On a problem of S. Ulam concerning Cartesian squares of 2-dimensional polyhedra

by

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Abstract. This paper contains the proof of the following theorem: *If K and L are compact connected 2-polyhedra and their Cartesian squares K×K and L×L are homeomorphic, then K and L are homeomorphic.*

1. Introduction. In 1933 the following problem was posed by S. Ulam [17]:

"Assume that A and B are topological spaces and $A^2 = A\times A$ and $B^2 = B\times B$ are homeomorphic. Is it true that A and B are homeomorphic?"

In general this problem has the negative answer. Let Q denote the Hilbert cube, A_k the disjoint union of the set $\{1, 2, \ldots, k\}\times Q$ and the set of natural numbers N and A_0 the disjoint union of the sets $N\times Q$ and N. The spaces A_i, A_j are not homeomorphic for $i \neq j$, $i, j = 0, 1, 2, \ldots$, but $A_i^2 \approx A_0$ for $i = 0, 1, 2, \ldots$ hence $A_i^2 \approx A_j^2$ for $i, j = 0, 1, 2, \ldots$ The problem does not have a trivial answer when the spaces A and B are compact or connected. But in this case the answer is negative, too. In 1947 R. H. Fox [7] gave an example of two nonhomeomorphic compact 4-manifolds the Cartesian squares of which are homeomorphic. In 1960 J. Glimm [9] gave an example with open 3-manifolds. Other examples can be found in: D. R. McMillan Jr. [14], K. W. Kwun [12], K. W. Kwun and F. Raymond [13], A. J. Boals [1], Z. Čerin [5], H. Toruńczyk [16].

However, the problem considered has the positive answer for 2-manifolds. This simple fact was proved in [7].

The more general problem of the uniqueness of the decomposition of finite-dimensional compacta into Cartesian product was considered by several authors [2], [15], [8], [4]. It was proved that this problem has the positive answer if the factors are 1-dimensional locally connected continua. If the factors are 2-polyhedra or bounded 2-manifolds, then the uniqueness of the decomposition does not hold.

We prove that the Ulam problem has the positive answer for compact connected 2-dimensional polyhedra, that is:

THEOREM A. *If K and L are compact connected 2-dimensional polyhedra such that K^2 and L^2 are homeomorphic, then K and L are homeomorphic.*

A space A is said to be a *Cartesian root* of X if X and $A\times A$ are homeomorphic. Thus we can formulate a somewhat more general version of Theorem A:

THEOREM B. *A compact connected 4-dimensional polyhedron X has at most one Cartesian root.*

This version is in fact equivalent to Theorem A.

Proof. We assume that $X\approx K^2\approx L^2$. Since either $\dim K^2 = 2\dim K$ or $\dim K^2 = 2\dim K-1$ (see [11], p. 18), and $\dim K^2 = 4$, we have $\dim K = 2$. A. Kosiński proved that any 2-dimensional factor of a polyhedron is a polyhedron [10]. Thus the spaces K and L are 2-polyhedra and Theorem A implies Theorem B. The fact that Theorem B implies Theorem A is obvious.

In the case $\dim X = 2n$ ($n\geqslant 3$) Theorem B is not true. Let A denote the Cartesian product of n circles S^1 ($n\geqslant 3$) and let J be the wild arc of Blankinship [3] in a cell contained in A. We denote the space A/J by B. The spaces A and B are not homeomorphic, but the Cartesian squares A^2 and B^2 are homeomorphic. The proof of this fact is analogous to the proof of Kwun's theorem [12].

An example of compact nonhomeomorphic 3-polyhedra A and B such that A^2 and B^2 are homeomorphic is not known. If there exists a 3-dimensional Poincaré fake cell (a 3-dimensional compact contractible manifold not homeomorphic to I^3 with the boundary equal to S^2), then — since its Cartesian square is a 6-cell — such an example would exist.

2. Outline of proof of Theorem A. First, we define some subsets of non-Euclidean points of a polyhedron P.

DEFINITION 2.1. If P is a k-dimensional polyhedron, then we define inductively the sets n_iP for $i = 0, 1, \ldots, k$.

(i) $n_0P = P$

(ii) n_iP denotes the subset of $n_{i-1}P$ consisting of the points which have no neighborhood homeomorphic to R^{k-i+1} or R_+^{k-i+1} in the set $n_{i-1}P$.

We denote the set n_1P by nP.

Remark. It is easy to see that every set n_iP is a polyhedron and $\dim n_iP\leqslant k-i$.

Now we present

Outline of proof of Theorem A: The proof of Theorem A is divided into three propositions. To prove these propositions we need some lemmas. It is assumed that K and L are compact connected 2-polyhedra such that $K^2\approx L^2$.

In the first proposition (Prop. 3.1) we consider the case where certain isolated points are distinguished in the polyhedron K. In Lemma 3.1 we study the structure of the polyhedron K^2 and we obtain the formulas:

$$n_i(K^2) = \bigcup \{n_pK\times n_qK:\ p+q = i,\ p, q\in\{0, 1, 2\}\}.$$

In Lemma 3.3 we prove, using the technical Lemma 3.2, that if K has a local cut point then $K\approx L$. Next, using Lemmas 3.1–3.3 and Borsuk's theorem [2] on the uniqueness of the decomposition into Cartesian product of 1-polyhedra we prove that if the condition $K\approx L$ does not hold then:

(*) each component X of the set nK is either an arc or a simple closed curve.

We also prove that if $K\not\approx L$ then:

(**) for every $x\in nK$ there are $n\in N$ and a neighborhood V of x in K such that $V\approx [0,1]\times \operatorname{cone}\{1, 2, \ldots, n\}$.

Let $F\colon K^2\to L^2$ be a fixed homeomorphism. If $K\not\approx L$ then we prove, using Lemma 3.2, that there exists a one-to-one correspondence $A\leftrightarrow A'$ between the components of $K\setminus nK$ and $L\setminus nL$ such that $F(K\times nK) = L\times nL$ and $F(K\times A) = L\times A'$, or $F(K\times nK) = nL\times L$ and $F(K\times A) = A'\times L$.

For every component A of the set $K\setminus nK$ we define some 2-manifold $M(A)$. This manifold is homeomorphic to the set A minus some open regular neighborhood of nK. The polyhedron K is built up of the manifolds $M(A)$.

In the next proposition (4.2) we prove that if conditions (*) and (**) hold and $K\times S^1\approx L\times S^1$ then $K\approx L$. Hence if $K^2\approx L^2$ and $K\not\approx L$ then all components of the set nK are arcs. The polyhedron K is the union of the manifolds $M(A)$ such that their intersections are the arcs lying in the boundaries of $M(A)$.

Let us notice that if J is an arc which is a component of the set nK, then its end-points are not distinguished by the stratification given in Definition 2.1. Let the end-point x_0 have an open neighborhood homeomorphic to the set $T_n\times[0,1)$ where $T_n = \{(t, it):\ i = 1, \ldots, n,\ t\in[0,1)\}$ and let $x\in J$ have an open neighborhood homeomorphic to $T_n\times(0,1)$. It is obvious that $T_n\times[0,1)\times R^2\approx T_n\times[0,1)\times R_+^2\approx T_n\times(0,1)\times R_+^2$. So if the end-point x_0 distinguished, the formula from Lemma 3.1 would not hold.

Now, we cannot use the methods similar to those used in the proof of the Proposition 3.1. We prove in Section 5 that if $A\leftrightarrow A'$ is the one-to-one correspondence between the components of $K\setminus nK$ and $L\setminus nL$, then the manifolds $M(A)$ and $M(A')$ are homeomorphic. In the last section we prove that the manifolds $M(A)$ are stuck to the set nK in K on the same way as the manifolds $M(A')$ are stuck to the set nL in L. So homeomorphisms between the manifolds $M(A)$ and $M(A')$ yield a homeomorphism $f\colon K\to L$. This part of the proof is the most complicated one.

3. Investigation of the non-Euclidean part nK of K. This section contains the proof of

PROPOSITION 3.1. *If K and L are compact, connected 2-polyhedra, $F\colon K^2\to L^2$ is a homeomorphism and $K\not\approx L$, then*

(*) *each component X of nK is either an arc or a simple closed curve,*

(**) *for each $x\in nK$ there are $n\in N$ and a neighborhood V of x in K such that $V\approx[0,1]\times \mathrm{cone}\{1, 2, \ldots, n\}$,*

(***) *$F(nK\times K) = nL\times L$ or $F(nK\times K) = L\times nL$,*

(****) *either for each component A of the set $K\setminus nK$ there exists a component A' of the set $L\setminus nL$ such that $F(A\times K) = A'\times L$ or* *for each component A of the set $K\setminus nK$ there exists a component A' of the set $L\setminus nL$ such that $F(A\times K) = L\times A'$.*

First, we prove some lemmas.

LEMMA 3.1. *If K is a 2-polyhedron, then*

$$n_i(K^2) = \bigcup \{n_p K\times n_q K\colon p+q = i,\ p, q\in\{0,1,2\}\}$$

Proof. We should consider the cases $i = 1, 2, 3, 4$.

(a) The case $i = 1$.

We should show that $x\in K^2$ has a Euclidean neighborhood iff $x\in K^2\setminus[(nK\times K)\cup(K\times nK)] = (K\setminus nK)\times(K\setminus nK)$, i.e. x belongs to the product of Euclidean parts of K. The proof is easy and the details are left to the reader.

(b) The case $i = 2$.

We have $[(K\times nK)\cup(nK\times K)]\setminus[(K\times n_2K)\cup(nK\times nK)\cup(n_2K\times K)] = [(K\setminus nK)\times(nK\setminus n_2K)]\cup[(nK\setminus n_2K)\times(K\setminus nK)]$, so this set consists of Euclidean points, and therefore $n_2(K^2)\subset(K\times n_2K)\cup(nK\times nK)\cup(n_2K\times K)$.

We will prove that if the point x belongs to the set $(K\times n_2K)\cup(nK\times nK)\cup(n_2K\times K)$, then $x\in n_2(K^2)$. Let $x = (x_1, x_2)$. If x_i ($i = 1$ or $i = 2$) is an isolated point or $\dim_{x_i} K = 1$ for both points x_1 and x_2, then a sufficiently small neighborhood of x in K^2 has dimension less than 3 and therefore $x\in n_2(K^2)$. Now we consider the remaining cases.

Let $(x_1, x_2)\in K\times n_2K$. If x_2 is isolated in nK then it easy to see that x_2 is a local cut point of K. Therefore a sufficiently small neighborhood of $x = (x_1, x_2)$ in $n(K^2)$ either has dimension less than 3 or there is an arc cutting this neighborhood into disjoint components. Hence $x\in n_2(K^2)$. Let x_2 be not isolated in nK. If $\dim_{x_1} K = 2$ then each neighborhood of (x_1, x_2) in $K\times nK$ contains the set homeomorphic to $I^2\times T$, where T is a triod and I is an arc. If $\dim_{x_2} K = 1$ then $x_1\in nK$ and $\dim_{x_1} K = 2$. Therefore, each neighborhood of (x_1, x_2) in $nK\times K$ contains a set homeomorphic to $I\times(T\times I)$. Since $I^2\times T$ is not embeddable into R^3, the point x does not have an Euclidean neighborhood in $n(K^2)$ and $K\times n_2K\subset n_2(K^2)$.

Similarly $n_2K\times K\subset n_2(K^2)$.

Let $(x_1, x_2)\in nK\times nK$ and $x_i\notin n_2K$ for $i = 1, 2$. If $\dim_{x_1} K = 2$ then each neighborhood of (x_1, x_2) in $K\times nK$ contains a set homeomorphic to $(T\times I)\times I$. This set is not embeddable in R^3; hence $(x_1, x_2)\in n_2(K^2)$. Hence, $nK\times nK\subset n_2(K^2)$.

(c) The cases $i = 3, 4$ can be proved using similar elementary considerations. These proofs are left to the reader.

DEFINITION 3.1. The collection of components of a set X will be denoted by $\square X$.

LEMMA 3.2. *Suppose DK and DL are nowhere dense subpolyhedra of compact connected 2-polyhedra K and L, respectively, and $F\colon K^2\to L^2$ is a homeomorphism such that*

(i) $F\big((K\times DK)\cup(DK\times K)\big) = (L\times DL)\cup(DL\times L)$,

(ii) $F(DK\times DK) = DL\times DL$.

Then $F(DK\times K) = DL\times L$ or $F(DK\times K) = L\times DL$.

Proof. By (i) and (ii) we obtain $F\big([(K\setminus DK)\times DK]\cup[DK\times(K\setminus DK)]\big) = [(L\setminus DL)\times DL]\cup[DL\times(L\setminus DL)]$. Therefore, if $A\in\square(K\setminus DK)$ and $X\in\square DK$, then $F(A\times X) = A'\times X'$ or $F(A\times X) = X'\times A'$, where $A'\in\square(L\setminus DL)$ and $X'\in\square DL$. Assume $F(A\times X) = A'\times X'$. Let B be another component of $K\setminus DK$ such that $\bar{A}\cap\bar{B}\neq\emptyset$. Suppose $F(B\times X) = Y'\times B'$, where $Y'\in\square DL$ and $B'\in\square(L\setminus DL)$. Since $\overline{A'\times X'}\cap\overline{Y'\times B'}\neq\emptyset$, there exist points $x'\in\bar{B}'\cap\bar{X}'$ and $y'\in\bar{Y}'\cap\bar{A}'$. If $b'\in B'$ and $a'\in A'$ then there exist arcs $\widehat{x'b'}$ and $\widehat{a'y'}$ such that their interiors are contained in B and A, respectively, hence, there exists an arc $\widehat{(a',x')(y',b')}$ such that its interior lies in $A'\times B'$, i.e. this interior is disjoint with the set $(L\times DL)\cup(DL\times L)$. We consider the arc $\widehat{(a,x)(b,y)} = F^{-1}\big(\widehat{(a',x')(y',b')}\big)$. Then $a\in A$, $b\in B$ and $x, y\in X$. But if $a\in A$, $b\in B$ and $x, y\in X$, then the interior of any arc $\widehat{(a,x)(b,y)}\subset K^2$ and the set $(K\times DK)\cup(DK\times K)$ are not disjoint. Hence, $F(B\times X) = B'\times Y'$. Since $\overline{A'\times X'}\cap\overline{B'\times Y'}\neq\emptyset$, X', Y' are closed in L and $X', Y'\in\square DL$, we have $X' = Y'$.

The polyhedron K is connected, the set DK is nowhere dense, thus for any component B of the set $K\setminus DK$ there exists a sequence of components $A = A_1, A_2, \ldots, A_n = B$ of the set $K\setminus DK$ such that $\bar{A}_i\cap\bar{A}_{i+1}\neq\emptyset$ for $i = 1, 2, \ldots, n-1$. Therefore, $F(B\times X) = B'\times X'$ for any $B\in\square(K\setminus DK)$ and $F(K\times X) = L\times X'$. Hence $F(K\times DK) = L\times DL$.

LEMMA 3.3. *Suppose K and L are compact connected 2-polyhedra, $F\colon K^2\to L^2$ is a homeomorphism and K has local cut points. Then K and L are homeomorphic.*

Proof. First we consider the case where there exists a point x of K such that $\dim_x K = 1$. We denote $P = \{x\in K\colon \dim_x K = 2\}$, $R = \{x\in K\colon \dim_x K = 1\}$, $DK = P\cap\bar{R}$, $P' = \{x\in L\colon \dim_x L = 2\}$, $R' = \{x\in L\colon \dim_x I = 1\}$ and $DL = P'\cap\bar{R}'$. The sets R and $D'K'$ are not empty. Observe that $K^2 = (P\times P)\cup(P\times R)\cup(R\times P)\cup(R\times R)$, $L^2 = (P'\times P')\cup(P'\times R')\cup(R'\times P')\cup(R'\times R')$ and $F(P\times P) = P'\times P'$, $F\big((P\times R)\cup(R\times P)\big) = (P'\times R')\cup(R'\times P')$ and $F(R\times R) = R'\times R'$, because homeomorphism preserves the local dimension. Hence $F\big((P\times DK)\cup(DK\times P)\big) = F\big((P\times P)\cap(\overline{P\times R}\cup\overline{R\times P})\big) = (P'\times P')\cap(\overline{P'\times R'}\cup\overline{R'\times P'}) = (P'\times DL)\cup(DL\times P')$ and $F\big((R\times DK)\cup(DK\times R)\big) = F\big((R\times R)\cap(\overline{P\times R}\cup\overline{R\times P})\big) = (R'\times DL)\cup(DL\times R')$. Therefore, $F\big((K\times DK)\cup(DK\times K)\big) = (L\times DL)\cup(DL\times L)$. Also $F(DK\times DK) = F\big((P\times P)\cap\overline{R\times R}\big) = (P'\times P')\cap\overline{R'\times R'} = DL\times DL$.

The sets DK and DL are finite, hence $F(K\times DK) = L\times DL$ or $F(K\times DK) = DL\times L$ and $K\approx L$.

Now, we consider the case $\dim_x K = 2$ for every $x\in K$. Then the set of local cut points of K is finite. Let DK denote this set. It is easy to see that the set $(DK\times K)\cup(K\times DK)$ is the set of such points $(x, y)\in K^2$ that there exists a neighborhood U of (x, y) such that for every connected neighborhood V of (x, y) contained in U there exists a 2-dimensional set separating the set V. Therefore $F\big((K\times DK)\cup(DK\times K)\big) = (L\times DL)\cup(DL\times L)$. The set $DK\times DK$ is the set of local cut points of the set $(K\times DK)\cup(DK\times K)$. Thus $F(DK\times DK)=DL\times DL$ and by Lemma 3.2 we obtain $K\approx L$.

LEMMA 3.4. *If K, L are compact connected 2-polyhedra, $F\colon K^2\to L^2$ is a homeomorphism and $n_2K\neq\emptyset$, then $K\approx L$.*

Proof. Because $F(n_2K\setminus n_3K)=n_2L\setminus n_3L$ we obtain

$$F\big([(K\setminus nK)\times n_2K]\cup[(nK\setminus n_2K)\times(nK\setminus n_2K)]\cup[n_2K\times(K\setminus nK)]\big)$$
$$=[(L\setminus nL)\times n_2L]\cup[(nL\setminus n_2L)\times(nL\setminus n_2L)]\cup[n_2L\times(L\setminus nL)].$$

Therefore if $A\in\square(K\setminus nK)$ and $x\in n_2K$, then $F(A\times x)=A'\times x'$, or $F(A\times x)=x'\times A'$, or $F(A\times x)=X'\times Y'$, where $A'\in\square(L\setminus nL)$, $x'\in n_2L$ and $X', Y'\in\square(nL\setminus n_2L)$.

Suppose there exist such $A\in\square(K\setminus nK)$ and $x\in n_2K$ that $F(A\times x)=A'\times x$.' We shall examine whether it is possible that $F(B\times x)=U'\times V'$ or $F(B\times x)=x'\times B'$ for another $B\in\square(K\setminus nK)$, where $U', V'\in\square(nL\setminus n_2L)$, $x'\in n_2L$, $B'\in\square(L\setminus nL)$. By Lemma 3.3 we may assume that the polyhedron K has no local cut point. Hence there exists a sequence of components $A=A_1, A_2, \ldots, A_n=B$ of the set $K\setminus nK$ such that $\dim\bar{A}_i\cap\bar{A}_{i+1}=1$ for $i=1,2,\ldots,n-1$. Therefore, it is enough to consider the case when $\dim\bar{A}\cap\bar{B}=1$.

Since the set $\overline{A'\times x'}\cap\overline{y'\times B'}$ for any $y'\in n_2L$, $B'\in\square(L\setminus nL)$ contains at most one point and $\overline{A\times x}\cap\overline{B\times x}$ has dimension 1, it is not true that $F(B\times x)=y'\times B'$. If $F(B\times x)=U'\times V'$, where $U', V'\in\square(nL\setminus n_2L)$, then $\bar{A}'\cap U'\neq\emptyset$ and $x'\in\bar{V}'$. Let $u'\in\bar{A}'\cap U'$. There exists a point $a'\in A'$ such that we can join a' with u' by an arc whose interior lies in A'. There exists a point $v'\in V'$ such that we can join x' with v' by an arc whose interior lies in some component C' of $L\setminus nL$. Hence, there exists an arc $\widehat{(a',x')(u',v')}$ whose interior lies in $A'\times C'$, and so is disjoint with the set $n(L^2)$. This is impossible because $(a,x)=F^{-1}(a',x')\in A\times x$, $(b,x)=F^{-1}(u',v')\in B\times x$ and the interior of every arc $\widehat{(a,x)(b,x)}$ lying in K^2 does not have empty intersection with $n(K^2)$. Hence, $F(K\times x)\subset L\times x'$. Similarly $F^{-1}(L\times x')\subset K\times x$; hence $F(K\times x)=L\times x'$.

We consider the case $F\big((K\times n_2K)\cup(n_2K\times K)\big)\subset nL\times nL$. Let $F(A\times x)=X'\times Y'$ where $A\in\square(K\setminus nK)$, $x\in n_2K$ and $X', Y'\in\square(nL\setminus n_2L)$. If $B\in\square(K\setminus nK)$, then $F(B\times x)=U'\times V'$, where $U', V'\in\square(nL\setminus n_2L)$.

For the components of the set $nL\setminus n_2L$ we define a relation: $X'\sim U'$ iff there is a sequence $X'=X'_0, X'_1, \ldots, X'_k=U'$ of the components of the set $nL\setminus n_2L$ such that for all $i=0,1,\ldots,k-1$, $\bar{X}'_i\cap\bar{X}'_{i+1}\neq\emptyset$ (may be $X'_i=X'_{i+1}$) and if $X'_i\neq X'_{i+1}$ then no component C' of the set $L\setminus nL$ satisfies $X'_i\cap\bar{C}'\neq\emptyset$ and $X'_{i+1}\cap\bar{C}'\neq\emptyset$.

We will prove that there exists a component $B\in\square(K\setminus nK)$ such that $F(B\times x)=U'\times V'$ iff $X'\sim U'$ and $Y'\sim V'$. We define $P'=\bigcup\{U'\colon U'\sim X'\}$ and $R'=\bigcup\{V'\colon V'\sim Y'\}$ and note that $F(K\times x)=P'\times R'$.

Now we show that if $F(B\times x)=U'\times V'$ then $U'\sim X'$ and $V'\sim Y'$. Since we may assume that K does not have local cut points and the relation "$\sim$" is transitive, we may assume that $\dim\bar{A}\cap\bar{B}=1$. We note that $F\big((\bar{A}\cap\bar{B})\times x\big)=(\bar{X}'\cap\bar{U}')\times(\bar{Y}'\cap\bar{V}')$, i.e. the set $(\bar{X}'\cap\bar{U}')\times(\bar{Y}'\cap\bar{V}')$ has dimension 1, hence either $X'=U'$ or $Y'=V'$. Assume that $X'=U'$. It is obvious that $\bar{Y}'\cap\bar{V}'\neq\emptyset$ and $Y'\neq V'$. It is easy to see that the interior of every arc which lies in K^2 with end-points in $A\times x$ and $B\times x$ has nonempty intersection with the set $n(K^2)$. If there is a component C' of the set $L\setminus nL$ such that $Y'\cap\bar{C}'\neq\emptyset$ and $V'\cap\bar{C}'\neq\emptyset$, then there is an arc having its interior in C' and its end-points in Y' and V', respectively. Hence, there is an arc lying in L^2 with end-points in $X'\times Y'$ and $U'\times V'=X'\times V'$ such that its interior is disjoint with the set $n(L^2)$.

Let $X'\sim U'$ and $Y'\sim V'$. It is enough to consider the case where $X'=U'$ and $\bar{Y}'\cap\bar{V}'\neq\emptyset$. We have assumed that $F^{-1}(X'\times Y')=A\times x$ and we know that $F^{-1}(U'\times V')\neq b\times B$, because $\dim\overline{X'\times Y'}\cap\overline{U'\times V'}=1$. If $F^{-1}(U'\times V')=U\times V$, where $U, V\in\square(nK\setminus n_2K)$, then there exist points $(a,x)\in A\times x$ and $(u,v)\in U\times V$, and an arc $\widehat{(a,x)(u,v)}\subset K\times K$ such that its interior is disjoint with the set $n(K^2)$. Since $F(a,x)\in X'\times Y'$, $F(u,v)\in U'\times V'$ and no component C' of the set $L\setminus nL$ satisfies $Y'\cap\bar{C}'\neq\emptyset$ and $V'\cap\bar{C}'\neq\emptyset$, the interior of each arc joining $F(a,x)$ with $F(u,v)$ in L^2 does not have an empty intersection with $n(L^2)$. Hence $F^{-1}(U'\times V')=B\times x$.

Hence $K\approx L$ or $K\approx P'\times R'$ and $L\approx P\times R$, where P, R, P', R' are graphs. If $K\approx P'\times R'$, $L\approx P\times R$ and $K^2\approx L^2$, then $K\approx L$ by Borsuk's theorem [2] on the uniqueness of the decomposition into Cartesian product of 1-polyhedra.

Proof of Proposition 3.1. By Lemma 3.4 the set n_2K is empty, hence condition $(*)$ holds. By Lemma 3.1, $n(K^2)=(K\times nK)\cup(nK\times K)$ and if $n_2K=\emptyset$ then $n_2(K^2)=nK\times nK$. Hence, $F\big((K\times nK)\cup(nK\times K)\big)=(L\times nL)\cup(nL\times L)$ and $F(nK\times nK)=nL\times nL$. By Lemma 3.3 the polyhedron K does not have local cut points. Hence, the set nK is nowhere dense in K. Analogously nL is nowhere dense in L. Lemma 3.2 implies $({*}{*}{*})$.

If $F(nK\times K)=nL\times L$ then $F\big((K\setminus nK)\times K\big)=(L\setminus nL)\times L$. Hence for each component A of the set $K\setminus nK$ there exists a component A' of the set $L\setminus nL$ such that $F(A\times K)=A'\times L$. Hence condition $\binom{**}{**}$ holds.

Let DK denote the set of points $x\in nK$ such that there do not exist $n\in N$ and a neighborhood V of x in K such that $V\approx[0,1]\times\operatorname{cone}\{1,\ldots,n\}$. The corresponding subset of nL is denoted by DL. Since K does not have local cut points, $\dim_xK=2$ for $x\in K$ and the sets DK and DL are finite. Since $F(K\times nK)=L\times nL$, we have $F\big((K\setminus nK)\times nK\big)=(L\setminus nL)\times nL$. The point $(x,y)\in(K\setminus nK)\times nK$ belongs to the set $(K\setminus nK)\times(nK\setminus DK)$ iff (x,y) has a neighborhood homeomorphic to the set $I^3\times\operatorname{cone}\{1,2,\ldots,n\}$ for some $n\in N$. Therefore, $F\big((K\setminus nK)\times DK\big)=(L\setminus nL)\times DL$. The sets nK and nL are nowhere dense in K and L, hence $F(K\times DK)=L\times DL$. If $DK\neq\emptyset$ then $K\approx L$. Hence condition $({*}{*})$ holds.

Remark. By $\binom{*}{**}$ we have one-to-one correspondences $A\leftrightarrow A'$ and $A\leftrightarrow A''$

between components of $K\setminus nK$ and $L\setminus nL$ such that:

$$F(A\times K) = A'\times L, F(K\times A) = L\times A'' \quad \text{and} \quad F(A\times B) = A'\times B''.$$

Remark. Condition $(\substack{**\\**})$ is true without the assumption $K\approx L$, but we shall not use this and omit the proof.

4. Cancelling an S^1-factor. In this section we will prove

PROPOSITION 4.1. *If K is a compact connected 2-polyhedron, $K^2\approx L^2$ and the set nK contains a simple closed curve then $K\approx L$.*

If one of the conditions $(*)$–$(\substack{**\\**})$ from Proposition 3.1 does not hold then $K\approx L$. We can consider the case when $(*)$–$(\substack{**\\**})$ hold. By $(\substack{*\\**})$, if $F\colon K^2\to L^2$ is a homeomorphism then $F(K\times nK) = L\times nL$ (or $F(K\times nK) = nL\times L$). Hence, if S is a component of nK homeomorphic to a simple closed curve then $F(K\times S) = L\times S'$ where $S'\in\square nL$. If S' is an arc then $K^2\times(S^1\times S^1)\approx L^2\times(I\times I)\underset{\text{h.e.}}{\simeq} K^2$. Hence S' is also a simple closed curve. Therefore, it is enough to prove

PROPOSITION 4.2. *If K is a compact connected 2-polyhedron which satisfies $(*)$ and $(**)$ of Prop. 3.1 and $K\times S^1\approx L\times S^1$ then $K\approx L$.*

Remark. Possibly, for every 2-polyhedron K, if $K\times S^1\approx L\times S^1$ then $K\approx L$. Nevertheless, we only consider the above-mentioned special case.

Before we will prove Proposition 4.2 we give a definition of manifolds $M(A)$ and some properties of these manifolds. Let K be 2-polyhedron and $U(nK)$ denote the regular neighborhood of the polyhedron nK in the polyhedron K. Then we can define $M(A)$ as the set $\overline{A\setminus U(nK)}$. For technical reasons, in the sequel we shall use another definition of $M(A)$.

DEFINITION 4.1. (1) We denote by $N(A)$ the set of all sequences $\{x_n\}$ in A which converge in K and are such that for every neighborhood U of the point $\lim x_n$ in K there exists $U_0\in\square(U\setminus nK)$ and a natural number n_0 such that for every $n>n_0$ we have $x_n\in U_0$.

(2) In the set $N(A)$ we define the equivalence relation "$\sim$". We have $\{x_n\}\sim\{y_n\}$ iff

(i) $\lim x_n = \lim y_n = x_0$ in K

(ii) for every neighborhood U of x_0 in K there exist $U_0\in\square(U\setminus nK)$ and a natural number n_0 such that for every $r>n_0$ we have $x_n\in U_0$ and $y_n\in U_0$.

(3) By $M(A)$ we denote the set $N(A)/\sim$.

(4) We define a basis for the topology of $M(A)$. Let $[\{x_n^0\}]\in M(A)$ and $\lim x_n^0 = x^0$. Let U be a neighborhood of the point x^0 in K and let U_0 denote the component of the set $U\setminus nK$ such that for almost all n we have $x_n^0\in U_0$. We denote by $V(U,[\{x_n^0\}])$ the set of $[\{x_n\}]\in M(A)$ such that $\lim x_n\in U$ and for almost all n, $x_n\in U_0$. The collection of the sets $V(U,[\{x_n^0\}])$ is a basis for the topology of $M(A)$.

The first definition is simpler then the second, but if we use the second definition, then the following properties are very simple.

Property 4.1. If $\lim x_n = x$ in A, then $[\{x_n\}] = [\{x\}]$ (where $\{x\}$ is the constant sequence).

Property 4.2. The function $h_A\colon A\to M(A)$ given by $h_A(x) = [\{x\}]$ is a topological embedding.

Property 4.3. If condition $(**)$ from Proposition 3.1 holds then the space $M(A)$ is a compact 2-manifold and $M(A)\setminus h_A(A)\subset\partial M(A)$.

Property 4.4. If condition $(**)$ from Proposition 3.1 holds and $g_A\colon M(A)\to\bar{A}$ is given by the formula $g_A([\{x_n\}]) = \lim x_n$, then $g_A|_{g_A^{-1}(nK\cap\bar{A})}$ is a covering.

Property 4.5. Let K and L be 2-polyhedra and let $G\colon K\times S^1\to L\times S^1$ be a homeomorphism. There exists a homeomorphism $G_A\colon M(A)\times S^1\to M(A')\times S^1$ such that $(g_{A'}\times\mathrm{id}_{S^1})\circ G_A = G|_{\bar{A}\times S^1}\circ(g_A\times\mathrm{id}_{S^1})$.

Remark. Denote by $P_1\colon A'\times S^1\to A'$ and $P_2\colon A'\times S^1\to S^1$ the projections on the first and the second factor respectively. The homeomorphism G_A is given by the formula:

$$G_A([\{x_n\}], t) = \big([\{P_1 G(x_n, t)\}], P_2(\lim G(x_n, t))\big).$$

Property 4.6. Let K and L be 2-polyhedra and let $F\colon K\times K\to L\times L$ be a homeomorphism. Let $F(A\times B) = A'\times B'$, where $A, B\in\square(K\setminus nK)$ and $A', B'\in\square(L\setminus nL)$. There exists a homeomorphism $F_{A,B}\colon M(A)\times M(B)\to M(A')\times M(B')$ such that $(g_{A'}\times g_{B'})\circ F_{A,B} = F|_{\bar{A}\times\bar{B}}\circ(g_A\times g_B)$.

Remark. Denote by $P_1\colon A'\times B'\to A'$ and $P_2\colon A'\times B'\to B'$ the projections on the first and the second factor respectively. The homeomorphism $F_{A,B}$ is given by the formula:

$$F_{A,B}([\{x_n\}], [\{y_n\}]) = \big([\{P_1 F(x_n, y_n)\}], [\{P_2 F(x_n, y_n)\}]\big).$$

The proofs of these properties are easy and left to the reader.

Now we prove Proposition 4.2

Proof of Proposition 4.2. Since $\dim(K\times S^1) = \dim(L\times S^1) = \dim L+1$, we have $\dim L = 2$ and by [10], the space L is a polyhedron.

Let the set nK consist of pairwise disjoint arcs $I_1 = \widehat{a_1b_1}, \ldots, I_k = \widehat{a_kb_k}$ and simple closed curves $S_1, \ldots, S_l$. It is easy to see that $n(K\times S^1) = nK\times S^1$. Hence, if $G\colon K\times S^1\to L\times S^1$ is a homeomorphism, then $G(nK\times S^1) = nL\times S^1$. Therefore, the set nL consists of pairwise disjoint arcs $I_1' = \widehat{a_1'b_1'}, \ldots, I_k' = \widehat{a_k'b_k'}$ and simple closed curves $S_1', \ldots, S_l'$ such that

(i) $G(I_i\times S^1) = I_i'\times S^1$, $G(a_i\times S^1) = a_i'\times S^1$ and $G(b_i\times S^1) = b_i'\times S^1$ for $i = 1, 2, \ldots, k$.

(ii) $G(S_i\times S^1) = S_i'\times S^1$ for $i = 1, 2, \ldots, l$.

Let $f\colon nK\to nL$ be a homeomorphism such that

(iii) $f(I_i) = I_i'$, $f(a_i) = a_i'$, $f(b_i) = b_i'$ for $i = 1, 2, \ldots, k$ and $f(S_i) = S_i'$ for $i = 1, 2, \ldots, l$.

It suffices to show that the homeomorphism f has an extension to a homeomorphism $\tilde{f}\colon K\to L$.

For every $A\in\Box(K\setminus nK)$ and $A'\in\Box(L\setminus nL)$ such that $G(A\times S^1)=A'\times S^1$ we have constructed the homeomorphism $G_A\colon M(A)\times S^1\to M(A')\times S^1$ (see Property 4.5). Since $M(A)$ and $M(A')$ are 2-manifolds, it follows that $M(A)$, $M(A')$ are homeomorphic.

In case $nK=\emptyset$ we have $M(A)=K$ and the conclusion of the theorem holds. So assume that nK is not empty.

Let $g_A\colon M(A)\to\bar A$ be given by $g_A([\{x_n\}])=\lim x_n$, as in Property 4 and let $g_A\colon M(A')\to\bar A'$ be defined analogously. For every $A\in\Box(K\setminus nK)$ we shall find a homeomorphism $f_A\colon g_A^{-1}(nK\cap\bar A)\to g_A^{-1}(nL\cap\bar A')$ such that

$$(+)\qquad f\circ g_A(x)=g_{A'}\circ f_A(x)\quad\text{for } x\in g_A^{-1}(nK\cap\bar A)\,.$$

Then we shall extend it to a homeomorphism $f_A\colon M(A)\to M(A')$. Next, we obtain a homeomorphism $\tilde f\colon K\to L$ given by the formula

$$(++)\qquad \tilde f(x)=g_{A'}\circ f_A(g_A^{-1}(x))\quad\text{for } x\in\bar A,\text{ where } A\in\Box(K\setminus nK)\,.$$

Since the diagram

$$\text{(iv)}\qquad\begin{array}{ccc} M(A)\times S^1 & \xrightarrow{\;G_A\;} & M(A')\times S^1\\ {\scriptstyle g_A\times\mathrm{id}}\big\downarrow & & \big\downarrow{\scriptstyle g_{A'}\times\mathrm{id}}\\ \bar A\times S^1 & \xrightarrow{\;G|_{\bar A\times S^1}\;} & \bar A\times S^1\end{array}$$

commutes

(see Property 4.5) it follows that for every $T\in\Box g_A^{-1}(S_i)$ and any $T'\in\Box g_{A'}^{-1}(S_i')$ such that $G_A(T\times S^1)=T'\times S^1$ the diagram

$$\text{(v)}\qquad\begin{array}{ccc} T\times S^1 & \xrightarrow{\;G_A|_{T\times S^1}\;} & T'\times S^1\\ {\scriptstyle g_A|_T\times\mathrm{id}}\big\downarrow & & \big\downarrow{\scriptstyle g_{A'|T'}\times\mathrm{id}}\\ S_i\times S^1 & \xrightarrow{\;G|_{S_i\times S^1}\;} & S_i'\times S^1\end{array}$$

commutes too.

Since the maps $g_{A|T}\times\mathrm{id}$ and $g_{A'|T'}\times\mathrm{id}$ are coverings and the maps $G_{A|T\times S^1}$, $G_{|S_i\times S^1}$ are homeomorphisms, the degrees of the coverings are equal and there exists a homeomorphism $f_T\colon T\to T'$ such that

$$\text{(vi)}\qquad f\circ g_A(x)=g_{A'}\circ f_T(x)\quad\text{for } x\in T\,.$$

The set $\Box g_A^{-1}(I_i)$ is a family of arcs. If J is one of them and J' is the corresponding component of $g_{A'}^{-1}(I_i')$, we have a homeomorphism $f_J\colon J\to J'$ such that

$$\text{(vii)}\qquad f\circ g_A(x)=g_{A'}\circ f_J(x)\quad\text{for } x\in J\,.$$

Let us define a homeomorphism $f_A\colon g_A^{-1}(nK\cap\bar A)\to g_{A'}^{-1}(nL\cap\bar A')$ by the formula

$$\text{(viii)}\qquad f_A(x)=\begin{cases} f_T(x) & \text{for } x\in T,\ T\in\Box g_A^{-1}(S_i),\ i=1,\dots,l\,,\\ f_J(x) & \text{for } x\in J,\ J\in\Box g_A^{-1}(I_i),\ i=1,\dots,k\,.\end{cases}$$

It is obvious that the homeomorphism f_A satisfies condition $(+)$.

Let $J\in\Box g_A^{-1}(I_i)$, $J'\in\Box g_{A'}^{-1}(I_i')$. By (i) and (iii), $G(I_i\times S^1)=I_i'\times S^1$ iff $f(I_i)=I_i'$. Hence by (vii) and (iv) $f_A(J)=J'$ iff $G_A(J\times S^1)=J'\times S^1$. Moreover, if $J=\widehat{cd}$ and $J'=\widehat{c'd'}$ then $f_A(c)=c'$ iff $G_A(c\times S^1)=c'\times S^1$.

This implies that the homeomorphism f_A is extendable to a homeomorphism $f_A'\colon\partial M(A)\to\partial M(A')$. If the manifold $M(A)$ is not orientable, then there exists an extension $\tilde f_A\colon M(A)\to M(A')$ of the map f_A' such that $\tilde f_A$ is a homeomorphism. If the manifold $M(A)$ is orientable, it suffices to show that the homeomorphism f_A' does not change the orientation.

We choose orientations of the curves S_i in any manner. Next, we choose orientations of the curves S_i' in such a manner that the maps $G_{|S_i\times S^1}\colon S_i\times S^1\to S_i'\times S^1$ preserve orientation. We can additionally require that

(ix) $f_{|S_i}\colon S_i\to S_i'$ preserves orientation for any $i=1,2,\dots,l$. We choose orientations of the manifolds $M(A)$ in any manner and we choose the orientations of the manifolds $M(A')$ in such a manner that the maps $G_A\colon M(A)\times S^1\to M(A')\times S^1$ preserve orientation. The orientations of $T\in\Box\partial M(A)$ and $T'\in\Box\partial M(A')$ we have fixed by the orientations of $M(A)$ and $M(A')$.

Since (v) $G_{|S_i\times S^1}\circ(g_{A|T}\times\mathrm{id})=(g_{A'|T'}\times\mathrm{id})\circ G_{A|T\times S^1}$, it follows that $g_{A|T}\times\mathrm{id}$ preserves orientation iff $g_{A'|T'}\times\mathrm{id}$ preserves orientation. Hence, $g_{A|T}$ preserves orientation iff $g_{A'|T'}$ preserves orientation. This and (vi) imply that $f_T\colon T\to T'$ preserves orientation for $T\in\Box g_A^{-1}(S_i)$.

Let $T\supset J\in\Box g_A^{-1}(I_i)$. Then, as we have mentioned above, if $J=\widehat{cd}$ and $J'=\widehat{c'd'}$ then $f_A(c)=c'$ iff $G_A(c\times S^1)=c'\times S^1$. Since G_A preserves orientation on $T\times S^1$, f_A' preserves orientation on T.

The lemma is thus proved.

In the next section we can consider polyhedra K such that nK does not contain simply closed curves.

5. A one-to-one correspondence between the manifolds $M(A)$ and $M(A')$. In Section 4 for every component $A\in\Box(K\setminus nK)$ we have defined the compact 2-manifold M with boundary fold (with boundary) $M(A)$. Here, for every compact 2-manifold M with boundary we will define a number $\sigma(M)$. In Lemma 5.1 we will prove that if $M\times N\approx P\times R$ then $\sigma(M)\sigma(N)=\sigma(P)\sigma(R)$. Next, in Lemma 5.2, we will prove that there exists a one-to-one correspondence between manifolds $M(A)$ and $M(A')$ such that $M(A)\approx M(A')$.

Definition 5.1. Let M be a compact 2-manifold with boundary $\partial M\neq\emptyset$. We define the number

$$\sigma(M)=\operatorname{rank}H_1(M)-\operatorname{rank}H_1(\partial M)+\varepsilon$$

where
$$\varepsilon=\begin{cases}0 & \text{for } M \text{ nonorientable}\,,\\ 1 & \text{for } M \text{ orientable}\,.\end{cases}$$

Remark. If the surfaces M and N are both orientable or both nonorientable, $\sigma(M) = \sigma(N)$ and $\operatorname{rank} H_1(M) = \operatorname{rank} H_1(N)$, then $M \approx N$.

LEMMA 5.1. *If M, N, P, R are compact 2-manifolds with boundary and the manifolds $M\times N$ and $P\times R$ are homeomorphic, then $\sigma(M)\sigma(N) = \sigma(P)\sigma(R)$.*

Proof. Let $H_1(M) = Z^m$, $H_1(N) = Z^n$, $H_1(P) = Z^p$, $H_1(R) = Z^r$.

From Künneth's formula we conclude that

$$Z^{pr} \approx H_2(P\times R) \approx H_2(M\times N) \approx Z^{mn},$$

$$Z^{p+r} \approx H_1(P\times R) \approx H_1(M\times N) \approx Z^{m+n}.$$

Hence, $p = m$ and $r = n$ or $p = n$ and $r = m$. We can assume that the first case holds.

Also, from Künneth's formula it follows that $H_1(M, \partial M) \approx H_1(P, \partial P)$ and $H_1(N, \partial N) \approx H_1(R, \partial R)$. Hence, both manifolds M and P are orientable or both are nonorientable and the same holds for N and R.

It is easy to compute the group $H_2(\partial(M\times N))$, using Künneth's formula and the Mayer–Vietoris sequence or the methods from [7].

If M and N are orientable, then $H_2(\partial(M\times N)) \approx Z^{(n+1)(m+1)-\sigma(M)\sigma(N)-1}$ and $H_2(\partial(P\times R)) \approx Z^{(n+1)(m+1)-\sigma(P)\sigma(R)-1}$. Therefore $\sigma(M)\sigma(N) = \sigma(P)\sigma(R)$.

Similarly, if M is orientable and N is nonorientable, then

$$H_2(\partial(M\times N)) \approx Z^{(m+1)n-\sigma(M)\sigma(N)-1} \oplus Z_2$$

and

$$H_2(\partial(P\times R)) \approx Z^{(m+1)n-\sigma(P)\sigma(R)-1} \oplus Z_2;$$

hence $\sigma(M)\sigma(N) = \sigma(P)\sigma(R)$.

If M and N are nonorientable, then $H_2(\partial(M\times N)) \approx Z^{mn-\sigma(M)\sigma(N)} \oplus Z_2$ and $H_2(\partial(P\times R)) \approx Z^{mn-\sigma(P)\sigma(R)} \oplus Z_2$, hence $\sigma(M)\sigma(N) = \sigma(P)\sigma(R)$.

Therefore the lemma is proved.

We recall that if $F\colon K\times K \to L\times L$ is a homeomorphism and $A, B \in \square(K\setminus nK)$ then $F(A\times B) = A'\times B'$, where $A', B' \in \square(L\setminus nL)$. In the case considered, by condition (****) from Proposition 3.1, $F(A\times K) = A'\times L$ and $F(K\times B) = L\times B'$ for every $A, B \in \square(K\setminus nK)$. By Property 4.6 we have $M(A)\times M(B) \approx M(A')\times M(B')$.

LEMMA 5.2. *Let K be a compact connected 2-polyhedron. Assume that all components of the set nK are arcs and the regular neighborhood of any $x\in nK$ is homeomorphic to the set $\operatorname{cone}\{1, \ldots, n\}\times I$.*

If $F\colon K^2 \to L^2$ is a homeomorphism and $F(A\times A) = A'\times A''$, where $A\in \square(K\setminus nK)$ and $A', A'' \in \square(L\setminus nL)$, then the manifolds $M(A)$, $M(A')$, $M(A'')$ are homeomorphic.

Proof. It is sufficient to show that either the manifolds $M(A)$, $M(A')$, $M(A'')$ are all orientable or all nonorientable, $\operatorname{rank} H_1(M(A)) = \operatorname{rank} H_1(M(A')) = \operatorname{rank} H_1(M(A''))$, and $\sigma(M(A)) = \sigma(M(A')) = \sigma(M(A''))$. From the topological equality $M(A)^2 \approx M(A')\times M(A'')$ we easily conclude that the first and second conditions hold.

Let

$$m = \max\{\sigma(M(B))\colon B\in \square(K\setminus nK)\}$$

and

$$m' = \max\{\sigma(M(B'))\colon B'\in \square(L\setminus nL)\}.$$

We can assume that $m \geqslant m'$. (If $m < m'$, we use the inverse homeomorphism.) If $\sigma(M(A)) = m$, then from Lemma 5.1 we have $m^2 = \sigma(M(A'))\sigma(M(A''))$. Since $\sigma(M(A'))$, $\sigma(M(A''))$ are less than or equal to m, $\sigma(M(A')) = \sigma(M(A'')) = m = m'$. Hence $M(A) \approx M(A') \approx M(A'')$.

Now, let $\sigma(M(A)) = n < m$ and suppose the conclusion holds for all $B\in \square(K\setminus nK)$ such that $\sigma(M(B)) > n$. By induction, we need to show that it holds for A.

If $F(A\times A) = A'\times A''$, then $F((K\times A)\cup(A\times K)) = (L\times A'')\cup(A'\times L)$. This fact follows from condition (****) from Proposition 3.1. Thus we have a one-to-one correspondence between the components A of the set $K\setminus nK$ and the components A' of the set $L\setminus nL$, and another one-to-one correspondence between A and A''.

By the induction assumption, if $\sigma(M(B)) > n$, then $\sigma(M(B')) > n$ and $\sigma(M(B'')) > n$. If $\sigma(M(A')) > n$ or $\sigma(M(A'')) > n$, then the number of $B\in \square(K\setminus nK)$ such that $\sigma(M(B)) > n$ is less than the number of $B'\in \square(L\setminus nL)$ such that $\sigma(M(B')) > n$. We consider the inverse homeomorphism and observe that this is impossible. In fact, if $\sigma(M(B')) > n$, $\sigma(M(B'')) > n$, $F^{-1}(B'\times B'') = B_1\times B_2$ and $\sigma(M(B_1)) \leqslant n$, $\sigma(M(B_2)) \leqslant n$, then $\sigma(M(B'))\sigma(M(B'')) > \sigma(M(B_1))\sigma(M(B_2))$, but $M(B')\times M(B'') \approx M(B_1)\times M(B_2)$. Hence $\sigma(M(A'')) \leqslant n$ and $\sigma(M(A'')) \leqslant n$. Since $n^2 = \sigma(M(A'))\sigma(M(A''))$, we have $\sigma(M(A')) = \sigma(M(A'')) = n$ and $M(A) \approx M(A') \approx M(A'')$.

6. The remaining case. We have a one-to-one correspondence between the manifolds $M(A)$ and $M(A')$ such that $M(A) \approx M(A')$. We will establish a family of homeomorphisms $\{f_A\colon A \to A'\}$ and correct these homeomorphisms, so that they will yield a homeomorphism $f\colon K \to L$.

PROPOSITION 6.1. *Let K be a compact connected 2-polyhedron such that all components of nK are arcs and there exists a regular neighborhood of any $x\in nK$ homeomorphic to $\operatorname{cone}\{1, \ldots, n\}\times I$. If K^2 is homeomorphic to L^2, then K is homeomorphic to L.*

Sketch of proof. If $nK = \emptyset$, then K is a manifold and by [7], $K \approx L$. Thus we assume that $nK \neq \emptyset$.

Let $F\colon K^2 \to L^2$ be a homeomorphism. Then $nK \approx nL$ and by Proposition 3 1 (***), $F(K\times I) = L\times I''$ or $F(K\times I) = I'\times L$, where $I\in \square nK$ and $I', I'' \in \square nL$. Let us assume that

(i) $$F(K\times I) = L\times I'' \quad\text{and}\quad F(I\times K) = I'\times L.$$

Let $F(A^2) = A'\times A''$, where $A\in \square(K\setminus nK)$ and $A', A'' \in \square(L\setminus nL)$. By Property 4.6 there exists a homeomorphism $F_{AA}\colon M(A)\times M(A) \to M(A')\times M(A'')$ and by Lemma 5.2 the manifolds $M(A)$, $M(A')$, $M(A'')$ are homeomorphic.

If g_A, $g_{A'}$ and $g_{A''}$ are given as in Property 4.4, then the diagram

$$\text{(ii)}\qquad \begin{array}{ccc} M(A)\times M(A) & \xrightarrow{F_{AA}} & M(A')\times M(A'') \\ \downarrow{\scriptstyle g_A\times g_A} & & \downarrow{\scriptstyle g_{A'}\times g_{A''}} \\ \bar{A}\times\bar{A} & \xrightarrow{F_{|\bar{A}\times\bar{A}}} & \bar{A}'\times\bar{A}'' \end{array}$$

commutes.

Hence, if $J\in\Box g_A^{-1}(nK\cap\bar{A})$, then

$$\text{(iii)}\qquad F_{AA}(M(A)\times J) = M(A')\times J'' \quad\text{and}\quad F_{AA}(J\times M(A)) = J'\times M(A''),$$
$$\text{where } J'\in\Box g_{A'}^{-1}(nL\cap\bar{A}') \quad\text{and}\quad J''\in\Box g_{A''}^{-1}(nL\cap\bar{A}'').$$

By Properties 4.3, 4.4, $J\subset\partial M(A)$, $J'\subset\partial M(A')$, $J''\subset\partial M(A'')$ and $g_{A|J}$ $g_{A'|J'}$, $g_{A''|J''}$ are homeomorphisms.

We shall prove

LEMMA 6.1. *If all the assumptions of Proposition* 6.1 *and* (iii) *hold, then there exist homeomorphisms* $f_{1,A}\colon M(A)\to M(A')$ *and* $f_{2,A}\colon M(A)\to M(A'')$ *such that*

$$(+)\qquad f_{1,A}(J) = J' \quad\textit{and}\quad f_{2,A}(J) = J''.$$

Next we shall prove

LEMMA 6.2. *Let all the assumptions of Proposition* 6.1 *and* (i)–(iii) *hold. Let* $I=\widehat{ab}$, $I'=\widehat{a'b'}$, $I''=\widehat{a''b''}$, $J=\widehat{cd}$, $J'=\widehat{c'd'}$, $J''=\widehat{c''d''}$ *be such that* $g_A(c)=a$, $g_{A'}(c')=a'$ *and* $g_{A''}(c'')=a''$. *The homeomorphism from Lemma* 6.1 *can be corrected in such a way that additionally the following condition holds*:

$(++)$ *either* $f_{1,A}(c)=c'$ *for all* $A\in\Box(K\setminus nK)$ *and all* $J\in\Box g_A^{-1}(I)$ *or* $f_{1,A}(c)=d'$ *for all* $A\in\Box(K\setminus nK)$ *and all* $J\in\Box g_A^{-1}(I)$, *and analogously for* $f_{2,A}$.

Since two homeomorphisms of arcs keeping the end-points are isotopic and the arcs J, J' have collars in $M(A)$ and $M(A')$, we can correct the homeomorphisms $f_{1,A}$ so that additionally the following condition holds:

$\binom{+}{++}$ $g_{A'}\circ f_{1,A}((g_{A|J_1})^{-1}(x)) = g_{B'}\circ f_{1,B}((g_{B|J_2})^{-1}(x))$ for $x\in I$, where $I\in\Box nK$, $A,B\in\Box(K\setminus nK)$ (which may be equal), $J_1\in\Box g_A^{-1}(I)$, $J_2\in\Box g_B^{-1}(I)$ and $A',B'\in\Box(L\setminus nL)$ are such that $F(A\times A)=A'\times A''$ and $F(B\times B)=B'\times B''$.

Thus we obtain a homeomorphism $f\colon K\to L$ which can be defined by the formula

$\binom{++}{++}$ $f(x) = g_{A'}\circ f_{1,A}(g_A^{-1}(x))$ where $x\in\bar{A}$, $A\in\Box(K\setminus nK)$, $A'\in\Box(L\setminus nL)$.

Condition $\binom{+}{++}$ yields the correctness of $\binom{++}{++}$.

Now, we need only prove Lemmas 6.1 and 6.2.

Proof of Lemma 6.1. We will prove that there exist homeomorphisms $f_{1,A}\colon M(A)\to M(A')$ and $f_{2,A}\colon M(A)\to M(A'')$ such that $(+)$ holds.

For $I_i\in\Box nK$ we define I_i', I_i'', J_i, J_i', J_i'' as above. By Lemma 5.2, the manifolds $M(A)$, $M(A')$ and $M(A'')$ are homeomorphic. It is sufficient to show that the following conditions hold:

(1) If $J_1, J_2\in\Box g_A^{-1}(nK\cap\bar{A})$ lie in one component $S\in\Box\partial M(A)$, then J_1', J_2' lie in one component $S'\in\Box\partial M(A')$, and J_1'', J_2'' lie in one component $S''\in\Box\partial M(A'')$.

(2) If some $J_1,J_2,J_3,J_4\in\Box g_A^{-1}(nK\cap\bar{A})$ lie in $S\in\Box\partial M(A)$ and the set $J_2\cup J_3$ lies in one component of the set $S\setminus(J_1\cup J_4)$, then the set $J_2'\cup J_3'$ lies in one component of the set $S'\setminus(J_1'\cup J_4')$, and the set $J_2''\cup J_3''$ lies in one component of the set $S''\setminus(J_1''\cup J_4'')$.

This means that the arcs J_i on S, J_i' on S' and J_i'' on S'' can be ordered in the same way. Hence, there exist homeomorphisms $\bar{f}_{1,A}\colon\partial M(A)\to\partial M(A')$ and $\bar{f}_{2,A}\colon\partial M(A)\to\partial M(A'')$ such that $\bar{f}_{1,A}(J)=J'$ and $\bar{f}_{2,A}(J)=J''$ for all $J\in\Box g_A^{-1}(nK\cap\bar{A})$.

The manifolds $M(A)$, $M(A')$, $M(A'')$ are homeomorphic. Hence, if the manifold $M(A)$ is nonorientable, we can extend the homeomorphisms $\bar{f}_{1,A}$, $\bar{f}_{2,A}$ to homeomorphisms $f_{1,A}\colon M(A)\to M(A')$ and $f_{2,A}\colon M(A)\to M(A'')$ so that condition $(+)$ holds.

(3) If the manifold $M(A)$ is orientable, we need to show that we can construct the homeomorphisms $\bar{f}_{1,A}$, $\bar{f}_{2,A}$ in such a manner that they preserve orientation on all components of $\partial M(A)$, or change orientation on all components of $\partial M(A)$. Then we extend the homeomorphisms $\bar{f}_{1,A}$, $\bar{f}_{2,A}$ onto the whole manifold $M(A)$.

Let $x_i\in(J_i)$, $x_i'\in(J_i')$, $x_i''\in(J_i'')$, $i=1,2,\ldots,6$ and $F_{AA}(x_i,x_j)=(x_i',x_j')$ in the proofs of (1), (2), (3). ((J) denotes the interior of J.)

Proof of (1). First, we will prove that if J_1 and J_2 lie in two different components S_1 and S_2 of $\partial M(A)$ then J_1', J_2' lie in two different components S_1', S_2' of $\partial M(A')$.

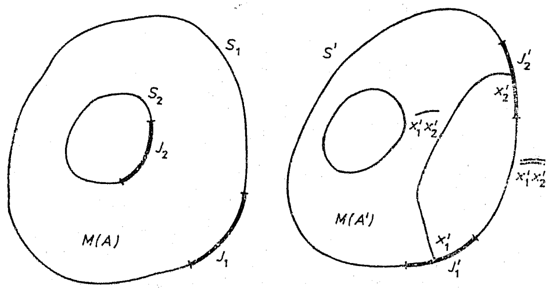

Fig. 1

Let us suppose $J_1', J_2'\subset S'\in\Box\partial M(A')$. Let $\widehat{x_1'x_2'}\subset M(A')$, $(\widehat{x_1'x_2'})\subset\operatorname{Int}M(A')$ and $\overline{x_1'x_2'}\subset S'$, $\widehat{x_1'x_2'}\simeq\overline{x_1'x_2'}$ rel$\{x_1',x_2'\}$ in $M(A')$. Let $\widehat{x_1''x_2''}\subset M(A'')$ be such that $(\widehat{x_1''x_2''})\subset\operatorname{Int}M(A'')$.

Let $T_1 = \partial(\widehat{x_1'x_2'}\times\widehat{x_1''x_2''}) = (\widehat{x_1'x_2'}\times x_1'')\cup(x_2'\times\widehat{x_1''x_2''})\cup(\widehat{x_1'x_2'}\times x_2'')\cup$

$\cup\,(x_1' \times \widehat{x_1''x_2''})$. The set T_1 is a simple closed curve contained in $\partial(M(A')\times M(A''))$.

If $T_2 = (\widehat{x_1'x_2'} \times x_1'') \cup (x_2' \times \widehat{x_1''x_2''}) \cup (\widehat{x_1'x_2'} \times x_2'') \cup (x_1' \times \widehat{x_1''x_2''})$, then $T_1 \simeq T_2$ in $\partial(M(A')\times M(A''))$. Since $T_2 = \partial(\widehat{x_1'x_2'} \times \widehat{x_1''x_2''})$ and $\widehat{x_1'x_2'} \times \widehat{x_1''x_2''} \subset \partial(M(A')\times M(A''))$, we have $T_1 \simeq T_2 \simeq 0$ in $\partial(M(A')\times M(A''))$.

Now, let us suppose that $J_1 \subset S_1$, $J_2 \subset S_2$, where $S_1 \neq S_2$ and $S_1, S_2 \in \square\partial M(A)$. We consider the simple closed curve $F_{AA}^{-1}(T_1) \subset \partial(M(A)\times M(A))$. By (iii), $F_{AA}^{-1}(T_1) \cap (S_p \times S_r) = (x_p, x_r)$ for $p, r = 1, 2$ and we can assume that the manifolds $F_{AA}^{-1}(T_1)$ and $S_p \times S_r$ are transversal in $\partial(M(A)\times M(A))$. Since $\dim S_p \times S_r = 2$, $\dim F_{AA}^{-1}(T_1) = 1$ and $\dim \partial(M(A)\times M(A)) = 3$, it follows that the embeddings of the sets $S_p \times S_r$ and $F_{AA}^{-1}(T_1)$ in $\partial(M(A)\times M(A))$ are homologically essential over Z_2 (see [6], VIII. 13). Hence, the curve $F_{AA}^{-1}(T_1)$ is not contractible in $\partial(M(A)\times M(A))$.

Therefore, if $J_1 \subset S_1$, $J_2 \subset S_2$ and $S_1 \neq S_2$, then $J_1' \subset S_1'$, $J_2' \subset S_2'$ and $S_1' \neq S_2'$, and analogously $J_1'' \subset S_1''$, $J_2'' \subset S_2''$ and $S_1'' \neq S_2''$.

If $J_1, J_2 \subset S \in \square\partial M(A)$, then $J_1', J_2' \subset S' \in \square\partial M(A')$ or $J_1'', J_2'' \subset S'' \in \square\partial M(A'')$. (In the opposite case we consider the homeomorphism $(F_{AA})^{-1}\colon M(A')\times M(A'') \to M(A)\times M(A)$.) Let us suppose that $J_1, J_2 \subset S$, $J_1', J_2' \subset S'$ and $J_1'' \subset S_1''$, $J_2'' \subset S_2''$, where $S_1'' \neq S_2''$. By Proposition 3.1 $(\ddagger\ddagger)$, $F((K\times A)\cup(A\times K)) = (L\times A'')\cup(A'\times L)$. Hence $F^{-1}(A''\times A'') = A_1 \times A$ or $F^{-1}(A''\times A'') = A\times A_1$, where $A_1 \in \square(K\setminus nK)$. Let $F^{-1}(A''\times A'') = A_1\times A$. We consider the homeomorphism $(F^{-1})_{A''A''}\colon M(A'')\times M(A'') \to M(A_1)\times M(A)$, so the arcs J_1, J_2 lie in different component of $\partial M(A)$. This is impossible, whence J_1'', J_2'' lie in one components of $\partial M(A'')$. Thus (1) is proved.

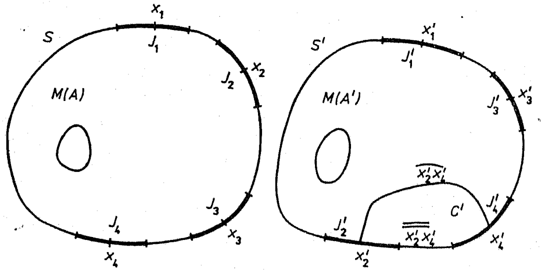

Fig. 2

Proof of (2). Let arcs J_1, J_2, J_3, J_4 be as in (2). Let $x_2 \in \widehat{x_1x_3} \subset S$. We consider the square $D_1 = \widehat{x_1x_3}\times\widehat{x_1x_3} \subset S\times S \subset \partial(M(A)\times M(A))$ and let $T_1 = \partial D_1$. Let the arcs $\widehat{x_2'x_4'}$, $\widehat{x_2'x_4'}$, $\widehat{x_2''x_4''}$, $\widehat{x_2''x_4''}$ be constructed as the arcs $\widehat{x_1'x_2'}$, $\widehat{x_1'x_2'}$ in (1) (Fig. 2).

We consider the simple closed curve $T_2' = \partial(\widehat{x_2'x_4'}\times\widehat{x_2''x_4''}) = (\widehat{x_2'x_4'}\times x_2'') \cup (x_2'\times\widehat{x_2''x_4''}) \cup (\widehat{x_2'x_4'}\times x_4'') \cup (x_4'\times\widehat{x_2''x_4''})$. Since $\widehat{x_2'x_4'} \cup \widehat{x_2'x_4'}$ is the boundary of disk $C' \subset M(A')$ and $\widehat{x_2''x_4''} \cup \widehat{x_2''x_4''}$ is the boundary of a disk $C'' \subset M(A'')$, T_2' is the boundary of the disk $D_2' = (\widehat{x_2'x_4'}\times\widehat{x_2''x_4''}) \cup (C'\times x_2'') \cup (C'\times x_4'') \cup (x_2'\times C'') \cup (x_4'\times C'') \subset \partial(M(A'))\times M(A''))$.

Let us denote $F_{AA}^{-1}(T_2') = T_2$. The simple closed curves T_1 and T_2 are linked because $T_2 \cap D_1 = \{(x_2, x_2)\}$ and T_2 intersects the disk transversally D_1 (Fig. 3).

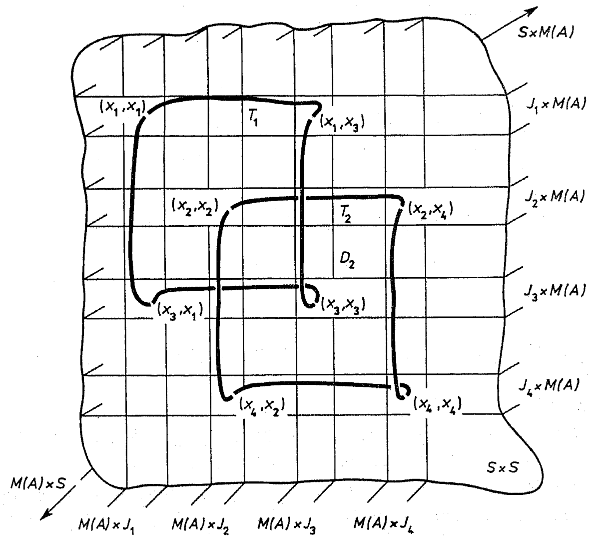

Fig. 3

If $J_1', J_2', J_3', J_4' \subset S' \in \square\partial M(A')$ and the set $J_2' \cup J_3'$ does not lie in one component of $S'\setminus(J_1'\cup J_4')$, then there exists an arc $\widehat{x_2'x_4'} \subset S'$ such that $x_1', x_3' \notin \widehat{x_2'x_4'}$. Let us observe that

$$D_2' \subset (J_2'\times M(A'')) \cup (J_4'\times M(A'')) \cup (M(A')\times J_2'') \cup (M(A')\times J_4'') \cup (\widehat{x_2'x_4'}\times\widehat{x_2''x_4''})$$

and $T_1' \subset (J_1'\times M(A'')) \cup (J_3'\times M(A'')) \cup (M(A')\times J_1'') \cup (M(A')\times J_3'')$. Hence $T_1' \cap D_2' = \varnothing$ (Fig. 4).

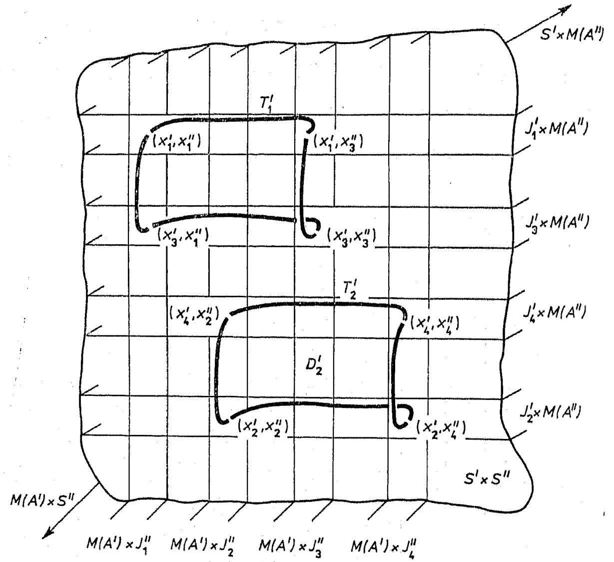

Fig. 4

So T_1' and T_2' are not linked. Therefore $J_2' \cup J_3'$ lies in one component of the set $S' \setminus (J_1' \cup J_4')$.

The proof for arcs $J_1'', J_2'', J_3'', J_4''$ is similar.

Therefore there exist homeomorphisms $\bar{f}_{1,A}\colon \partial M(A) \to \partial M(A')$ and $\bar{f}_{2,A}\colon \partial M(A) \to \partial M(A'')$ such that condition (+) holds.

Proof of (3). If the manifold $M(A)$ is nonorientable, then the homeomorphisms $\bar{f}_{1,A}$, $\bar{f}_{2,A}$ extend to homeomorphisms $f_{1,A}\colon M(A) \to M(A')$ and $f_{2,A}\colon M(A) \to M(A'')$.

Now, suppose $M(A)$ is orientable. Let us choose the orientation of $M(A)$ and $M(A')$. Suppose, contrary to (3), that $\bar{f}_{1,A}$ preserves the orientation on S_1 and changes the orientation on S_2, where $S_1, S_2 \in \square \partial M(A)$. If one of S_1 or S_2 contains less than three components of $g_A^{-1}(nK \cap \bar{A})$, then they do not fix orientation and the homeomorphism $\bar{f}_{1,A}$ can be corrected. Let $J_1, J_2, J_3 \subset S_1$ and $J_4, J_5, J_6 \subset S_2$.

We proceed to show that our assumption contradicts (iii). We will prove that the homeomorphism F_{AA} maps the curve $S_1 \times x_1$ onto the set $F_{AA}(S_1 \times x_1)$ homologic to $S_1' \times x_1''$ in $M(A') \times J_1''$ and it does not change the orientation, whereas F_{AA} maps the curve $S_2 \times x_1$ onto the set $F_{AA}(S_2 \times x_1)$ homologic to $S_2' \times x_1''$ in $M(A') \times J_1''$, and it changes the orientation. This is impossible because, by (iii), $F_{AA}(M(A) \times J_1) = M(A') \times J_1''$ and the surfaces $M(A) \approx M(A')$ are orientable.

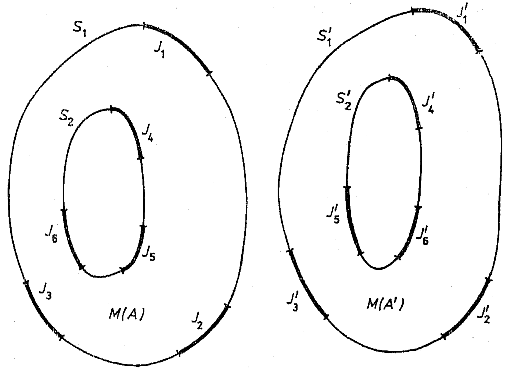

Fig. 5

We recall that $F_{AA}(x_i, x_j) = (x_i', x_j'') \in J_i' \times J_j''$ for $i, j = 1, \ldots, 6$. Let $\widehat{x_i x_j} \subset S_1 \setminus J_k$, where $i \neq j \neq k$, $i, j, k \in \{1, 2, 3\}$ and $\widehat{x_i x_j} \subset S_2 \setminus J_k$, where $i \neq j \neq k$, $i, j, k \in \{4, 5, 6\}$. The analogous arcs in $M(A')$ are denoted by $\widehat{x_i' x_j'}$. It suffices to show that all curves $F_{AA}(\widehat{x_i x_j} \times x_1) \cup \widehat{x_i' x_j'} \times x_1''$ are homologically trivial in $\partial(M(A') \times J_1'')$.

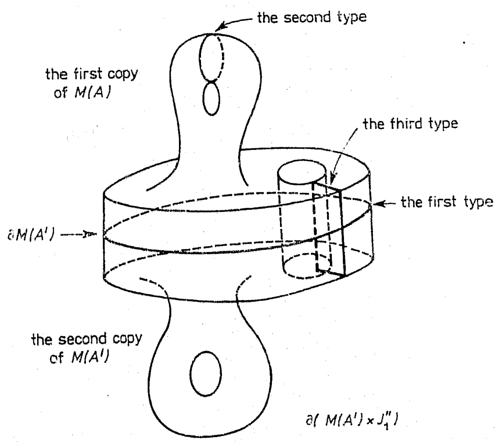

Fig. 6

Let us observe that $\partial(M(A')\times J_1'')$ is homeomorphic to the union of two copies of $M(A')$ with common boundary. From the Mayer–Vietoris exact sequence it follows that $H_1(\partial(M(A')\times J_1''))$ has three types of generators. The first ones are given by generators lying in the common boundary of both copies of $M(A')$, the second are given by the other generators of two copies of $M(A')$, the third are taken from $\operatorname{im}\partial$, where $\partial\colon H_1(\partial(M(A')\times J_1''))\to H_0(\partial M(A'))$ is the homomorphism from the Mayer–Vietoris exact sequence.

Let us suppose that the curve $X_1' = F_{AA}(\widehat{x_1x_2}\times x_1)\cup \widehat{x_1'x_2'}\times x_1''$ is not homologically trivial in $\partial(M(A')\times J_1'')$. Let us consider the curves $T_1 = \partial(\widehat{x_1x_2}\times\widehat{x_1x_2})$, $T_2 = \partial(\widehat{x_1x_2}\times\widehat{x_2x_3})$, $T_3 = \partial(\widehat{x_1x_2}\times\widehat{x_3x_1})$, and their images $T_i' = F_{AA}(T_i)$. The curves T_i' are contractible in $N' = \partial(M(A')\times M(A''))\setminus\bigcup_{i=1}^{3}\operatorname{Int}(M(A')\times J_i'')$, so they are also homologically trivial in N'. Let us denote $X_i' = F_{AA}(\widehat{x_1x_2}\times x_i)\cup \cup\widehat{x_1'x_2'}\times x_i''$, $i=1,2,3$, and $P_i' = F_{AA}(x_i\times\widehat{x_1x_2})\cup x_i'\times\widehat{x_1''x_2''}$, $Q_i' = F_{AA}(x_i\times\widehat{x_2x_3})\cup \cup x_i'\times\widehat{x_2''x_3''}$, $R_i' = F_{AA}(x_i\times\widehat{x_3x_1})\cup x_i'\times\widehat{x_3''x_1''}$, $i=1,2$. Let us observe that

$$[T_1'] = [X_1']+[P_2']-[X_2']-[P_1'],$$
$$[T_2'] = [X_2']+[Q_2']-[X_3']-[Q_1'],$$
$$[T_3'] = [X_3']+[R_2']-[X_1']-[R_1'],$$

where [ ] denotes a homology class in N'.

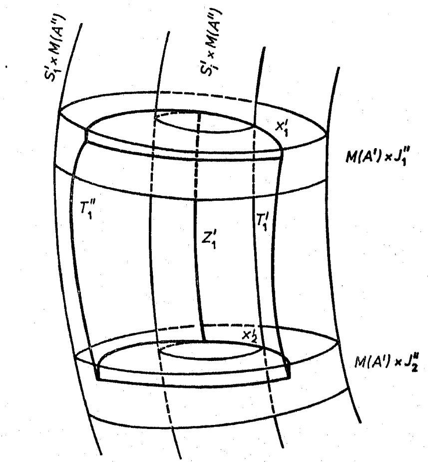

Fig. 7

Since $X_i'\subset M(A')\times J_i''$, $P_i', Q_i', R_i'\subset J_i'\times M(A'')$, it is easy to see that if $T_i'\sim 0$ in N' then $X_1'\sim X_2'\sim X_3'$ in N'. It is also easy to see that if there is a generator of the third type in $[X_1']$, then X_1' is not homologous to X_2' and X_3' in N' and if there is a generator of the second type in $[X_1']$, then X_1' is not homologous to X_2' or X_3' in N.

Now, suppose that there is a generator of the first type in $[X_1']$. If it is a generator given by the curve $S_i'\times x_1''$ then the curve T_1' is not contractible in $N'\setminus(S_i'\times M(A''))$. So we can assume that there is an arc Z_1' joining the curves $F_{AA}(\widehat{x_1x_2}\times x_1)$ and $F_{AA}(\widehat{x_1x_2}\times x_2)$ and lying in $F_{AA}(\widehat{x_1x_2}\times\widehat{x_1x_2})\cap(S_i'\times M(A''))$. The analogous arcs Z_2' joining $F_{AA}(\widehat{x_1x_2}\times x_2)$ with $F_{AA}(\widehat{x_1x_2}\times x_3)$ and Z_3' joining $F_{AA}(\widehat{x_1x_2}\times x_3)$ with $F_{AA}(\widehat{x_1x_2}\times x_1)$, lie in $F_{AA}(\widehat{x_1x_2}\times\widehat{x_2x_3})\cap(S_i'\times M(A''))$ and $F_{AA}(\widehat{x_1x_2}\times\widehat{x_3x_1})\cap \cap(S_i'\times M(A''))$. The arc Z_i' divides the curve T_i' into two curves T_i'' and T_i'''. It is easy to see that at least two of the cycles given by the curves T_i'' have a generator being an image by inclusion of a generator of the third type. So it is not homologically trivial in N'. But T_i'' are contractible in N'.

The proof of (3) is completed.

Proof of Lemma 6.2. We will show that the homeomorphisms $f_{1,A}$, $f_{2,A}$ can be corrected in such a manner that condition $(++)$ holds.

Let us choose orientations for all orientable manifolds $M(A)$ and for components S of $\partial M(A)$ for nonorientable manifolds $M(A)$. Let us fix orientations of all orientable manifolds $M(A')$, $M(A'')$ in such a manner that the homeomorphisms $f_{1,A}$, $f_{2,A}$ preserve orientation. Let us fix orientations of all components S', S'' of $\partial M(A')$, $\partial M(A'')$ for nonorientable $M(A')$, $M(A'')$ in such a manner that the homeomorphisms $f_{1,A|S}$, $f_{2,A|S}$ preserve orientations.

Condition $(++)$ may be formulated as follows:

If $I\in\square nK$, $J_1\in\square g_A^{-1}(I)$, $J_2\in\square g_B^{-1}(I)$, where $A, B\in\square(K\setminus nK)$, then the homeomorphism $h = (g_{B|J_2})^{-1}\circ(g_{A|J_1})\colon J_1\to J_2$ preserves orientation iff so do the homeomorphisms

$$h' = (g_{B'|J_2'})^{-1}\circ(g_{A'|J_1'})\colon J_1'\to J_2'$$

and

$$h'' = (g_{B''|J_1''})^{-1}\circ(g_{A''|J_1''})\colon J_1''\to J_2''$$

The proof will be divided into 3 cases:

(1) $A = B$.
(2) $\bar{A}\cap\bar{B}\neq\emptyset$.
(3) The general case.

Case 1. On the manifold $M(A)$ we define a relation

$$x\sim y\Leftrightarrow\begin{cases}h(x) = y,\\ x = y\end{cases}\quad\text{where } x\in J_1,\ y\in J_2 \text{ and } J_1, J_2\in\square g_A^{-1}(I),\ I\in\square nK$$

Analogously, we define relations for the manifolds $M(A')$ and $M(A'')$:

$$x \sim y \Leftrightarrow \begin{cases} h'(x) = y, & \text{where } x \in J_1', \ y \in J_2' \text{ and } J_1', J_2' \in \square g_A^{-1}(I'),\ I' \in \square nL \\ x = y \end{cases}$$

$$x \sim y \Leftrightarrow \begin{cases} h''(x) = y, & \text{where } x \in J_1'', \ y \in J_2'' \text{ and } J_1'', J_2'' \in \square g_A^{-1}(I''),\ I'' \in \square nL \\ x = y \end{cases}$$

The spaces $M(A)/_\sim$, $M(A')/_\sim$, $M(A'')/_\sim$ are manifolds, too. By (ii) $F_{|\bar{A}\times\bar{A}} \circ (g_A \times g_A) = (g_{A'} \times g_{A''}) \circ F_{AA}$, hence

(iv) $$M(A)/_\sim \times M(A)/_\sim \approx M(A')/_\sim \times M(A'')/_\sim$$

If the manifold $M(A)$ is orientable, then h preserves orientation iff so do h' and h'', because $M(A)/_\sim$ is orientable iff so are $M(A')/_\sim$ and $M(A'')/_\sim$. If the manifold $M(A)$ is nonorientable, we only consider arcs J_1, J_2 lying in one component S of $\partial M(A)$ (Fig. 8).

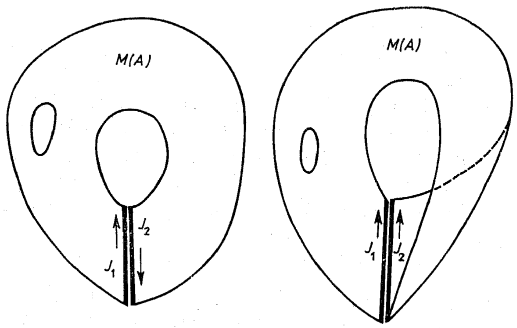

Fig. 8

We observe that $\operatorname{rank} H_1(M(A)/_\sim) = \operatorname{rank} H_1(M(A))+1$. If h changes orientation, the number of elements of $\square\partial(M(A)/_\sim)$ is greater by one than the number of elements of $\square\partial M(A)$. If h does not change orientation, they are the same. Hence, if h changes orientation, $\sigma(M(A)/_\sim) = \sigma(M(A))$, and if h does not change orientation, $\sigma(M(A)/_\sim) = \sigma(M(A))+1$ (see Def. 5.1). From (iv) and Lemma 5.1 it follows that $\sigma(M(A)/_\sim)^2 = \sigma(M(A')/_\sim)\sigma(M(A'')/_\sim)$. By Lemma 5.2 the manifolds $M(A)$, $M(A')$, $M(A'')$ are homeomorphic.

If $\sigma(M(A)/_\sim) = n > 0$, then $n^2 \neq (n+1)n \neq (n+1)^2$ and the conclusion in this case holds. If $\sigma(M(A)/_\sim) = 0$, then $\sigma(M(A')/_\sim) = 0$ or $\sigma(M(A'')/_\sim) = 0$. Let $\sigma(M(A')/_\sim) = 0$ and $\sigma(M(A'')/_\sim) = 1$. By Proposition 3.1 (****), $F^{-1}(A'' \times A'') = A \times A_1$, where $A_1 \in \square(K \setminus nK)$. By the formula $M(A'')/_\sim \times M(A'')/_\sim \approx M(A)/_\sim \times M(A_1)/_\sim$ and by Lemma 5.1, $1 = \sigma(M(A'')/_\sim)^2 = \sigma(M(A)/_\sim)\sigma(M(A_1)/_\sim) = 0$, which is impossible. Hence $\sigma(M(A')/_\sim) = \sigma(M(A'')/_\sim) = 0$.

The proof of case 1 is complete.

Case 2. Let $I \in \square nK$, $J_1 \in \square g_A^{-1}(I)$, $J_2 \in \square g_B^{-1}(I)$, $A \neq B$, $A, B \in \square(K \setminus nK)$ and $h\colon J_1 \to J_2$ be given as above. Let us denote $M(A \cup B) = M(A) \underset{h}{\cup} M(B)$. In the same way we define $M(A' \cup B') = M(A') \underset{h'}{\cup} M(B')$ and $M(A'' \cup B'') = M(A'') \underset{h''}{\cup} M(B'')$. Let us observe that they are manifolds.

We will consider the following cases:

(a) Assume that the manifold $M(A)$ (or $M(B)$) is nonorientable, $J_1 \subset S_1 \in \square\partial M(A)$ and at most two components of $g_A^{-1}(nK \cap \bar{A})$ lie in S_1. Then these components do not fix an orientation of S_1. Hence we can correct the homeomorphisms $f_{1,A}, f_{2,A}$ in such a manner that conditions (+) and (++) for J_1, J_2 hold.

(b) $M(A)$ is orientable and in no component S of $\partial M(A)$ there are more than two components of $g_A^{-1}(nK \cap \bar{A})$. Then the homeomorphisms $f_{1,A}, f_{2,A}$ may be corrected.

(c) Let $J_1, J_3, J_4 \subset S_1 \in \square\partial M(A)$, $J_2, J_5, J_6 \subset S_2 \in \square\partial M(B)$ (Fig. 9).

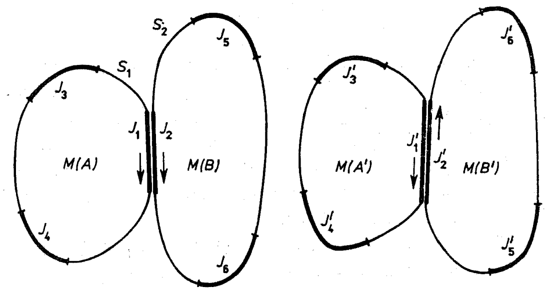

Fig. 9

If the homeomorphism $h\colon J_1 \to J_2$ does not change orientation and the homeomorphism $h'\colon J_1' \to J_2'$ changes orientation, then the ordering of the arcs J_3, J_4, J_5, J_6 on a component of $\partial M(A \cup B)$ and the ordering of the arcs J_3', J_4', J_5', J_6' on a component of $\partial M(A' \cup B')$ are different.

By (ii) and (iii) there exists a homeomorphism

$$F_{A\cup B, A\cup B}\colon M(A \cup B) \times M(A \cup B) \to M(A' \cup B') \times M(A'' \cup B'')$$

such that

(v) $F_{A\cup B, A\cup B}(J_i \times M(A \cup B)) = J_i' \times M(A'' \cup B'')$ and $F_{A\cup B, A\cup B}(M(A \cup B) \times J_i) = M(A' \cup B') \times J_i''$ for $i = 3, 4, 5, 6$.

Using to the manifold $M(A \cup B)$ a reasoning similar to that of (2) in the proof of Lemma 6.1, we obtain a contradiction.

(d) Let the manifolds $M(A)$ and $M(B)$ be orientable, $J_1 \subset S_1 \in \square\partial M(A)$,

$J_2 \subset S_2 \in \square\partial M(B)$, $J_3, J_4, J_5 \subset S_3 \in \square\partial M(A)$ and $J_6, J_7, J_8 \subset S_4 \in \square\partial M(B)$. If $S_3 = S_1$ and $S_4 = S_2$, the proof of (d) is the same as that of (c).

We will consider the case when $S_1 \neq S_3$ and $S_2 \neq S_4$ (Fig. 10).

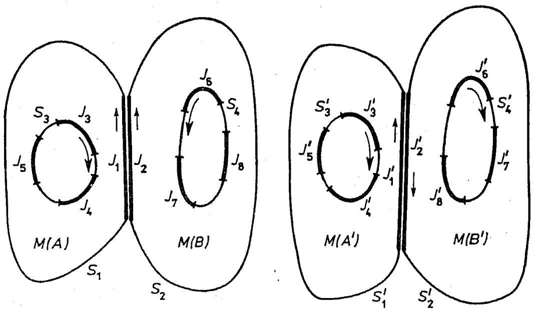

Fig. 10

Let us observe that the manifolds $M(A \cup B)$, $M(A' \cup B')$ and $M(A'' \cup B'')$ are orientable. If the homeomorphism $h\colon J_1 \to J_2$ does not change orientation and the homeomorphism $h'\colon J_1' \to J_2'$ changes orientation, then the ordering of the arcs $J_3, J_4, J_5 \subset S_3$ and $J_6, J_7, J_8 \subset S_4$ agrees with the orientation, and the ordering of the arcs $J_3', J_4', J_5' \subset S_3'$ or $J_6', J_7', J_8' \subset S_4'$ does not agree with the orientation. Using to the manifold $M(A \cup B)$ a reasoning similar to that of (3) in the proof of Lemma 6.1, we obtain a contradiction.

Similarly for the cases $S_1 = S_3$ and $S_2 \neq S_4$, or $S_1 \neq S_3$ and $S_2 = S_4$.

(e) If we match next arcs $J_3 \subset \partial M(A)$ and $J_4 \subset \partial M(B)$, we can reason as in case 1 considering the manifold $M(A \cup B)$.

Case 3. For any $A, B \in \square(K \setminus nK)$, there exists a sequence of components $A = A_1, A_2, \ldots, A_n = B$ of the set $K \setminus nK$ such that $\bar{A}_i \cap \bar{A}_{i+1} \neq \emptyset$ for $i = 1, 2, \ldots, n-1$. By (v) and induction it suffices to consider the case when $\bar{A} \cap \bar{B} \neq \emptyset$, which is given in (2).

Therefore, the proof of Lemma 6.2 is complete.

Lemmas 6.1 and 6.2 complete the proof of Proposition 6.1.

Propositions 3.1, 4.1 and 6.1 imply Theorem A.

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