y|$.

Then take $E_y = \{y\} \cup \bigcup_{k=1}^{\infty} E_k$. This E_y is perfect and f restricted to E_y differentiates to g on E_y. That is, y belongs to V. Therefore W is a nonempty nowhere perfect set. However, this also is a contradiction. Another application of the Baire category theorem with the sequence $W \cap X_n$ assures there is an open interval (a, b) and an integer N such that $(a, b) \cap W$ is nonempty and contained in X_N. Since W is nowhere dense, we may assume that $(a, b) \cap W$ is perfect. Further, since this set is contained in X_N, each point in $(a, b) \cap W$ satisfies the conclusion of the lemma. But this implies that $(a, b) \subset V$, which is a contradiction. Thus $V = R$, which finishes the proof of the lemma. It also ends the proof of the proposition.

Now we can establish a converse to Theorem 1.

THEOREM 2. *If $f\colon R \to R$ is compositely differentiable to $g\colon R \to R$ and, for each x, $g(x)$ is a derived number of f at x, then there is a path system $E = \{E_x\colon x \in R\}$ having the multiple intersection property such that g is the path derivative of f relative to E.*

Proof. We may assume the existence of perfect sets P_n, $n = 1, 2, \ldots$, such that

(u) $\varnothing = P_0 \neq P_1 \subset P_2 \subset \ldots \subset P_n \subset \ldots$,

(v) $\bigcup_{n=1} P_n = R$,

(w) f restricted to P_n differentiates to g on P_n for all n.

Let $d(x, Y)$ denote the distance between the point x and the set Y and define $d(x, \varnothing) = 1$ for all x. For $x \in R$, let $n(x) = \inf\{n\colon x \in P_n\}$. Note $n(x) \geqslant 1$. Let $E_x = P_{n(x)}$. It is clear that all that remains is to show that the system $E = \{E_x\colon x \in R\}$ has the multiple intersection property by defining an appropriate function δ. If x is a two-sided limit point of $E_{n(x)}$, let $\delta(x) = \frac{1}{2} d(x, P_{n(x)-1})$. If x is a one-sided limit point of $E_{n(x)}$, let I be the interval contiguous to $P_{n(x)}$ having x as endpoint. Let $\delta(x) = \min(\delta_1(x), \frac{1}{2}$ length of $I)$ where $\delta_1(x) = \frac{1}{2} d(x, P_{n(x)-1})$.

We claim that if $n(a) \neq n(b)$ then $|b-a| > \min(\delta(a), \delta(b))$. For this, assume without loss of generality that $a < b$ and $n(a) < n(b)$. This means that $a \in P_{n(b)-1}$ but $b \notin P_{n(b)-1}$. Hence $|a-b| \geqslant d(b, P_{n(b)-1}) > \delta_1(b) \geqslant \delta(b) \geqslant \min(\delta(a), \delta(b))$. From the definition of $\delta_1(x)$ and $\delta(x)$, the only time we would apply the condition of the multiple intersection property to a triple $x \leqslant y \leqslant z$ is when $n(x) = n(y) = n(z)$. Then $E_x = E_y = E_z = P_{n(x)}$. Further, we must have $(x, y) \cap P_{n(x)} \neq \varnothing \neq (y, z) \cap P_{n(x)}$, so the system E has the multiple intersection property relative to $\delta(x)$.

We now finish with an obvious result about the thickness of such path systems.

PROPOSITION 2. *If $E = \{E_x\colon x \in R\}$ is a path system having the multiple intersection property relative to a function $\delta\colon R \to R$, then there is a dense open set U such that for each x in U there is a neighborhood of x in which E_x is dense.*

Proof. Let $A_n = \{x\colon \delta(x) > n^{-1}\}$. Then $\bigcup_{n=1}^{\infty} A_n = R$. Let n be fixed and suppose (a, b) is an interval in which A_n is dense. Let x_0 belong to (a, b). Let $\delta_1(x_0) = \min(|x_0-a|, |x_0-b|, \delta(x_0), n^{-1})$ and consider $(x_0-\delta_1(x_0), x_0+\delta_1(x_0))$. We claim E_{x_0} is dense in this neighborhood of x_0. Let y and z be any two points of A_n in this neighborhood. The multiple intersection property can be applied to the triple x_0, y, z, so that E_{x_0} has points in $[y, z]$; so E_{x_0} is dense in $(x_0-\delta_1(x_0), x_0+\delta_1(x_0))$. Now, the union of intervals (a, b) such that there is an n with A_n dense in (a, b) forms the desired dense open set U.

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Received 30 January 1985;
in revised form 10 January 1986

FUNDAMENTA
MATHEMATICAE
128 (1987)

Fix-finite and fixed point free approximations of symmetric product maps

by

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Abstract. Let X be a locally finite simplicial complex with the weak topology. It is shown that every symmetric product map $f\colon X \to X^n/G$ is homotopic to a symmetric product map $f'\colon X \to X^n/G$ so that all fixed points of f' are isolated and so that the fixed point set of f' is finite (countable) if X is a finite (countable) complex. In the former case f' can be chosen so that, for *every* $m \geqslant 1$, its set of periodic points of period $m \leqslant 1$ is finite as well. If X is a noncompact manifold, then f can be homotoped to a fixed point free symmetric product map.

1. Introduction. Let X be a topological space and X^n be the n-fold Cartesian product of X with the product topology. Any (proper or improper) subgroup G of the symmetric group S_n of all permutations of $\{1, 2, \ldots, n\}$ acts on X^n as a group of homeomorphisms by permuting its factors. Let X^n/G be the orbit space with the quotient topology induced by the quotient map $q\colon X^n \to X^n/G$. Then a map (i.e. a single-valued continuous function) $f\colon X \to X^n/G$ is called a *symmetric product map*, and a point $x \in X$ is called a *fixed point* of the symmetric product map f if $f(x) = q(z)$, where $z \in X^n$, implies that x is a coordinate of z. Fixed points of symmetric product maps have been studied e.g. by S. Kwasik [5], C. N. Maxwell [8], [9], S. Masih [6], [7], Nancy Rallis [11] and C. Vora [17], [18]. Periodic points of symmetric product maps (see the definition in § 3) have been considered by Nancy Rallis [12].

In this paper we extend to symmetric product maps the Hopf approximation theorem which states that every selfmap of a compact polyhedron is homotopic to a fix-finite one (see e.g. [2], Ch. VIII A, Theorem 2, p. 118), and also prove related results for noncompact polyhedra. In Theorem 1 we show that every symmetric product map $f\colon X \to X^n/G$ is homotopic to a symmetric product map $f'\colon X \to X^n/G$, which has a finite (countable) fixed point set, if X is a finite (countable) polyhedron. Theorem 2 states that if X is finite, then f' can be chosen so that it has, for *every* $m \geqslant 1$, at most finitely many points of period $\leqslant m$. Finally, Theorem 1 is used to show, in Theorem 3, that if X is a noncompact manifold, then f is homotopic to a fixed point free symmetric product map.

As $X^n/G = X$ if $n = 1$, these theorems extend the Hopf approximation theorem, an approximation theorem for periodic points by Boju Jiang [4], p. 62, and, for