On a problem of S. Ulam concerning Cartesian squares of 2-dimensional polyhedra

by

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Abstract. This paper contains the proof of the following theorem: if $K$ and $L$ are compact connected 2-polyhedra and their Cartesian squares $K \times K$ and $L \times L$ are homeomorphic, then $K$ and $L$ are homeomorphic.

1. Introduction. In 1933 the following problem was posed by S. Ulam [17].

"Assume that $A$ and $B$ are topological spaces and $A^2 = A \times A$ and $B^2 = B \times B$ are homeomorphic. Is it true that $A$ and $B$ are homeomorphic?"

In general this problem has the negative answer. Let $Q$ denote the Hilbert cube, $A_i$ the disjoint union of the set $\{1, 2, ..., k\} \times Q$ and the set of natural numbers $N$ and $A_0$ the disjoint union of the sets $N \times Q$ and $N$. The spaces $A_i, A_j$ are not homeomorphic for $i \neq j, i, j = 0, 1, 2, ..., but A_0 \cong A_i$ for $i = 0, 1, 2, ...$, hence $A_i \cong A_j$ for $i, j = 0, 1, 2, ...$. The problem does not have a trivial answer when the spaces $A$ and $B$ are compact or connected. But in this case the answer is negative, too. In 1947 R. H. Fox [7] gave an example of two non-homeomorphic compact 4-manifolds the Cartesian squares of which are homeomorphic. In 1960 J. Glimm [9] gave an example with open 3-manifolds. Other examples can be found in: D. R. McMillan Jr. [14], K. W. Kwun [12], K. W. Kwun and F. Raymond [13], A. J. Boals [1], Z. Čerin [5], H. Toruńczyk [16].

However, the problem considered has the positive answer for 2-manifolds. This simple fact was proved in [7].

The more general problem of the uniqueness of the decomposition of finite-dimensional compacta into Cartesian product was considered by several authors [2], [15], [8], [4]. It was proved that this problem has the positive answer if the factors are 1-dimensional locally connected continua. If the factors are 2-polyhedra or bounded 2-manifolds, then the uniqueness of the decomposition does not hold.

We prove that the Ulam problem has the positive answer for compact connected 2-dimensional polyhedra, that is:

**Theorem A.** If $K$ and $L$ are compact connected 2-dimensional polyhedra such that $K^2$ and $L^2$ are homeomorphic, then $K$ and $L$ are homeomorphic.
A space $A$ is said to be a Cartesian root of $X$ if $X$ and $A \times A$ are homeomorphic. Thus we can formulate a somewhat more general version of Theorem A:

**Theorem B.** A compact connected 4-dimensional polyhedron $X$ has at most one Cartesian root.

This version is in fact equivalent to Theorem A.

**Proof.** We assume that $X \approx K^2 \approx L^3$. Since either $\dim K^2 = 2 \dim K$ or $\dim K^2 = 2 \dim K - 1$ (see [11], p. 18), and $\dim K \geq 2$, we have $\dim K = 2$. A. Kosinski proved that any 2-dimensional factor of a polyhedron is a polyhedron [10]. Thus the spaces $K$ and $L$ are 2-polyhedra and Theorem A implies Theorem B. The fact that Theorem B implies Theorem A is obvious.

In the case $\dim X = 2n (n \geq 3)$ Theorem B is not true. Let $A$ denote the Cartesian product of $n$ circles $S^1 (n \geq 3)$ and let $T$ be the wild arc of Blankinship [3] in a cell contained in $A$. We denote the space $A/\sim$ by $B$. The spaces $A$ and $B$ are not homeomorphic, but the Cartesian squares $A^2$ and $B^2$ are homeomorphic. The proof of this fact is analogous to the proof of Kwun's theorem [12].

An example of compact non-homeomorphic 3-polyhedra $A$ and $B$ such that $A^2$ and $B^2$ are homeomorphic is not known. If there exists a 3-dimensional Poincaré fake cell (a 3-dimensional compact contractible manifold not homeomorphic to $I^n$ with the boundary equal to $S^{n-1}$), then — since its Cartesion square is a 6-cell — such an example would exist.

2. Outline of proof of Theorem A. First, we define some subsets of non-Euclidean points of a polyhedron $P$.

**Definition 2.1.** If $P$ is a $k$-dimensional polyhedron, then we define inductively the sets $n_i P$ for $i = 0, 1, \ldots, k$:

(i) $n_0 P = P$

(ii) $n_i P$ denotes the subset of $n_{i-1} P$ consisting of the points which have no neighborhood homeomorphic to $R^{k-i+1} \times R^i$ or $R^{k-i+1}$ in the set $n_{i-1} P$.

We denote the set $n_k P$ by $\mathcal{N}$.

**Remark.** It is easy to see that every set $n_i P$ is a polyhedron and $\dim n_i P = k-i$.

Now we present

**Outline of proof of Theorem A:** The proof of Theorem A is divided into three propositions. To prove these propositions we need some lemmas. It is assumed that $K$ and $L$ are compact connected 2-polyhedra such that $K^2 \approx L^3$.

In the first proposition (Prop. 3.1) we consider the case where certain isolated points are distinguished in the polyhedron $K$. In Lemma 3.1 we study the structure of the polyhedron $K^2$ and we obtain the formulas:

$$n_2(K^2) = \bigcup \{ n_i K \times n_q K : p+q = i, p, q \in \{0, 1, 2\} \}.$$

In Lemma 3.3 we prove, using the technical Lemma 3.2, that if $K$ has a local cut point then $K \approx L$. Next, using Lemmas 3.1-3.3 and Borsuk's theorem [2] on the uniqueness of the decomposition into Cartesian product of 1-polyhedra we prove that if the condition $K \approx L$ does not hold then:

(-) each component $X$ of the set $nK$ is either an arc or a simple closed curve.

We also prove that if $K \approx L$ then:

(++) for every $x \in nK$ there are $n \in N$ and a neighborhood $V$ of $x$ in $K$ such that $V \approx [0, 1] \times \{1, 2, \ldots, n\}$.

Let $F : K^2 \approx L^2$ be a fixed homeomorphism. If $K \approx L$ then we prove, using Lemma 3.2, that there exists a one-to-one correspondence $A \approx A'$ between the components of $K \times nK$ and $L \times nL$ such that $F(K \times nK) = L \times nL$ and $F(K \times A) = L \times A'$, or $F(K \times nK) = nL \times L$ and $F(K \times A) = A' \times L$.

For every component $A$ of the set $K \times nK$ we define some 2-manifold $M(A)$.

This manifold is homeomorphic to the set $A$ minus some open regular neighborhood of $nK$. The polyhedron $K$ is built up of the manifolds $M(A)$.

In the next proposition (4.2) we prove that if conditions (-) and (++) hold and $K \approx L$ then all components of the set $nK$ are arcs.

The polyhedron $K$ is the union of the manifolds $M(A)$ such that their intersections are the arcs lying in the boundaries of $M(A)$.

Let us notice that if $J$ is an arc which is a component of $nK$, then its endpoints are not distinguished by the stratification given in Definition 2.1. Let the endpoint $x_0$ have an open neighborhood homeomorphic to the set $T \times [0, 1)$ where $T = \{(t, 0) : t = 1, \ldots, n, \ t \in [0, 1)\}$ and let $x \in J$ have an open neighborhood homeomorphic to $T \times (0, 1)$. It is obvious that $T \times [0, 1] \times R^2 \approx T \times [0, 1] \times R^2 = T \times (0, 1) \times R^2$. So if the endpoint $x_0$ distinguished, the formula from Lemma 3.1 would not hold.

Now, we cannot use the methods similar to those used in the proof of the Proposition 3.1. We prove in Section 5 that if $A \approx A'$ is the one-to-one correspondence between the components of $K \times nK$ and $L \times nL$, then the manifolds $M(A)$ and $M(A')$ are homeomorphic. In the last section we prove that the manifolds $M(A)$ are stuck to the set $nK$ in $K$ on the same way as the manifolds $M(A')$ are stuck to the set $nL$ in $L$. So homeomorphisms between the manifolds $M(A)$ and $M(A')$ yield a homeomorphism $f : K \approx L$. This part of the proof is the most complicated one.

3. Investigation of the non-Euclidean part $nK$ of $K$. This section contains the proof of:

**Proposition 3.1.** If $K$ and $L$ are compact, connected 2-polyhedra, $F : K^2 \approx L^2$ is a homeomorphism and $K \approx L$, then:

(-) each component $X$ of $nK$ is either an arc or a simple closed curve,

(++) for each $x \in nK$ there are $n \in N$ and a neighborhood $V$ of $x$ in $K$ such that $V \approx [0, 1] \times \{1, 2, \ldots, n\}$,

(***$F(nK \times nK) = nL \times nL$ or $F(nK \times nK) = L \times nL$,

(****) either each component $A$ of the set $K \times nK$ there exists a component $A'$ of the set $L \times nL$ such that $F(A \times nK) = A' \times L$ or
for each component $A$ of the set $K \setminus nK$ there exists a component $A'$ of the set $L \setminus nL$ such that $F(A \times L) = L \times A'$.

First, we prove some lemmas.

**Lemma 3.1.** If $K$ is a $2$-polyhedron, then

$$n_i(K^2) = \sum_{p \in A, q \in B} p + q = i, p, q \in \{0, 1, 2\}$$

**Proof.** We consider the cases $i = 1, 2, 3, 4$.

(a) The case $i = 1$.

We should show that $x \in K^2$ has a Euclidean neighborhood if $x \in K \setminus ((nK \times nK) \cup (nK \times nK)) = (K \setminus nK) \times (K \setminus nK)$, i.e. $x$ belongs to the product of Euclidean parts of $K$. The proof is easy and the details are left to the reader.

(b) The case $i = 2$.

We have

$$[(K \times nK) \cup (nK \times K)] \cup [(nK \times nK) \cup (nK \times nK) \cup n_2(K \times K) = [K \times (nK \setminus nK) \cup (nK \setminus nK)] \cup [(nK \setminus nK) \times (nK \setminus nK)] \cup n_2(K \times K) \cup n_2(K \times K).$$

We will show that if the point $x$ belongs to the set $(K \times nK) \cup (nK \times nK) \cup (n_2(K \times K), x \in n_2(K^2))$, then $x \in n_2(K^2)$.

Let $x = (x_1, x_2)$. If $x_1 = (i = 1) or i = 2$ is an isolated point or $dim_{K^2} = 1$ for both points $x_1$ and $x_2$, then a sufficiently small neighborhood of $x$ in $K^2$ has dimension less than 3 and $x \in n_2(K^2)$.

Now we consider the remaining cases.

Let $(x_1, x_2) \in K \times nK$. If $x_2$ is isolated in $nK$ then it is easy to see that $x_2$ is a local cut point of $K$. Therefore a sufficiently small neighborhood of $x_2$ in $K$ is (in $K^2)$ either has dimension less than 3 or there is an arc cutting this neighborhood into disjoint parts. Hence $x \in n_2(K^2)$.

Similarly, if $x_1 \in nK \times K$ contains the set homeomorphic to $I^2 \times T$, where $T$ is a triod and $T$ is an arc.

Therefore each neighborhood of $(x_1, x_2) \in K \times nK$ contains a set homeomorphic to $I \times (T \setminus I)$. Since $I \times T$ is not embeddable in $R^4$, the point $x$ that does not have an Euclidean neighborhood in $(K \times nK, n_2(K^2))$.

(c) The cases $i = 3, 4$ can be proved using similar elementary considerations. These proofs are left to the reader.

**Definition 3.1.** The collection of components of a set $X$ will be denoted by $\square X$.

**Lemma 3.2.** Suppose $DK$ and $DL$ are nowhere dense subpolyhedra of compact connected $2$-polyhedron $K$ and $L$, respectively, and $F : K^2 \to L^2$ is a homeomorphism such that

(i) $F((K \times DK) \cup (DK \times K)) = (L \times DL) \cup (DL \times L)$,

(ii) $F(DK \times DK) = DL \times DL$.

Then $F(DK \times K) = DL \times L$ or $F(DK \times K) = L \times DL$.

**Proof.** By (i) and (ii) we obtain $F((K \times DK) \cup DK \times K)) = (L \times DL) \cup (DL \times L)$.

Now, if $A \in \square (K \times DK)$ and $X \in \square DK$, then $F(A \times X) = A' \times X'$ or $F(A \times X) = X' \times A'$, where $A' \in \square (L \times DL)$ and $X' \in \square DL$.

Assume $A \times X = A' \times X'$. Let $B$ be another component of $K \times DK$ such that $A \times B \neq \emptyset$. Suppose $F(B \times X) = Y' \times B'$, where $Y' \in \square DL$ and $B' \in \square (L \times DL)$.

Since $A' \times X' \cap Y' \times B' \neq \emptyset$, there exist points $x' \in B' \cap X'$ and $y' \in F' \cap A'$.

If $B \in B'$ and $A' \in A'$ then there exist arcs $x'y'$ and $y'x'$ such that their interiors are contained in $B$ and $A'$, respectively; hence, there exists an arc $(x', y')(y', b')$ such that its interior lies in $A' \times B'$ i.e. this interior is disjoint with the set $(L \times DL) \cup (DL \times L)$. We consider the arc $(a, x)(b, y) = F^{-1}((a', x')(y', b'))$. Then $a' \in A$, $b' \in B$ and $x, y \in x$. But if $a, b \in A$ and $x, y \in x$, then the interior of any arc $(a, x)(b, y) \in K^2$ and the set $(K \times DK) \cup (DK \times K)$ is not disjoint. Hence, $F(B \times X) = B' \times Y'$. Since $A' \times X' \cap B' \times Y' \neq \emptyset$, $X'$, $Y'$ are closed in $L$ and $X', Y' \in \square DL$, we have $X' = Y'$.

The polyhedron $K$ is connected, the set $DK$ is nowhere dense, thus for any component $B$ of the set $\square DK$ there exists a sequence of components $A = A_1, A_2, \ldots, A_n = B$ of the set $\square K \setminus DK$ such that $A_n \cap A_{n+1} = \emptyset$ for $i = 1, 2, \ldots, n-1$. Therefore, $F(B \times X) = B' \times X'$ for any $B \in \square (K \times DK)$ and $F(K \times X) = L \times X'$.

Hence $F(K \times X) = L \times DL$.

**Lemma 3.3.** Suppose $K$ and $L$ are compact connected $2$-polyhedra, $F : K^2 \to L^2$ is a homeomorphism and $K$ has local cut points. Then $K$ and $L$ are homeomorphic.

**Proof.** First we consider the case where there exists a point $x$ of $K$ such that $dim_{K} = 1$. We denote $P = \{x \in K : dim_{K} = 1\}$, $DK = K \setminus P$, $P' = \{x \in K : dim_{K} = 0\}$ and $DL = L \setminus P'$. The sets $R$ and $D'K$ are empty. Observe that $K^2 = (P \times P) \cup (P \times R) \cup (R \times P) \cup (R \times R)$, $L^2 = (P' \times P') \cup (P' \times R') \cup (R' \times P') \cup (R' \times R')$, $F(P \times P) = (P' \times P') \cup (P' \times R') \cup (R' \times P')$, $F(P \times R) = (P' \times R') \cup (R' \times R')$, $F(R \times P) = (P' \times P') \cup (P' \times R') \cup (R' \times P')$, $F(R \times R) = (R' \times R')$.

Therefore, $K^2 = (P \times P) \cup (P \times R) \cup (R \times P) \cup (R \times R)$, $L^2 = (P' \times P') \cup (P' \times R') \cup (R' \times P') \cup (R' \times R')$, $F(K \times K) = F(K \times DK) \cup (DK \times K) = (P \times P) \cup (P \times R) \cup (R \times P) \cup (R \times R)$, $F(K \times L) = F(DK \times L) \cup (L \times DL)$. Also $F(DK \times K) = F(K \times DK) \cup (DK \times K) = (P' \times P') \cup (P' \times R') \cup (R' \times P') \cup (R' \times R')$.

The sets $DK$ and $DL$ are finite, hence $F(K \times DK) = L \times DL$ or $F(K \times DK) = DL \times L$ and $K \cong L$.

Now, we consider the case $dim_{K} = 2$ for every $x \in K$. Then the local set cut points of $K$ is finite. Let $DK$ denote this set. It is easy to see that the set $(DK \times K) \cup (K \times DK)$ is the set of such points $(x, y) \in K^2$ that there exists a neighborhood $U$ of $(x, y)$ such that for every connected neighborhood $V$ of $(x, y)$ contained in $U$ there exists a 2-dimensional set separating the set $V$. Therefore $F((DK \times K) \cup (K \times DK)) = (L \times DL) \cup (DL \times L)$. The set $DK \times K$ is the set of local cut points.
of the set \((K \times DK) \cup (DK \times K)\). Thus \(F(DK \times DK) = DL \times DL\) and by Lemma 3.2 we obtain \(K \equiv L\).

**Lemma 3.4.** If \(K, L\) are compact connected 2-polyhedra, \(F: K \rightarrow L\) is a homeomorphism and \(n_2K \neq \emptyset\), then \(K \equiv L\).

**Proof.** Because \(F(n_2K \times n_2K) = n_2L \times n_2L\) we obtain

\[
F([K \times n_2K] \times [n_2K \times K]) = ([L \times n_2L] \times [n_2L \times L]) \cup ([n_2L \times L] \times [L \times n_2L]).
\]

Therefore if \(A \in \square(K \times n_2K)\) and \(x \in n_2K\), then \(F(A \times x) = A' \times x'\), or \(F(A \times x) = x' \times A'\), or \(F(A \times x) = X' \times Y', \quad A' \in \square(L \times n_2L), \quad x' \in n_2L\), and \(X', Y' \in \square(n_2L \times L)\).

Suppose there exist such \(A \in \square(K \times n_2K)\) and \(x \in n_2K\) that \(F(A \times x) = A' \times x'\). We shall examine whether it is possible to find \(B \in \square(K \times n_2K)\), \(x' \in n_2L\) and \(B' \in \square(L \times n_2L)\) by Lemma 3.3 we may assume that the polyhedron \(K\) has no local cut point. Hence there exists a sequence of components \(A = A_1, A_2, ..., A_n = B\) of the set \(K \times n_2K\) such that \(A_i \cap A_{i+1} = 1\) for \(i = 1, 2, ..., n-1\). Therefore, it is enough to consider the case when \(n = 1\).

Since the set \(A \times x \times y' \times B'\) for any \(y' \in n_2L\), \(B' \in \square(L \times n_2L)\) contains at most one point and \(A \times x \times B \times x\) has dimension 1, it is not true that \(F(B \times x) = x' \times B'\). If \(F(B \times x) = U' \times V'\), then \(U' \in \square(L \times n_2L)\), \(x' \in n_2L\), and \(x' \in U' \times V'\). Let \(x' \in U' \cap V'\). There exists a point \(v' \in U'\) such that we can join \(x'\) with \(v'\) by an arc whose interior lies in \(U'\). There exists a point \(v' \in V'\) such that we can join \(x'\) with \(v'\) by an arc whose interior lies in some component \(C' \times L\). Hence, there exists an arc \((x', v')(u, v)\) whose interior lies in \(A' \times C'\), and so is disjoint with the set \(n(L)\). This is impossible because \((a, x) = F^{-1}(u, v)(x) \in A \times (b, x) = F^{-1}(u, v)(x) \in E \times B \times x\) and the interior of every arc \((a, x)(b, x)\) lying in \(K\) does not have empty intersection with \((K \times n_2K)\).

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Now we show that if \(F(B \times x) = U' \times V'\) then \(U' \cap U \times V' \cap V\). Since we may assume that \(K\) does not have local cut points and the relation \(\sim\) is transitive, we may assume that \(\dim A \cap B = 1\). We note that if \(A \cap B \times x = (X' \times Y' \cap (Y' \cap V'))\), i.e. the set \((X' \times Y' \times (Y' \cap V'))\) has dimension 1, hence either \(x' = U'\) or \(y' = V'\). Assume that \(x' = U'\). It is obvious that \(Y' \cap V' \cap B \neq \emptyset\) and \(Y' \cap B \neq V'\). It is easy to see that the interior of every arc which lies in \(K\) with end-points in \(A \times x \times B \times x\) has nonempty intersection with the set \(n(K)\). If there is a component \(C' \times L\) of the set \(A \times n_2L\) such that \(y' \cap C' \neq \emptyset\) and \(V' \cap C' \neq \emptyset\), then there is an arc having its interior in \(C'\) and its end-points in \(Y'\) and \(V'\), respectively. Hence, there is an arc lying in \(L\) with end-points in \(X' \times Y'\) and \(U' \times V' = X' \times Y'\) such that its interior is disjoint with the set \(n(L)\).

Let \(X' \cap U' \cap V' \cap B\). It is enough to consider the case where \(X' = U'\) and \(Y' \cap B' \cap B' \neq \emptyset\). We have assumed that \(F^{-1}(X' \times X') = A \times x\) and we know that \(F^{-1}(U' \times V') \neq B \times X\), because \(Y' \cap X' \cap B' \cap B' \neq \emptyset\). If \(F^{-1}(U' \times V') = \emptyset\), then there exist points \((a, x) \in A \times x\) and \((u, v) \in U \times V\), and an arc \((a, x)(u, v)\) which is not in \(K\) such that its interior is disjoint with the set \(n(K)\). Since \((a, x) \in X' \times Y', (u, v) \in U' \times V' \times B\) and no component \(C' \times L\) of the set \(A \times n_2L\) satisfies \(y' \cap C' = \emptyset\) and \(V' \cap C' \neq \emptyset\), the interior of each arc joining \((a, x)\) with \((u, v)\) in \(L\) does not have an empty intersection with \(n(L)\). Hence \(F^{-1}(U' \times V') = B \times X\).

Hence \(K = n_2L\) or \(K = A \times x' \times B' \neq \emptyset\). Where \(n \equiv P \times R\), and \(P, R, P', R'\) are graphs. If \(K \equiv P \times R, L \equiv P \times R\) and \(K \equiv L\), then \(K\) is by Borsuk's theorem [2] on the uniqueness of the decomposition into Cartesian product of 1-polyhedra.

**Proof of Proposition 3.1.** By Lemma 3.4 the set \(n_2K\) is empty, hence condition \((***)\) holds. By Lemma 3.1, \(n(K) = (K \times n_2K) \cup (n_2K \times K)\) and if \(n_2K = \emptyset\) then \(n_2K) = n_2K \times K\). Hence, \(F(K \times n_2K) = (K \times n_2L) \cup (n_2K \times n_2L) = n_2K \times n_2L\). By Lemma 3.3 the polyhedron \(K\) does not have local cut points. Hence, the set \(n_2K\) is nowhere dense in \(K\). Analogously \(n_2L\) is nowhere dense in \(L\). Lemma 3.2 implies \((***)\).

If \(F(K \times n_2L) = (L \times n_2L) \times L\). Then \(F(K \times n_2K) = (K \times n_2L) \times L\). Hence for each component \(A\) of the set \(K \times n_2K\) there exists a component \(A'\) of the set \(L \times n_2L\) such that \(F(A \times x) = A' \times x\). Hence condition \((***)\) holds.

Let \(DK\) denote the set of points \(x \times K\) such that there does not exist \(y \in n_2L\) and a neighborhood \(V\) of \(x\) in \(K\) such that \(V \equiv [0, 1] \times n_2L \cap \{x\} \times \{y\} \times n_2L\). The corresponding subset of \(n_2L\) is denoted by \(DL\). Since \(K\) does not have local cut points, \(n_2K\) is a 2 manifold, the sets \(DK\) and \(DL\) are finite. Since \(F(K \times n_2K) = L \times n_2L\), we have \(F(K \times n_2K) = (L \times n_2L) \times n_2L\). The point \((a, y) \in (K \times n_2K) \times n_2K\) belongs to the set \((K \times n_2K) \times (n_2K)\) iff \((a, y) \in n_2K \times n_2L\). If \(L \equiv n_2L\), therefore \(F(K \times n_2K) = L \times n_2L \times n_2L\).

The sets \(n_2K\) and \(n_2L\) are nowhere dense in \(K\) and \(L\), hence \(F(K \times DK) = L \times DL\times DL\).

If \(DK \neq 0\) then \(K \equiv L\). Hence condition \((***)\) holds.

Remark. By \((***)\) we have one-to-one correspondences \(A \leftrightarrow A'\) and \(A \leftrightarrow A''\).
between components of $K \times nL$ and $L \times nl$ such that:

$$F(A \times K) = A' \times L, \quad F(K \times A) = L \times A'$$

and

$$F(A \times B) = A' \times B'.$$

Remark. Condition (**) is true without the assumption $K \approx L$, but we shall not use this and omit the proof.

4. Cancellation of $S'$-factor. In this section we will prove

**Proposition 4.1.** If $K$ is a compact connected 2-polyhedron, $K \approx L^2$ and the set $nK$ contains a simple closed curve then $K \approx L$.

If one of the conditions (**) or (***) from Proposition 3.1 does not hold then $K \approx L$.

We consider the case when (**) holds. By (**), if $K \rightarrow L^2$ is a homeomorphism then $F(K \times nK) = L \times nL$ (or $F(K \times nK) = nL \times L$). Hence, if $S'$ is a component of $nK$ homomorphic to a simple closed curve then $F(K \times S') = L \times S'$ where $S' \subset \partial nL$. If $S'$ is an arc then $K^2 \times (S' \times S') \approx L^2 \times (I \times I)$ in $\approx K^3$. Hence $S'$ is also a simple closed curve. Therefore, it is enough to prove (**).

**Proposition 4.2.** If $K$ is a compact connected 2-polyhedron which satisfies (*) and (**) of Prop. 3.1 and $K \times S' \approx L \times S'$ then $K \approx L$.

Remark. Possibly, for every 2-polyhedron $K$, if $K \times S' \approx L \times S'$ then $K \approx L$. Nevertheless, we only consider the above-mentioned special case.

Before we will prove Proposition 4.2, we give a definition of manifolds $M(A)$ and some properties of these manifolds. Let $K$ be a 2-polyhedron and $nK$ denote the regular neighborhood of the polyhedron $nK$ in the polyhedron $K$. Then we can define $M(A)$ as the set $A \times U(nK)$. For technical reasons, in the sequel we shall use another definition of $M(A)$.

**Definition 4.1.** (i) We denote by $N(A)$ the set of all sequences $x_n \in A$ which converge in $A$ and such that for every neighborhood $U$ of the point $x_n$, if $x_n \in K$ there exist $U \cap \partial (U \cap nK)$ and a natural number $n_0$ such that for every $n > n_0$ we have $x_n \in U$.

(ii) In the set $N(A)$ we define the equivalence relation $\sim$. We have $\{x_n \sim y_n \}$ if

(i) $lim_{n \to \infty} x_n = lim_{n \to \infty} y_n = x_0$ in $K$,

(ii) for every neighborhood $U$ of $x_0$ in $K$ there exist $U \subset \partial (U \cap nK)$ and a natural number $n_0$ such that for every $n > n_0$ we have $x_n \in U$ and $y_n \in U$.

(3) By $M(A)$ we denote the set $N(A)/\sim$.

We define a basis for the topology of $M(A)$. Let $\{n_2i\} \subset M(A)$ and $\lim_{n \to \infty} x_n = x_0$. Let $U$ be a neighborhood of the point $x_0$ in $K$ and let $U_0$ denote the component of the set $U \times nK$ such that for almost all $n$ we have $x_n \in U_0$. We denote by $V(n_2) \subset M(A)$ the union of all $n_2i$ such that $lim_{n \to \infty} x_n \in U$ and for almost all $n$, $x_n \in U_0$. The collection of the sets $V(n_2)$ is a basis for the topology of $M(A)$.

The first definition is simpler than the second, but if we use the second definition, then the following properties are very simple.

**Property 4.1.** If $\lim_{n \to \infty} x_n = x$ in $A$, then $\{\lim_{n \to \infty} x_n = \{x\}$ (where $\{x\}$ is the constant sequence).

**Property 4.2.** The function $h: A \rightarrow M(A)$ given by $h(x) = \{\lim_{n \to \infty} x_n = \{x\}$ is a topological embedding.

**Property 4.3.** If condition (**) from Proposition 3.1 holds then the space $M(A)$ is a compact 2-manifold and $M(A) \cap h(A) = \partial M(A)$.

**Property 4.4.** If condition (**) from Proposition 3.1 holds and $g: M(A) \rightarrow A$ is given by the formula $g(x) = \lim_{n \to \infty} x_n$, then $h\circ g: \partial M(A) \rightarrow \partial A$ is a covering.

**Property 4.5.** Let $K$ and $L$ be 2-polyhedra and let $G: K \times S' \rightarrow L \times S'$ be a homeomorphism. There exists a homeomorphism $G_{A}: (A \times S') \rightarrow (A \times S')$ such that $G_{A} \times g_{A} = G_{A} \times g_{A}$.

Remark. Denote by $P_{1} : A \times S' \rightarrow A$ and $P_{2} : A \times S' \rightarrow S'$ the projections on the first and the second factor respectively. The homeomorphism $G_{A}$ is given by the formula:

$$G_{A}(\{x_n\}, t) = \{(P_{1}F_{\times}(S_{n}, y_n)), \{P_{2}F_{\times}(S_{n}, y_n)\}\}.$$ 

The proofs of these properties are easy and left to the reader.

Now we prove Proposition 4.2.

**Proof of Proposition 4.2.** Since $dim(K \times S') = dim(L \times S') = dim L + 1$, we have $dim L = 2$ and by [10], the space $L$ is a polyhedron.

Let the set $nK$ consist of pairwise disjoint arcs $I_i = a_iB_i$, $L_k = \bar{a}_kB_i$ and simple closed curves $S_1, \ldots, S_k$. It is easy to see that $n(K \times S') = nK \times S'$. Hence, if $G: K \times S' \rightarrow L \times S'$ is a homeomorphism, then $G(nK \times S') = nL \times S'$.

Therefore, the set $nL$ consists of pairwise disjoint arcs $I_i = a_iB_i$, $L_k = \bar{a}_kB_i$ and simple closed curves $S_1, \ldots, S_k$ such that

(i) $G(I_i \times S') = I_i \times S'$ and $G(I_i \times S') = I_i \times S'$ for $i = 1, 2, \ldots, k$;

(ii) $G(S_i \times S') = S_i \times S'$ for $i = 1, 2, \ldots, l$.

Let $f: nK \rightarrow nL$ be a homeomorphism such that

(iii) $f(I_i) = I_i$, $f(L_k) = L_k$ for $i = 1, 2, \ldots, k$ and $f(S_i) = S_i$ for $i = 1, 2, \ldots, l$.

It suffices to show that the homeomorphism $f$ has an extension to a homeomorphism $f: K \rightarrow L$. 

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For every $A \in \square(K \times \kappa)$ and $A' \in \square(L \times \kappa)$ such that $G(A \times S^1) = A' \times S^1$ we have constructed the homeomorphism $G_A : M(A) \rightarrow M(A')$.\footnote{4} Since $M(A)$ and $M(A')$ are 2-manifolds, it follows that $M(A), M(A')$ are homeomorphic.

In case $nK = \emptyset$ we have $M(A) = K$ and the conclusion of the theorem holds. So assume that $nK$ is not empty.

Let $g_A : M(A) \rightarrow A$ be given by $g_A([x]) = \lim_{y \to x} y$, as in Property 4.5 and let $g_A : M(A) \rightarrow A$ be defined analogously. For every $A \in \square(K \times nK)$ we shall find a homeomorphism $f_A : g_A^{-1}(nK \cap A) \rightarrow g_A^{-1}(nL \cap A)$ such that

$$f \circ g_A(x) = g_A \circ f_A(x)$$ for $x \in g_A^{-1}(nK \cap A)$.

Then we shall extend it to a homeomorphism $f_A : M(A) \rightarrow M(A')$. Next, we obtain a homeomorphism $f : K \rightarrow L$ given by the formula

$$f(x) = g_A \circ f_A(g_A^{-1}(x))$$ for $x \in \overline{A}$, where $A \in \square(K \times nK)$.

Since the diagram

$$\begin{array}{ccc}
M(A) \times S^1 & \xrightarrow{G_A} & M(A') \times S^1 \\
g_A \times \text{id} & \downarrow & \text{id} \\
\overline{A} \times S^1 & \xrightarrow{\text{id}} & \overline{A} \times S^1
\end{array}$$

commutes (see Property 4.5) it follows that for every $T \in \square g_A^{-1}(S)$ and any $T' \in \square g_A^{-1}(S)$ such that $G_A(T \times S^1) = T' \times S^1$ the diagram

$$\begin{array}{ccc}
T \times S^1 & \xrightarrow{G_A \times \text{id}} & T' \times S^1 \\
g_T \times \text{id} & \downarrow & \text{id} \\
S_T \times S^1 & \xrightarrow{\text{id}} & S_T \times S^1
\end{array}$$

commutes too.

Since the maps $g_A \times \text{id}$ and $g_A \times \text{id}$ are coverings and the maps $G_A \times \text{id}$, $G_A \times \text{id}$ are homeomorphisms, the degrees of the coverings are equal and there exists a homeomorphism $f : T \rightarrow T'$ such that

$$f \circ g_A(x) = g_A \circ f_A(x)$$ for $x \in T$.

The set $\square g_A^{-1}(t)$ is a family of arcs. If $J$ is one of them and $J'$ is the corresponding component of $g_A^{-1}(t)$, we have a homeomorphism $f_J : J \rightarrow J'$ such that

$$f \circ g_A(x) = g_A \circ f_A(x)$$ for $x \in J$.

Let us define a homeomorphism $f_A : g_A^{-1}(nK \cap A) \rightarrow g_A^{-1}(nL \cap A)$ by the formula

$$f_A(x) = \begin{cases}
 f_T(x) & \text{for } x \in T, T \in \square g_A^{-1}(S), t = 1, \ldots, l, \\
 f_J(x) & \text{for } x \in J, J \in \square g_A^{-1}(\{l\}), l = 1, \ldots, k.
\end{cases}$$

It is obvious that the homeomorphism $f_A$ satisfies condition (3).

Let $J \in \square g_A^{-1}(\{l\})$, $J' \in \square g_A^{-1}(\{l\})$. By (i) and (iii), $G(A \times S^1) = J' \times S^1$ iff $f(J) = J'$.

By (iv) and (v) $f_J(x) = J'$ iff $G_J(J \times S^1) = J' \times S^1$. Moreover, if $J = c_1$ and $J' = c_2$ then $f_J(c) = c'$ iff $G_J(c \times S^1) = c' \times S^1$.

This implies that the homeomorphism $f_A$ is extendable to a homeomorphism $f : \partial M(A) \rightarrow \partial M(A')$. If the manifold $M(A)$ is not orientable, there exists an extension $f_A : M(A) \rightarrow M(A')$ of the map $f_A$ such that $f_A$ is a homeomorphism. If the manifold $M(A)$ is orientable, it suffices to show that the homeomorphism $f_A$ does not change the orientation.

We choose orientations of the curves $S_T$ in any manner. Next, we choose orientations of the curves $S_J$ in such a manner that the maps $G_A \times \text{id}$ preserve orientation. We can additionally require that

$$f_T : S_T \rightarrow S_T$$ preserves orientation for any $t = 1, 2, \ldots, l$. We choose orientations of the manifolds $M(A)$ in any manner and we choose the orientations of the manifolds $M(A')$ in such a manner that the maps $G_A : M(A) \times S^1 \rightarrow M(A') \times S^1$ preserve orientation. The orientations of $T \in \square \partial M(A)$ and $T' \in \square \partial M(A')$ we have fixed by the orientations of $M(A)$ and $M(A')$.

Since (v) $G_A \times \text{id}$ preserves orientation iff $g_A \times \text{id}$ preserves orientation, hence $g_A \times \text{id}$ preserves orientation iff $f_T$ preserves orientation. This and (vi) imply that $f_T : T \rightarrow T'$ preserves orientation for $T \in \partial g_A^{-1}(S)$. \(\square\)

Let $T \cap J = \square g_A^{-1}(l)$. Then, as we have mentioned above, if $J = c_1$ and $J' = c_2$ then $f_J(c) = c'$ iff $G_J(c \times S^1) = c' \times S^1$. Since $G_A$ preserves orientation on $T \times S^1$, $f_J$ preserves orientation on $T$.

The lemma is thus proved.

In the next section we can consider polyhedra $K$ such that $nK$ does not contain simply closed curves.

5. A one-to-one correspondence between the manifolds $M(A)$ and $M(A')$. In Section 4 for every component $A \in \square(K \times nK)$ we have defined the compact 2-manifold (with boundary) $M(A)$. Here, for every compact 2-manifold $M$ with boundary we will define a number $\sigma(M)$. In Lemma 5.1 we will prove that if $M \times \mathbb{N} \times \mathbb{P} \times \mathbb{R}$ then $\sigma(M \times \mathbb{P}) = \sigma(M \times \mathbb{R})$. Moreover, in Lemma 5.2, we will prove that there exists a one-to-one correspondence between manifolds $M(A)$ and $M(A')$ such that $M(A) \rightarrow M(A')$.

**Definition 5.1.** Let $M$ be a compact 2-manifold with boundary $\partial M \neq \emptyset$. We define the number

$$\sigma(M) = \text{rank} H_0(M) - \text{rank} H_1(\partial M) + \epsilon$$

where

$$\epsilon = \begin{cases}
0 & \text{for } M \text{ nonorientable}, \\
1 & \text{for } M \text{ orientable}.
\end{cases}$$
Remark: If the surfaces $M$ and $N$ are both orientable or both nonorientable, $\sigma(M) = \sigma(N)$ and $\text{rank} H_1(M) = \text{rank} H_1(N)$, then $M = N$.

**Lemma 5.1.** If $M$, $P$, $R$ are compact 2-manifolds with boundary and the manifolds $M \times N$ and $P \times R$ are homeomorphic, then $\sigma(M)(N) = \sigma(P)(R)$.

**Proof.** Let $H_2(M) = Z^p$, $H_2(N) = Z^q$, $H_2(P) = Z^r$, $H_2(R) = Z^s$.

From Künneth's formula we conclude that

$$Z^{p+q} \approx H_2(P \times R) \approx H_2(M \times N) \approx Z^{ps},$$

$$Z^{r+s} \approx H_2(P \times R) \approx H_1(M \times N) \approx Z^{qs}.$$

Hence, $p = m$ and $r = n$ or $p = n$ and $r = m$. We can assume that the first case holds.

Also, from Künneth's formula it follows that $H_2(M, \partial M) \approx H_2(P, \partial P)$ and $H_2(N, \partial N) \approx H_2(R, \partial R)$. Hence, both manifolds $P$ and $R$ are orientable or both are nonorientable and the same holds for $M$ and $N$.

It is easy to compute the group $H_2(\partial(M \times N))$, using Künneth's formula and the Mayer-Vietoris sequence or the methods from [7].

If $M$ and $N$ are orientable, then $H_2(\partial(M \times N)) \approx Z^{(m+1)(n-1)+1-\varepsilon}(\partial(M \times N))$ and $H_2(\partial(P \times R)) \approx Z^{(p+1)(n-1)+1-\varepsilon}(\partial(P \times R))$. Therefore $H_2(\partial(M \times N)) = H_2(\partial(P \times R))$. Similarly, if $M$ is orientable and $N$ is nonorientable, then

$$H_2(\partial(M \times N)) \approx Z^{(m+1)(n-1)+1-\varepsilon}(\partial(M \times N)) \oplus Z_2,$$

$$H_2(\partial(P \times R)) \approx Z^{(p+1)(n-1)+1-\varepsilon}(\partial(P \times R)) \oplus Z_2,$$

hence $\sigma(M)(\partial(N)) = \sigma(P)(\partial(R))$.

If $M$ and $N$ are nonorientable, then $H_2(\partial(M \times N)) \approx Z^{m-\varepsilon}(\partial(M \times N)) \oplus Z_2$ and $H_2(\partial(P \times R)) \approx Z^{p-\varepsilon}(\partial(P \times R)) \oplus Z_2$, hence $\sigma(M)(\partial(N)) = \sigma(P)(\partial(R))$.

Therefore the lemma is proved.

We recall that if $F: K \times L \rightarrow X$ is a homeomorphism and $A, B \in \partial(K \times L)$ then $F(A \times B) = A' \times B'$, where $A', B' \in \partial(L \times X)$. In the case considered, by condition (***) from Proposition 3.1, $F(K \times L) = A' \times L$ and $F(K \times L) = B' \times L$ for every $A, B \in \partial(K \times L)$. By Property 4.6 we have $M(A) \times M(B) \approx M(A') \times M(B')$.

**Lemma 5.2.** Let $K$ be a compact connected 2-polyhedron. Assume that all components of the set $K \times L$ are arcs and the regular neighborhood of any $x \in K \times L$ homeomorphic to the set cone $[a, \ldots, n] \times x$.

If $F: K \rightarrow L$ is a homeomorphism and $F(A \times X) = A' \times X$, where $A, A' \in \partial(K \times L)$ and $A', A'' \in \partial(L \times X)$, then the manifolds $M(A)$, $M(A')$, $M(A'')$ are homeomorphic.

**Proof.** It is sufficient to show that the manifolds $M(A)$, $M(A')$, $M(A'')$ are all orientable or all nonorientable, rank $H_1(M(A)) = \text{rank} H_1(M(A')) = \text{rank} H_1(M(A''))$, and $\sigma(M(A)) = \sigma(M(A')) = \sigma(M(A''))$. From the topological equality $M(A') \approx M(A) \times M(A'')$ we easily conclude that the first and second conditions hold.

**Let**

$$m = \max \{ \sigma(M(B)): B \in \partial(K \times L) \},$$

and

$$m' = \max \{ \sigma(M(B')): B' \in \partial(L \times X) \}.$$
If $g_A$, $g_B$, and $g_{\lambda}$ are given as in Property 4.4, then the diagram

\[
\begin{array}{ccc}
M(A) \times M(\lambda) & F_{A\lambda} & M(A) \times M(\lambda) \\
\downarrow g_A \times g_{\lambda} & & \downarrow g_A \times g_{\lambda} \\
A \times \lambda & F_{A\lambda} & A \times \lambda
\end{array}
\]

commutes.

Hence, if $J \in \square g_{\lambda}^{-1}(nK \cap \lambda)$, then

\[F_{A\lambda}(M(A) \times J) = M(A) \times J', \quad F_{A\lambda}(M(\lambda) \times J) = M(\lambda) \times J', \quad F_{A\lambda}(J \times M(A)) = J' \times M(A), \quad F_{A\lambda}(J \times M(\lambda)) = J' \times M(\lambda), \]

where $J' \in \square g_{\lambda}^{-1}(nL \cap \lambda)$ and $J' \in \square g_{\lambda}^{-1}(nL \cap \lambda)$.

By Properties 4.3, 4.4, $J \cap \partial M(A) = J' \cap \partial M(A')$, $J' \cap \partial M(A')$ and $g_{A\lambda}$.

We shall prove

**Lemma 6.1.** If all the assumptions of Proposition 6.1 and (ii) hold, then there exist homeomorphisms $f_{1, A}: A \times M(A) \rightarrow M(A')$ and $f_{2, A}: A \times M(\lambda) \rightarrow M(\lambda')$ such that

\[f_{1, A}(J) = J' \quad \text{and} \quad f_{2, A}(J) = J''\]

Next we shall prove

**Lemma 6.2.** Let all the assumptions of Proposition 6.1 and (i)-(iii) hold. Let $I = \partial J$, $I' = \partial J'$, $J'' = \partial J''$; $J = \partial J$, $J' = \partial J'$, $J'' = \partial J''$ be such that $g_A(c) = a$, $g_{\lambda}(c) = b$, and $g_{\lambda}(c) = b'$. The homeomorphism from Lemma 6.1 can be corrected in such a way that additionally the following condition holds:

\[+1\] either $f_{1, A}(c) = c'$ for all $A \in \square g_{\lambda}^{-1}(1)$ or $f_{2, A}(c) = c''$ for all $A \in \square g_{\lambda}^{-1}(1)$ and $J = \partial g_{\lambda}^{-1}(1)$, and analogously for $f_{2, A}$.

Since two homeomorphisms of arcs keeping the end-points are isotopic and the arcs $J$, $J'$ have collars in $M(A)$ and $M(A')$, we can correct the homeomorphisms $f_{1, A}$ so that additionally the following condition holds:

\[+1\] $g_A \circ f_{1, A} \circ (g_{\lambda})^{-1}(c) = g_A \circ f_{1, A} \circ (g_{\lambda})^{-1}(c)$ for $x \in I$, where $I \in \square g_{\lambda}$, $A, B \in \square (K \times nK)$ (which may be equal), $f_{1, A} \in \square g_{\lambda}^{-1}(1)$ and $f_{2, A} \in \square g_{\lambda}^{-1}(1)$. $A', B' \in \square (K \times nL)$ are such that $F(A \times A) = A' \times A'$ and $F(B \times B) = B' \times B'$.

Thus we obtain a homeomorphism $f: K \rightarrow L$ which can be defined by the formula

\[+1\] $f(x) = g_A \circ f_{1, A} \circ (g_{\lambda}^{-1}(c))$ where $x \in A$, $A \in \square (K \times nK)$, $A' \in \square (L \times nL)$.

Condition $(\ast \ast)$ yields the correctness of $(\ast \ast)$.

Now, we need only prove Lemmas 6.1 and 6.2.

**Proof of Lemma 6.1.** We will prove that there exist homeomorphisms $f_{1, A}: A \times M(A) \rightarrow M(A')$ and $f_{2, A}: A \times M(\lambda) \rightarrow M(\lambda')$ such that $(\ast \ast)$ holds.

For $I \in \square g_{\lambda}$ we define $I_1$, $I_2$, $J_1$, $J_2$ as above. By Lemma 5.2, the manifolds $M(A)$, $M(A')$ and $M(\lambda')$ are homeomorphic. It is sufficient to show that the following conditions hold:

1. If $J_1, J_2 \in \square g_{\lambda}^{-1}(nK \cap \lambda)$ lie in one component $S \in \square g_{\lambda}^{-1}(nK \cap \lambda)$, then $J_1', J_2'$ lie in one component $S' \in \square g_{\lambda}^{-1}(nK \cap \lambda)$.
2. If some $J_1, J_2, J_3, J_4 \in \square g_{\lambda}^{-1}(nK \cap \lambda)$ lie in $S \in \square g_{\lambda}^{-1}(nK \cap \lambda)$ and the set $J_2 \cup J_4$ lies in one component of the set $S \cup \{J_1 \cup J_2 \}$, then the set $J_2' \cup J_4'$ lies in one component of the set $S' \cup \{J_1' \cup J_2' \}$, and the set $J_2' \cup J_4'$ lies in one component of the set $S' \cup \{J_1' \cup J_2' \}$.

This means that the arcs $J_1, J_1'$ on $S'$ and $J_2'$ on $S''$ can be ordered in the same way. Hence, there exist homeomorphisms $f_{1, A}: \partial M(A) \rightarrow \partial M(A')$ and $f_{2, A}: \partial M(\lambda) \rightarrow \partial M(\lambda')$ such that $f_{1, A}(J) = J'$ and $f_{2, A}(J) = J''$ for all $J \in \square g_{\lambda}^{-1}(nK \cap \lambda)$.

The manifolds $M(A)$, $M(A')$, $M(\lambda')$ are homeomorphic. Hence, if the manifold $M(A)$ is orientable, we can extend the homeomorphisms $f_{1, A}$, $f_{2, A}$ to homeomorphisms $f_{1, A}: M(A) \rightarrow M(A')$ and $f_{2, A}: M(\lambda) \rightarrow M(\lambda')$ so that condition $(\ast \ast)$ holds.

3. If the manifold $M(A)$ is orientable, we need to show that we can construct the homeomorphisms $f_{1, A}$, $f_{2, A}$ in such a manner that they preserve orientation on all components of $\partial M(A)$, or change orientation on all components of $\partial M(A)$.

Then we extend the homeomorphisms $f_{1, A}$, $f_{2, A}$ onto the whole manifold $M(A)$.

Let $x_i \in (J_i)$, $x_i \in (J_i')$, $x_i \in (J_i'')$, $i = 1, 2, \ldots, g, F_{A\lambda}(x_i, x_j) = (x_i, x_j)$ in the proofs of (1), (2), (3). ($\partial$ denotes the interior of $J$).

**Proof of (1).** First, we will prove that if $I_1$ and $I_2$ lie in two different components $S_1$ and $S_2$ of $\partial M(A)$ then $I_1'$, $I_2'$ lie in two different components $S_1'$, $S_2'$ of $\partial M(A')$.

Let us suppose $J_1, J_2 \in S' \in \square M(A')$. Let $x_1, x_2 \in M(A')$, $(x_1, x_2) \in \text{Int}(M(A'))$ and $y_1, y_2 \in S'$, $x_1 \times y_1 = x_2 \times y_2$ rel $(x_1, x_2)$ in $M(A')$. Let $x_1', x_2' \in M(A')$ be such that $(x_1', x_2') \in \text{Int}(M(A'))$.

Let $T_1 = \delta(x_1', x_2' \times x_1, x_2') = (x_1', x_2' \times x_1, x_2') \cup (x_1', x_2' \times x_1, x_2') \cup (x_1', x_2' \times x_1, x_2')$.

![Fig. 1](image)
The set $T_1$ is a simple closed curve contained in $\partial(M(A) \times M(A'))$. If $T_2 = (x_1' \times x_1' \times x_2') \cup (x_1' \times x_1' \times x_2') \cup (x_1' \times x_1' \times x_2') \cup (x_1' \times x_1' \times x_2')$, then $T_1 \approx T_2$ in $\partial(M(A) \times M(A'))$. Since $T_2 = \partial(x_1' \times x_1' \times x_2')$ and $x_1' \times x_1' \times x_2' \subset \partial(M(A) \times M(A'))$, we have $T_1 \approx T_2 \approx 0$ in $\partial(M(A) \times M(A'))$.

Now, let us suppose that $J_1 = S_1 \times J_2 = S_2$, where $S_1 \neq S_2$ and $S_1, S_2 \in \partial M(A)$. We consider the simple closed curve $F_{x_1}(T_2) \subset \partial(M(A) \times M(A))$. By (iii), $F_{x_1}(T_2) \cap (S_x \times S_x) = (x_1, x_1)$ for $p = 1, 2$ and we can assume that the manifolds $F_{x_1}(T_2)$ and $S_x \times S_x$ are transversal in $\partial(M(A) \times M(A))$. Since $\dim S_x \times S_x = 2$, $\dim F_{x_1}(T_2) = 1$ and $\dim(\partial(M(A) \times M(A)) = 3$, it follows that the embeddings of the sets $S_x \times S_x$ and $F_{x_1}(T_2)$ in $\partial(M(A) \times M(A))$ are homologically essential over $Z_2$ (see [6], VIII. 13). Hence, the curve $F_{x_1}(T_2)$ is not contractible in $\partial(M(A) \times M(A))$.

Therefore, if $J_1 = S_1, J_2 = S_2$ and $S_1 \neq S_2$, then $J_1 \subset (S_1 \cap S_2)' \subset S_1$ and $S_2 \neq S_2'$ and analogously $J_1' \subset S_1', J_2' \subset S_2'$ and $S_1' \neq S_2'$. If $J_1, J_2 \subset S \cap \partial M(A)$, then $J_1, J_2 \subset S' \subset \partial M(A)$. In the opposite case we consider the homeomorphism $(F_{x_1})^{-1} : M(A) \times M(A') \rightarrow M(A) \times M(A')$. Let us suppose that $J_1, J_2 \subset S, J_1', J_2' \subset S'$ and $J_1' \subset S_1', J_2' \subset S_2'$, where $S_1' \neq S_2'$. By Proposition 3.1 (2), $F(K \times A) \cup (A \times K) = (L \times A') \cup (A \times L)$, hence $F^{-1}(M(A) \times M(A)) = A_1 \times A_1$. Let $F^{-1}(M(A) \times M(A)) = A_1 \times A_1$. We consider the homeomorphism $F^{-1}_{x_1} : M(A) \times M(A') \rightarrow M(A) \times M(A')$, so the arcs $J_1, J_3$ lie in different components of $\partial M(A)$. This is impossible, whence $J_1', J_2'$ lie in one component of $\partial M(A)$. Thus (1) is proved.

Proof of (2). Let arcs $J_1, J_2, J_3, J_4$ be as in (3). Let $x_3 \times x_4' \times x_5 \subset S$. We consider the square $D_1 = x_1 \times x_3 \times x_4 \times x_5 \subset S \times S \subset \partial(M(A) \times M(A))$ and let $T_1 = \partial D_1$. Let the arcs $x_1' \times x_2', x_2' \times x_3', x_3' \times x_4', x_4' \times x_5'$ be constructed as the arcs $x_1' \times x_2, x_2' \times x_3, x_3' \times x_4, x_4' \times x_5$ in (1) (Fig. 2).

We consider the simple closed curve $T_2 = \partial(x_1' \times x_2' \times x_3') = (x_1' \times x_2' \times x_3') \cup (x_1' \times x_2' \times x_3') \cup (x_1' \times x_2' \times x_3') \cup (x_1' \times x_2' \times x_3')$. Since $x_1' \times x_2' \times x_3'$ is the boundary of disk $C' \cap M(A) \times M(A')$ and $x_1' \times x_2' \times x_3'$ is the boundary of a disk $C'' \subset M(A')$, $T_2$ is the boundary of the disk $D_1 = (x_2' \times x_3' \times x_4') \cup (C' \times x_3') \cup (C'' \times x_2') \cup (x_1' \times C'') \cup (x_1' \times C'') \subset \partial(M(A') \times M(A'))$.

Let us denote $F_{x_1}(T_2) = T_2$. The simple closed curves $T_1$ and $T_2$ are linked because $T_1 \cap D_1 = \{(x_1, x_2')\}$ and $T_2$ intersects the disk transversally $D_1$ (Fig. 3).

If $J_1, J_2, J_3, J_4 \subset S \cap \partial M(A)$ and the set $J_1 \cup J_2 \cup J_3 \cup J_4$ does not lie in one component of $S \cap (J_1 \cup J_2 \cup J_3 \cup J_4)$, then there exists an arc $x_1' \times x_2' \times x_3' \subset S$ such that $x_1', x_2', x_3' \neq x_5'$. Let us observe that $D_3 = (x_1' \times M(A')) \cup (x_2' \times M(A')) \cup (x_3' \times M(A')) \cup (x_4' \times M(A')) \cup (x_5' \times M(A'))$ and $T_4 \subset (x_1' \times M(A')) \cup (x_2' \times M(A')) \cup (x_3' \times M(A')) \cup (x_4' \times M(A')) \cup (x_5' \times M(A'))$. Hence $T_1 \cap D_3 = \emptyset$ (Fig. 4).
So $T'_1$ and $T'_2$ are not linked. Therefore $J'_2 \cup J'_3$ lies in one component of the set $S' \setminus (J'_1 \cup J'_2)$.

The proof for arcs $I'_1, I'_2, I'_3, I'_4$ is similar.

Therefore there exist homeomorphisms $f_{1,4} : \partial M(A) \rightarrow \partial M(A')$ and $f_{2,4} : \partial M(A') \rightarrow \partial M(A'')$ such that condition (1+) holds.

Proof of (3). If the manifold $M(A)$ is nonorientable, then the homeomorphisms $f_{1,4}, f_{2,4}$ extend to homeomorphisms $f_{1,4} : M(A) \rightarrow M(A')$ and $f_{2,4} : M(A') \rightarrow M(A'')$.

Now, suppose $M(A)$ is orientable. Let us choose the orientation of $M(A)$ and $M(A')$. Suppose, contrary to (3), that $f_{1,4}$ preserves the orientation on $S_1$ and changes the orientation on $S_2$, where $S_1, S_2 \in \partial M(A')$. If one of $S_1$ or $S_2$ contains less than three components of $\partial M(A)$, then they do not fix orientation and the homeomorphism $f_{1,4}$ can be corrected. Let $J_1, J_2, J_3 \subseteq S_1$ and $J_4, J_5, J_6 \subseteq S_2$.

We proceed to show that our assumption contradicts (ii). We will prove that the homeomorphism $F_{\lambda}$ maps the curve $S_1 \times x_1$ onto the set $F_{\lambda}(S_1 \times x_1)$ homologic to $S_1 \times x'_1$ in $M(A') \times J''_1$ and it does not change the orientation, whereas $F_{\lambda}$ maps the curve $S_1 \times x_1$ onto the set $F_{\lambda}(S_1 \times x_1)$ homologic to $S_1 \times x'_1$ in $M(A') \times J''_1$, and it changes the orientation. This is impossible because, by (ii), $F_{\lambda}(M(A) \times J'_1) = M(A') \times J''_1$ and the surfaces $M(A) \approx M(A')$ are orientable.

We recall that $F_{\lambda}(x'_1, x_2) = (x'_1, x'_2) \in J'_1 \times J''_1$ for $i, j = 1, \ldots, 6$. Let $\bar{x}_i \bar{x}_k \subseteq S_2 \setminus S_1$, where $i \neq j \neq k, i, j, k \in \{1, 2, 3\}$ and $\bar{x}_i \bar{x}_k \subseteq S_2 \setminus S_1$, where $i \neq j \neq k, i, j, k \in \{4, 5, 6\}$. The analogous arcs in $M(A')$ are denoted by $\bar{x}_i \bar{x}_k$. It suffices to show that all curves $F_{\lambda}(\bar{x}_i \bar{x}_k \times x_1) \cup \bar{x}_i \bar{x}_k \times x'_1$ are homologically trivial in $\partial (M(A') \times J''_1)$.
Let us observe that \( \partial(M(A) \times J') \) is homeomorphic to the union of two copies of \( M(A) \) with common boundary. From the Mayer-Vietoris exact sequence it follows that \( H_1(\partial(M(A) \times J')) \) has three types of generators. The first ones are given by generators lying in the common boundary of both copies of \( M(A) \), the second are given by the other generators of two copies of \( M(A) \), the third are taken from \( \im \delta \), where \( \delta: H_1(\partial(M(A) \times J')) \to H_2(\partial(M(A))) \) is the homomorphism from the Mayer-Vietoris exact sequence.

Let us suppose that the curve \( X' = F_{ad}(x_1, x_2, x_3, x_4) \cup x_1 x_2 x_3 x_4 \) is not homologically trivial in \( \partial(M(A) \times J') \). Let us consider the curves \( T_1 = \delta(x_1, x_2, x_3, x_4) \), \( T_2 = \delta(x_1, x_2, x_3, x_4) \), \( T_3 = \delta(x_1, x_2, x_3, x_4) \), and their images \( T_1' = F_{ad}(T_2) \), \( T_2' = F_{ad}(T_3) \).

The curves \( T_1' \) are contractible in \( N' = \partial(M(A) \times M(A')) \cup \text{Int}(M(A) \times J') \), so they are also homologically trivial in \( N' \). Let us denote \( X_1' = F_{ad}(x_1, x_2, x_3, x_4) \cup x_1 x_2 x_3 x_4, i = 1, 2, 3, \) and \( P_i' = F_{ad}(x_1, x_2, x_3, x_4) \cup x_1 x_2 x_3 x_4, Q_i' = F_{ad}(x_1, x_2, x_3, x_4) \cup x_1 x_2 x_3 x_4, K_i' = F_{ad}(x_1, x_2, x_3, x_4) \), \( i = 1, 2, 3 \).

Let us observe that

\[
\begin{align*}
[T_1'] &= [X_1'] + [P_1'] - [X_1] - [P_1], \\
[T_2'] &= [X_2'] + [Q_2'] - [X_2] - [Q_2], \\
[T_3'] &= [X_3'] + [K_3'] - [X_3] - [K_3],
\end{align*}
\]

where \([ \cdot ]\) denotes a homology class in \( N' \).

Since \( X_1' \subset M(A) \times J', P_1', Q_2', K_3' \subset M(A) \times J' \), it is easy to see that if \( T_1' \sim 0 \) in \( N' \) then \( X_1' \subset X_1 \), \( X_1' \subset X_1 \). It is also easy to see that if there is a generator of the third type in \( [X_1] \), then \( X_1' \) is not homologous to \( X_1' \) or \( X_1' \) in \( N' \).

Now, suppose that there is a generator of the first type in \( [X_1] \). It is a generator given by the curve \( S_1 \times x_1' \) then the curve \( T_1' \) is not contractible in \( N' \setminus (S_1 \times M(A')) \). So we can assume that there is an arc \( Z_1' \) joining the curves \( F_{ad}(x_1, x_2, x_3, x_4) \) and \( F_{ad}(x_1, x_2, x_3, x_4) \) and lying in \( F_{ad}(x_1, x_2, x_3, x_4) \cap (S_1 \times M(A')) \). The analogous arcs \( Z_2' \) joining \( F_{ad}(x_1, x_2, x_3, x_4) \) with \( F_{ad}(x_1, x_2, x_3, x_4) \) and \( Z_3' \) joining \( F_{ad}(x_1, x_2, x_3, x_4) \) with \( F_{ad}(x_1, x_2, x_3, x_4) \), lie in \( F_{ad}(x_1, x_2, x_3, x_4) \cap (S_1 \times M(A')) \) and \( F_{ad}(x_1, x_2, x_3, x_4) \cap (S_1 \times M(A')) \). The arc \( Z_i' \) divides the curve \( T_i' \) into two curves \( T_1' \) and \( T_1'' \). It is easy to see that at least two of the cycles given by the curves \( T_i' \) have a generator being an image by inclusion of a generator of the third type. So it is not homologically trivial in \( N' \). But \( T_i'' \) are contractible in \( N' \).

The proof of (3) is completed.

Proof of Lemma 6.2. We will show that the homeomorphisms \( f_{1, A}, f_{2, A} \) can be corrected in such a manner that condition \((+++)\) holds.

Let us choose orientations for all orientable manifolds \( M(A) \) and for components \( S \) of \( \partial M(A) \) for nonorientable manifolds \( M(A) \). Let us fix orientations of all orientable manifolds \( M(A), M(A') \) in such a manner that the homeomorphisms \( f_{1, A}, f_{2, A} \) preserve orientation. Let us fix orientations of all components \( S', S'' \) of \( \partial M(A), \partial M(A') \) for nonorientable \( M(A), M(A') \) in such a manner that the homeomorphisms \( f_{1, A}, f_{2, A} \) preserve orientations.

Condition \((+++\ast)\) may be formulated as follows:

If \( J \in \square K, J_1 \in \square S' \), \( J_2 \in \square S'' \), \( A, B \in \square K \times K \), then the homeomorphism \( h = (g_{A,B})^{-1} \cdot (g_{B,A}) : J_1 \to J_2 \) preserves orientation iff the homeomorphisms

\[ h' = (g_{1,B})^{-1} \cdot (g_{B,1}) : J_1 \to J_2 \]

and

\[ h'' = (g_{A,2})^{-1} \cdot (g_{2,A}) : J_1 \to J_2 \]

The proof will be divided into 3 cases:

1. \( A = B \).
2. \( A \cap B \neq \emptyset \).
3. The general case.

Case 1. On the manifold \( M(A) \) we define a relation

\[ x \sim y \iff \left\{ \begin{array}{ll}
\delta(x) = y, & \text{where } x \in J_1, y \in J_2 \text{ and } J_1, J_2 \in \square S' \text{ or } J_1, J_2 \in \square S'' \text{ or } J_1, J_2 \in \square K.
\end{array} \right. \]
Analogously, we define relations for the manifolds $M(A')$ and $M(A'')$:

$$x \sim y \iff \begin{cases} h'(x) = y, & \text{where } x \in J'_1, \ y \in J'_2 \text{ and } J'_1, J'_2 \subseteq \partial g_A^{(n/2)}(I'), I' \subseteq I \cup L, \\ x = y \end{cases}$$

$$x \sim y \iff \begin{cases} h''(x) = y, & \text{where } x \in J''_1, \ y \in J''_2 \text{ and } J''_1, J''_2 \subseteq \partial g_A^{(n/2)}(I''), I'' \subseteq I \cup L, \\ x = y \end{cases}$$

The spaces $M(A')_\sim$, $M(A'')_\sim$, $M(A')''_\sim$ are manifolds, too. By (ii) $F_{J_1 \times J_2} \circ (g_A \times g_A) = (g_A \times g_A) \circ F_{J_1 \times J_2}$, hence (iv)

$$M(A')_\sim \times M(A'')_\sim = M(A')_\sim \times M(A'')_\sim$$

If the manifold $M(A)$ is orientable, then $h$ preserves orientation iff so do $h'$ and $h''$, because $M(A)_\sim$ is orientable iff so are $M(A')_\sim$ and $M(A'')_\sim$. If the manifold $M(A)$ is nonorientable, we only consider arcs $J_1$, $J_2$ lying in one component $S$ of $\partial M(A)$ (Fig. 8).

![Fig. 8](image)

We observe that $\text{rank } H^1(M(A)_\sim) = \text{rank } H^1(M(A)_\sim) + 1$. If $h$ changes orientation, the number of elements of $\partial^1(M(A)_\sim)$ is greater by one than the number of elements of $\partial^1(M(A)_\sim)$. If $h$ does not change orientation, they are the same. Hence, if $h$ changes orientation, $\sigma(M(A)_\sim) = \sigma(M(A)_\sim)$, and if $h$ does not change orientation, $\sigma(M(A)_\sim) = \sigma(M(A)_\sim) + 1$ (see Def. 5.1). From (iv) and Lemma 5.1 it follows that $\sigma(M(A)_\sim)^2 = \sigma(M(A')_\sim) \sigma(M(A'')_\sim)$. By Lemma 5.2 the manifolds $M(A)$, $M(A')$, $M(A'')$ are homeomorphic.

If $\sigma(M(A)_\sim) = n > 0$, then $n^2 = (n+1)^2$ and the conclusion in this case holds. If $\sigma(M(A)_\sim) = 0$, then $\sigma(M(A')_\sim) = 0$ or $\sigma(M(A'')_\sim) = 0$. Let $\sigma(M(A')_\sim) = 0$ and $\sigma(M(A'')_\sim) = 1$. By Proposition 3.1 (essentially), $F^{-1}(A'' \times A') = A \times A_1$, where $A_1 \subseteq (R \times nK)$. By the formula $M(A'')_\sim \times M(A')_\sim = M(A')_\sim \times M(A)_\sim$ and by Lemma 5.1, $1 = \sigma(M(A''))^2 = \sigma(M(A')_\sim) \sigma(M(A)_\sim) = 0$, which is impossible. Hence $\sigma(M(A')_\sim) = \sigma(M(A)_\sim) = 0$.

The proof of case 1 is complete.

Case 2. Let $J_1 \subseteq \partial g_A^{(n/2)}(I)$, $J_2 \subseteq \partial g_A^{(n/2)}(I)$, $A \neq B$, $A, B \subseteq (R \times nK)$ and $h : J_1 \rightarrow J_2$ be given as above. Let us denote $M(A \cup B) = M(A) \cup M(B)$. In the same way we define $M(A' \cup B') = M(A') \cup M(B')$ and $M(A'' \cup B'') = M(A'') \cup M(B'')$. Let us observe that they are manifolds.

We will consider the following cases:

(a) Assume that the manifold $M(A)$ (or $M(B)$) is nonorientable, $J_1 \subseteq S_1 \subseteq \partial M(A)$ and at most two components of $g_A^{(n/2)}(K \cap A)$ lie in $S_1$. Then these components do not fix an orientation of $S_1$. Hence we can correct the homeomorphisms $f_{J_1 \times J_2}$, $f_{J_2 \times A}$, in such a manner that conditions $(\ast)$ and $(\ast \ast)$ for $J_1$, $J_2$ hold.

(b) $M(A)$ is orientable and in no component $S$ of $\partial M(A)$ there are more than two components of $g_A^{(n/2)}(K \cap A)$. Then the homeomorphisms $f_{J_1 \times J_2}$, $f_{J_2 \times A}$ may be corrected.

(c) Let $J_1$, $J_2$, $J_3$, $J_4 \subseteq S_1 \subseteq \partial M(A)$, $J_2$, $J_4$, $J_6 \subseteq S_2 \subseteq \partial M(B)$ (Fig. 9).

![Fig. 9](image)

If the homeomorphism $h : J_1 \rightarrow J_2$ does not change orientation and the homeomorphism $h' : J_2 \rightarrow J_3$ changes orientation, then the ordering of the arcs $J_3$, $J_4$, $J_5$, $J_6$ on a component of $\partial M(A \cup B)$ and the ordering of the arcs $J_2$, $J_4$, $J_5$, $J_6$ on a component of $\partial M(A' \cup B')$ are different.

By (ii) and (iii) there exists a homeomorphism $F_{A \cup B, A \cup B'} : M(A \cup B) \times M(A \cup B') \rightarrow M(A' \cup B') \times M(A'' \cup B'')$ such that

$$F_{A \cup B, A \cup B'}(J_1 \times M(A \cup B)) = J_1 \times M(A'' \cup B'')$$

and $F_{A \cup B, A \cup B'}(M(A \cup B) \times J_1) = M(A' \cup B') \times J_3$.

Using the manifold $M(A \cup B)$ a reasoning similar to that of (2) in the proof of Lemma 6.1, we obtain a contradiction.

(d) Let the manifolds $M(A)$ and $M(B)$ be orientable, $J_1 \subseteq S_1 \subseteq \partial M(A)$,
$J_2 \subset S_3 \subset \partial M(B)$, $J_2 \subset S_3 \subset \partial M(A)$ and $J_2 \subset S_3 \subset \partial M(B)$. If $S_3 = S_4$ and $S_2 = S_4$, the proof of (d) is the same as that of (c).

We will consider the case when $S_1 \neq S_2$ and $S_1 \neq S_4$ (Fig. 10).

Fig. 10

Let us observe that the manifolds $M(A \cup B)$, $M(A' \cup B')$ and $M(A'' \cup B'')$ are orientable. If the homeomorphism $h: J_1 \rightarrow J_2$ does not change orientation and the homeomorphism $k: J'_1 \rightarrow J'_2$ changes orientation, then the ordering of the arcs $J_1, J_4, J_3 \subset S_2$ and $J_1, J_4, J_3 \subset S_4$ agrees with the orientation, and the ordering of the arcs $J_5, J_6, J_7 \subset S_3$ or $J_5, J_6, J_7 \subset S_6$ does not agree with the orientation.

Using the manifold $N(A \cup B)$ a reasoning similar to that of (3) in the proof of Lemma 6.1, we obtain a contradiction.

Similarly for the cases $S_1 = S_2$ and $S_1 = S_4$, or $S_1 = S_3$ and $S_3 = S_4$.

e) If we match next arcs $J_2 \subset \partial M(A)$ and $J_2 \subset \partial M(B)$, we can reason as in case 1 considering the manifold $N(A \cup B)$.

Case 3. For any $A, B \subset \partial(K \cap K')$, there exists a sequence of components $A = A_1, A_2, \ldots, A_n = B$ of the set $K \cap K$ such that $A_i \cap A_{i+1} \neq \emptyset$ for $i = 1, 2, \ldots, n-1$. By (c) and induction it suffices to consider the case when $A \cap B \neq \emptyset$, which is given in (2).

Therefore, the proof of Lemma 6.2 is complete.

Lemmas 6.1 and 6.2 complete the proof of Proposition 6.1. Propositions 3.1, 4.1 and 6.1 imply Theorem A.

References