On locally nonexpansive mappings and local isometries

by

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Abstract. The orbit structure of some locally nonexpansive mappings is studied. Some decomposition theorems are obtained. Several implications of these results are then considered, specifically to obtain conditions on a space under which local isometries are isometries. Finally, some connections of the results with a problem of A. D. Aleksandrov are discussed.

1. Introduction. Let f be a mapping of a metric space (M, ϱ) into a metric space (N, σ). Then f is said to be a *locally nonexpansive mapping* (resp. a *local isometry*) provided that for each $z \in M$ there exists a number $\varepsilon > 0$ such that

$$\sigma(f(x), f(y)) \leqslant \varrho(x, y) \tag{1}$$

(resp.

$$\sigma(f(x), f(y)) = \varrho(x, y)\ ) \tag{2}$$

for all $x, y \in K_\varrho(z, \varepsilon) = \{p \in M\colon \varrho(z, p) < \varepsilon\}$. If the number ε does not depend on $z \in M$, then f is called *ε-locally nonexpansive* (resp. an *ε-local isometry*). If for every bounded subset A of M there exists a number $\varepsilon_A > 0$ such that the restriction $f|A$ is an ε_A-locally nonexpansive mapping (resp. an ε_A-local isometry), then f is called *uniformly locally nonexpansive* (resp. a *uniform local isometry*). We observe that a *nonexpansive mapping* (resp. an *isometry*) can be regarded as an ∞-locally nonexpansive mapping (resp. an ∞-local isometry).

Now let f be a mapping of M into itself. A point $x \in A \subset M$ is said to belong to the *f-closure of A*, $x \in A^f$, if there exists a sequence of integers $\{n_i\}_{i=0}^{\infty}$, $0 < n_i < n_{i+1}$ for $i = 0, 1, \ldots$, so that $\{f^{n_i}(x)\}_{i=0}^{\infty}$ converges to x. Note that this definition is more restrictive than Edelstein's definition given in [8]. However, these definitions, in the case of ε-locally nonexpansive mappings, are equivalent (see [8, Proposition 1] and also Remark 4 below).

In this paper we are concerned with the following questions:

A. *Under what conditions on a metric space (M, ϱ) is the restriction of every locally nonexpansive (resp. uniformly locally nonexpansive; ε-locally nonexpansive)* *mapping* $f\colon M\to M$ *to the* f*-closure of* M, M^f, *a local isometry* (*resp. an uniform local isometry*; *an* ε*-local isometry*)?

B. *Under what conditions on a metric space* (M,ϱ) *is every local isometry* (*resp. uniform local isometry*; ε*-local isometry*) $f\colon M\to M$ *an isometry*?

Question A is motivated by a result of Edelstein (see [8, Theorem 1]) which states that if f is an ε-locally nonexpansive mapping of a metric space (M,ϱ) into itself, then for every point $x\in M^f$ the restriction of f to the set $\{f^n(x)\}_{n=0}^{\infty}$ is an ε-local isometry. Question B has been investigated by Busemann [2]–[3], Kirk [9]–[11] and Szenthe [13]–[15] in the special case where (M,ϱ) is a G-space and f is an open and surjective local isometry, and by the author in the case where f is a local isometry and (M,ϱ) is compact [4], or finitely compact [5], and in the case where f is an ε-local isometry and (M,ϱ) is totally bounded [6].

In § 2 of this paper we give some notation and preliminary remarks. In § 3 we give some examples of locally nonexpansive mappings and local isometries. In § 4 we collect the necessary information concerning uniformly locally nonexpansive mappings.

In § 5, starting from a description of the set M^f for uniformly locally nonexpansive mappings of *finitely totally bounded metric* spaces (i.e. such spaces that every bounded subset is totally bounded) into themselves (Proposition (5.3)), we give an answer to question A (Corollaries (5.6), (5.7) and Theorems (5.9), (5.12)). We observe that part (a) of Theorem (5.9) extends Theorem (5.6) of [7] and enables us to answer, in more generality than was asked for, a question posed by Kirk in [11].

§ 6 contains some decomposition theorems for uniform local isometries and ε-local isometries of finitely totally bounded metric spaces into themselves (Theorems (6.6), (6.9) and Corollaries (6.7), (6.8) and (6.10)).

Applications of the results of §§ 4, 5 and 6 to obtain an answer to question B are given in § 7 (Theorems (7.7)–(7.12)). These results extend considerably the results of [4], [5] and [6] as well as the results of [9], [10] and [11]. We conclude (§ 8) with some related questions and some remarks concerning a problem of A. D. Aleksandrov (see [12]).

Throughout the paper, cl will be used to denote closure; the completion of a metric space (M,ϱ) will be denoted by $(\overline{M},\bar{\varrho})$ and M will be considered as a subset of $\overline{M}$. If A, B are nonempty subsets of M, then $\varrho(A,B)=\inf\{\varrho(x,y)\colon x\in A,\ y\in B\}$ and

$$\operatorname{diam}_{\varrho}(A)=\sup\{\varrho(x,y)\colon x,y\in A\},$$

$$K_{\varrho}(A,r)=\{x\in M\colon \varrho(A,x)<r\},\qquad \text{for each } r>0.$$

If f is a mapping of M into itself, then $f^0=\mathrm{id}_M$, $f^{n+1}=f\circ f^n$ for each $n=0,1,\ldots$ The set of real numbers will be denoted by R, and $[x,y]=\{t\in R\colon x\leqslant t\leqslant y\}$, for all $x,y\in R$, $x\leqslant y$.

2. Basic concepts and preliminary remarks.

The following definitions and remarks will be needed.

(2.1) NOTATION. Let f be a mapping of a metric space (M,ϱ) into itself. Then for every $A\subset M$,

(a) A^f, the f*-closure of* A, will be $\{x\in A\colon x\in w_f(x)\}$, where

$$w_f(x)=\bigcap_{n\geqslant 0}\operatorname{cl}\{f^i(x)\colon i\geqslant n\};$$

(b) $b^f(A)=\{x\in A\colon \operatorname{diam}_{\varrho}(\{f^n(x)\}_{n=0}^{\infty})<\infty\}$;

(c) the function $\varrho_f(x,y)=\sup\limits_{n\geqslant 0}\varrho(f^n(x),f^n(y))$ defined for all $x,y\in A$ will be called the *induced metric on* A.

(2.2) Remark. Let f be a mapping of a metric space (M,ϱ) into itself. Then

(a) $A^f=A\cap M^f$ and $b^f(A)=A\cap b^f(M)$ for $A\subset M$,

(b) if $A\subset M$ is invariant (i.e., $f(A)\subset A$), then the sets $b^f(A)$ and $A\setminus b^f(A)$ are also invariant,

(c) if $A\subset M$ is invariant and if f is continuous, then the f-closure of A, A^f, is also invariant and $f(A^f)$ is a dense subset of A^f,

(d) in general the induced metric ϱ_f may have "infinite values" (hence it is not a metric in the usual sense); however,

$$\varrho_f(x,y)<\infty\qquad\text{for all } x,y\in b^f(M).$$

(2.3) DEFINITION. A metric space (M,ϱ) is said to be *finitely totally bounded* (resp. *finitely compact*) if each bounded (resp. bounded and closed) subset of M is totally bounded (resp. compact).

(2.4) Remark. A metric space (M,ϱ) is finitely totally bounded if and only if $(\overline{M},\bar{\varrho})$ is finitely compact. A mapping f of a finitely totally bounded metric space (M,ϱ) into a metric space (N,σ) is uniformly locally nonexpansive (resp. an uniform local isometry) if and only if f can be extended to a locally nonexpansive mapping (resp. a local isometry) of $(\overline{M},\bar{\varrho})$ into $(\overline{N},\bar{\sigma})$.

(2.5) DEFINITION. Let ϱ_i, $i=0,1$, be metrices on a set M and let $A\subset M$. We will say that ϱ_0 and ϱ_1 are *locally identical* (resp. ε*-locally identical*) *on* A if the identity mapping, id_A, is a local isometry (resp. an ε-local isometry) of (A,ϱ_i) into (A,ϱ_j) for all $i,j=0,1$.

(2.6) Remark. Let ϱ_i, $i=0,1$, be metrices on a set M. If ϱ_0 and ϱ_1 are locally identical, then they are topologically equivalent. If ϱ_0 and ϱ_1 are locally identical (resp. ε-locally identical) and $\varrho_1\geqslant\varrho_0$ and if (M,ϱ_0) is finitely compact (resp. finitely totally bounded), then (M,ϱ_1) is also finitely compact (resp. finitely totally bounded).

(2.7) Remark. Let f be a mapping of a metric space (M,ϱ) into itself and let $A\subset M$ be such that $\varrho_f(x,y)<\infty$ for all $x,y\in A$. Then, for every integer $n=0,1,\ldots$, the induced metric ϱ_f is a metric on the set $f^n(A)$ such that

(a) $\varrho_f\geqslant\varrho$;

(b) the restriction of f to $f^n(A)$ is a nonexpansive mapping of $(f^n(A),\varrho_f)$ into $(f^{n+1}(A),\varrho_f)$;

(c) if f is an ε-locally nonexpansive mapping, then ϱ_f and ϱ are ε-locally identical on $f^n(A)$;

(d) for each $x \in A$, $\operatorname{diam}_{\varrho_f}(\{f^k(x)\}_{k=0}^{\infty}) = \operatorname{diam}_{\varrho}(\{f^k(x)\}_{k=0}^{\infty})$.

(2.8) Definition. Let (M, ϱ) be a metric space and let $\varepsilon > 0$ be a given number. Then

(a) given a family A_t, $t \in T$, of subsets of M, we will say that the sets A_t, $t \in T$, are *ε-separated* if for all $t, s \in T$ the conditions $A_t \neq \emptyset \neq A_s$ and $A_t \neq A_s$ imply that $\varrho(A_t, A_s) \geqslant \varepsilon$;

(b) a finite sequence of points $z_0, z_1, \ldots, z_k$ of M is said to be an *ε-chain from x to y* if $z_0 = x$, $z_k = y$ and $\varrho(z_i, z_{i+1}) < \varepsilon$ for all $i = 0, \ldots, k-1$. For every $x \in M$, the set

$$C_\varepsilon(x) = \{y \in M\colon \text{ there is an } \varepsilon\text{-chain from } x \text{ to } y\}$$

will be called the *ε-component at x*. The space (M, ϱ) is said to be *ε-chainable* if $C_\varepsilon(x) = M$ for every $x \in M$.

(c) the space (M, ϱ) is said to be *ε-convex* if it is ε-chainable and if, for all $x, y \in M$, the number

$$(*) \qquad \varrho_\varepsilon(x, y) = \inf \sum_{i=0}^{k-1} \varrho(z_i, z_{i+1})$$

(where the infimum is taken over all possible ε-chains $z_0, z_1, \ldots, z_k$ from x to y) is equal to $\varrho(x, y)$.

(2.9) Remark. Let (M, ϱ) be a metric space and let $\varepsilon > 0$ be a given number. Then

(a) for every $x \in M$, the ε-component at x, $C_\varepsilon(x)$, is an open, closed and ε-chainable subspace of M containing the point x; moreover, $C_\varepsilon(x)$, $x \in M$, is a decomposition of M into ε-separated sets,

(b) for every $x \in M$, the function ϱ_ε defined by $(*)$ is a metric on $C_\varepsilon(x)$ such that (i) $\varrho_\varepsilon \geqslant \varrho$, (ii) ϱ_ε and ϱ are $\varepsilon/2$-locally identical (hence topologically equivalent), and (iii) the space $(C_\varepsilon(x), \varrho_\varepsilon)$ is ε-convex.

Recall that a metric space (M, ϱ) is *convex* (in the sense of Menger) provided that for each two distinct points $x, y \in M$ there exists a point $z \in M$, $z \neq x, y$, such that $\varrho(x, y) = \varrho(x, z) + \varrho(z, y)$. A relation between the convexity and the ε-convexity of the space is given by

(2.10) Remark. Let (M, ϱ) be a metric space.

(a) If $(\overline{M}, \bar{\varrho})$ is convex, then (M, ϱ) is ε-convex for every $\varepsilon > 0$.

(b) If (M, ϱ) is finitely totally bounded and ε-convex for every $\varepsilon > 0$, then $(\overline{M}, \bar{\varrho})$ is convex.

Proof. Assume that $(\overline{M}, \bar{\varrho})$ is convex and let $\varepsilon > 0$ be given. Let $x, y \in M$, $x \neq y$. It follows by a theorem of Menger (cf. [1, p. 41]), that there exists a metric segment $L \subset \overline{M}$ whose end-points are x and y, i.e., a subset isometric with an interval of length $\varrho(x, y)$. Thus there exists an ε-chain $z_0, z_1, \ldots, z_k \in L$ from x to y such that $\varrho(x, y) = \sum_{i=0}^{k-1} \bar{\varrho}(z_i, z_{i+1})$. For every $i = 1, \ldots, k-1$, we can choose a sequence of points z_i^n, $n = 0, 1, \ldots$, of M with $\lim_{n\to\infty} z_i^n = z_i$. Hence

$$\varrho_\varepsilon(x, y) = \lim_{n\to\infty} \sum_{i=0}^{k-1} \varrho(z_i^n, z_{i+1}^n) = \sum_{i=0}^{k-1} \bar{\varrho}(z_i, z_{i+1}) = \varrho(x, y),$$

which shows that (M, ϱ) is ε-convex. Since $\varepsilon > 0$ was chosen arbitrary, this proves (a).

In order to prove (b), let us assume that (M, ϱ) is finitely totally bounded and ε-convex for every $\varepsilon > 0$. Let $x, y \in \overline{M}$, $x \neq y$. It is easy to see that there exists a sequence of points x_n, $n = 1, 2, \ldots$, of M such that $\bar{\varrho}(x, x_n) \leqslant \frac{1}{2}\bar{\varrho}(x, y) + \frac{1}{n}$ and $\bar{\varrho}(x_n, y) \leqslant \frac{1}{2}\bar{\varrho}(x, y) + \frac{1}{n}$. Since $(\overline{M}, \bar{\varrho})$ is finitely compact (cf. Remark 2.4), we can assume that the sequence x_n, $n = 1, 2, \ldots$, converges to a point $z \in \overline{M}$. Thus $\bar{\varrho}(x, z) = \bar{\varrho}(z, y) = \frac{1}{2}\bar{\varrho}(x, y)$, which shows that $(\overline{M}, \bar{\varrho})$ is convex. This completes the proof.

3. Some examples. The examples of this section will be discussed further in the later sections.

First we give an example of a uniformly locally nonexpansive mapping f of a finitely totally bounded metric space (M, ϱ) into itself for which the restriction of f to M^f is not a uniform local isometry.

(3.1) Example. For every $n = 1, 2, \ldots$, and $i = 0, 1, \ldots, n$, let $\eta_i^n \geqslant 0$ be a number such that $\eta_0^n = 0$ and

$$\frac{1}{n} - \eta_i^n > \frac{1}{(n+1)} \quad \text{and} \quad \eta_i^n \leqslant \eta_{i+1}^n.$$

Let, in the euclidean plane,

$$M_0 = \{x_i^n\colon n = 1, 2, \ldots,\ i = 0, 1, \ldots, n\}, \quad \text{and}$$
$$M_1 = \{x_i\colon i = 0, 1, \ldots\},$$

where $x_i^n = \left(\frac{1}{n} - \eta_i^n, i\right)$ and $x_i = (0, i)$.

Let $M = M_0 \cup M_1$ with the euclidean metric ϱ. Then define a mapping $f\colon M \to M$ by

$$f(x_i^n) = \begin{cases} x_{i+1}^n & \text{if } i < n, \\ x_0^n & \text{if } i = n \end{cases}$$

and $f(x_i) = x_{i+1}$. It is easy to verify that (a) (M, ϱ) is finitely compact, (b) f is a locally nonexpansive mapping (hence, by Remark 2.4, it is uniformly locally nonexpansive), and if $\eta_i^n = 0$ for $n = 1, 2, \ldots$, $i = 0, 1, \ldots, n$, then f is a local isometry, while if $\eta_i^n < \eta_{i+1}^n$ for $n = 1, 2, \ldots, i = 0, 1, \ldots, n$, then the restriction of f to M_0 is not a uniform local isometry, and (c) $M^f = b^f(M) = M_0$.

The following is an example of a local isometry f of a connected finitely compact metric space (M, ϱ) into itself, for which $b^f(M)$ is a connected and dense subset of M, and $b^f(M) \neq M$.

(3.2) EXAMPLE. Let $M_0 = \{x_i^n\colon n = 1, 2, \ldots, \quad i = 0, 1, \ldots, 2^n - 1\}$, where $x_i^n = \left(\frac{1}{n}, i\right)$ and let $M_1 = \{(0, t)\colon t \geqslant 0\}$ and $M_2 = \{(1, t)\colon 0 \leqslant t \leqslant 1\}$. Let $A = M_0 \cup M_1 \cup M_2$ and let ϱ^* denote the euclidean metric on A.

Let $a_n = \varrho^*(x_0^n, x_{2^n}^{n+1})$ for $n = 1, 2, \ldots$ For each $n = 1, 2, \ldots$ and $i = 0, 1, \ldots, 2^{n+1} - 1$, consider the interval A_i^n of length a_n, identified with the set

$$A_i^n = \{(t, n, i)\colon 0 \leqslant t \leqslant a_n\},$$

and define the distance function on A_i^n by $|x_1 - x_2| = |t_1 - t_2|$, for $x_1 = (t_1, n, i)$, $x_2 = (t_2, n, i) \in A_i^n$.

Define M to be the space obtained by taking the disjoint sum

$$A \cup \bigcup_{n=1}^{\infty} \bigcup_{i=1}^{2^{n+1}-1} A_i^n$$

and then identifying $(0, n, i)$ with

$$\begin{cases} x_i^n & \text{if } 0 \leqslant i \leqslant 2^n - 1, \\ x_{i-2^n}^n & \text{if } 2^n \leqslant i \leqslant 2^{n+1} - 1, \end{cases}$$

and (a_n, n, i) with x_i^{n+1} for each $n = 1, 2, \ldots$, and $i = 0, 1, \ldots, 2^{n+1} - 1$ (see Fig. 1).

Let $f\colon M \to M$ be defined by

$$f(x) = \begin{cases} (t, n, s^n(i)) & \text{if } x = (t, n, i) \in A_i^n, \\ (0, t+1) & \text{if } x = (0, t) \in M_1, \\ (1, 1-t) & \text{if } x = (1, t) \in M_2, \end{cases}$$

where $s^n(i) = i+1$ for $i < 2^{n+1} - 1$ and $s^n(2^{n+1} - 1) = 0$.

Observe that $f(x_i^n) = x_{s^n(i)}^n$ for $n = 1, 2, \ldots, i = 0, 1, \ldots, 2^n - 1$, and thus the restriction $f|A$ is a local isometry with respect to the metric ϱ^* (cf. Example (3.1)).

Now, using the metric ϱ^* on A and the distance functions on the sets A_i^n, we proceed to define an extension ϱ of the metric ϱ^* to M, satisfying both of the following conditions:

(a) f is a local isometry of (M, ϱ) into itself,

(b) (M, ϱ) is a finitely compact metric space.

Let b_n, $n = 1, 2, \ldots$, be a sequence of integers such that $b_n \geqslant 2$ and $a_n/b_n \to 0$ as $n \to \infty$ (for example, such a sequence is $b_n = n \cdot 2^n$). For each $n = 1, 2, \ldots$, $i = 0, 1, \ldots, 2^{n+1} - 1$ and $k = 1, \ldots, b_n - 1$, let

$$p_{i,k}^n = \left(k \cdot \frac{a_n}{b_n}, n, i\right)$$

and

$$q_{i,k}^n = \begin{cases} i & \text{if } 0 \leqslant i \leqslant 2^n - 1, \\ i + k \cdot \dfrac{2^n}{b_n} & \text{if } 2^n \leqslant i \leqslant 2^{n+1} - 1. \end{cases}$$

Note that $p_{i,k}^n \in A_i^n$ and $q_{i,k}^n \in M_1$.

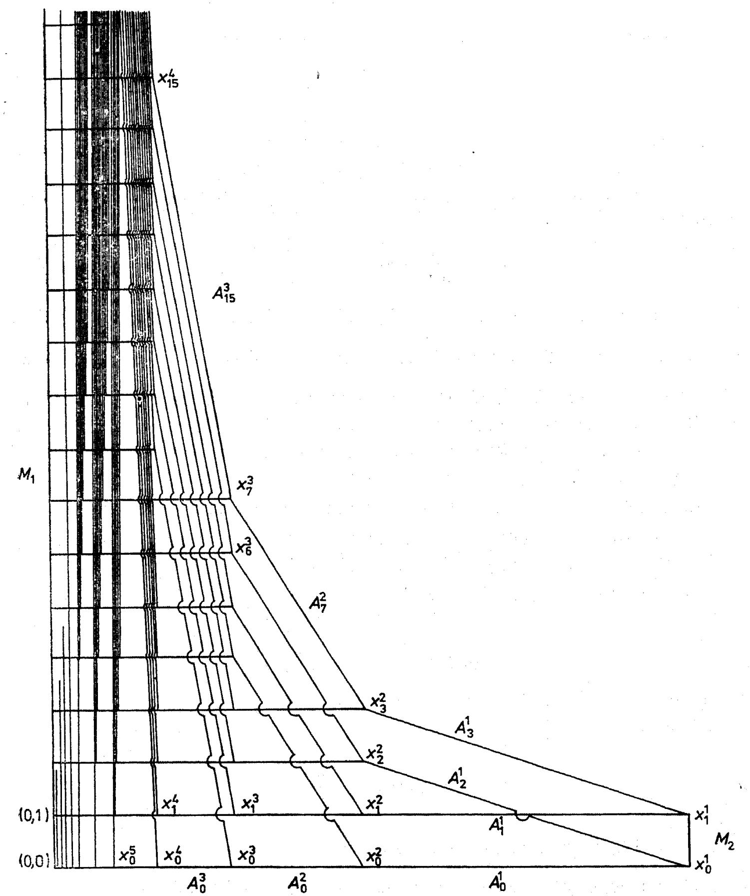

Fig. 1

Let Q be the sum of $A\times A$ and all $A_i^n\times A_i^n$ and all pairs (x, y), where $x = p_{i,k}^n$ and $y = q_{i,k}^n$, or $x = q_{i,k}^n$ and $y = p_{i,k}^n$, for $n = 1, 2, \ldots, i = 0, 1, \ldots, 2^{n+1}-1$ and $k = 1, \ldots, b_n-1$. Then define a function α on Q by

$$\alpha(x, y) = \begin{cases} \varrho^*(x, y) & \text{if } x, y\in A, \\ |x-y| & \text{if } x, y\in A_i^n, \\ 1 & \text{if } x = p_{i,k}^n \text{ and } y = q_{i,k}^n, \quad \text{or} \\ \overline{n} & \text{if } x = q_{i,k}^n \text{ and } y = p_{i,k}^n. \end{cases}$$

The desired extension ϱ of ϱ^* to M may thus be defined as follows:

$$\varrho(x, y) = \inf \sum_{i=0}^{k-1} \alpha(z_i, z_{i+1}),$$

where the infimum is taken over all possible finite chains of points $z_0, z_1, \ldots, z_k$ of M such that $z_0 = x$, $z_k = y$ and $(z_i, z_{i+1})\in Q$ for each $i = 0, \ldots, k-1$.

It can easily be verified that ϱ is an extension of the metric ϱ^* to M and that f maps the set

$$M\setminus \bigcup_{n=1}^{\infty} (A_{2^n-1}\cup A_{2^{n+1}-1})$$

isometrically into M. Since the metric ϱ is locally identical with the distance functions on each of A_i^n, we infer that ϱ satisfies condition (a). Since $\frac{a_n}{b_n}\to 0$ as $n\to\infty$, each bounded sequence of points of (M, ϱ) has a convergent subsequence, i.e., ϱ satisfies condition (b). It can also be easily seen that $b^f(M) = M^f = M\setminus M_1$ (and, moreover, M^f is the set of all periodic points of f) and that it is a path-connected, dense and open subset of M. Hence ϱ satisfies also the following condition:

(c) $b^f(M) = M^f$ is a connected, dense and open subset of M, and $b^f(M)\neq M$.

The following example shows that there exists a local isometry f of a finitely compact metric space (M, ϱ) into itself for which $M^f\setminus b^f(M)\neq \emptyset$.

(3.3) Example. Let C_0 be the Cantor set:

$$C_0 = \left\{\sum_{i=1}^{\infty} \frac{t_i}{3^i}: t_i = 0 \text{ or } 2\right\}.$$

For each $n = 1, 2, \ldots$ let

$$C_n = \left\{2n+\frac{x}{3^{n-1}}: x\in C_0\right\}.$$

Let $M = \bigcup_{n=0}^{\infty} C_n$ with absolute value distance ϱ.

If A is a nonempty and bounded subset of the real line R and if $B = \{x+a: x\in A\}$, then the (unique) translation of A onto B will be denoted by $T(A, B)$. Consider the following subsets of the real line R:

$$C_0^n = C_0\cap \left[\frac{2}{3^n}, \frac{1}{3^{n-1}}\right],$$

$$C_n^0 = C_n\cap \left[2n, 2n+\frac{1}{3^n}\right],$$

$$C_n^1 = C_n\setminus C_n^0,$$

for $n = 1, 2, \ldots$ Then we define a mapping $f: M\to M$ by: $f|_{C_0} = T(C_0, C_1)$, $f|_{C_n^0} = T(C_n^0, C_0^n)$ and $f|_{C_n^1} = T(C_n^1, C_{n+1})$.

The following facts may easily be verified: (a) (M, ϱ) is finitely compact, (b) f is a local isometry, and (c) $0\in M^f\setminus b^f(M)$.

Next we give an example of a local isometry f of a finitely compact metric space (M, ϱ) into itself for which $M^f\neq \bigcup \{w_f(x): x\in M\}$ (cf. (a) of (2.1)).

(3.4) Example. Let (M, ϱ), C_n, C_n^0, C_n^1 for $n = 1, 2, \ldots$, and let T be as in Example (3.3). For every $n = 1, 2, \ldots$, let

$$D_0^n = C_0\cap \left[1-\frac{1}{3^n}, 1\right].$$

Define a mapping $f: M\to M$ by: $f|_{C_0} = T(C_0, C_1)$, $f|_{C_n^0} = T(C_n^0, D_0^n)$ and $f|_{C_n^1} = T(C_n^1, C_{n+1})$.

It is easy to see that (a) (M, ϱ) is finitely compact, (b) f is a local isometry, and (c) $1\in w_f(0)$ (i.e., a subsequence of $\{f^n(0)\}_{n=0}^{\infty}$ converges to 1) while $w_f(1) = \emptyset$ (i.e., no subsequence of $\{f^n(1)\}_{n=0}^{\infty}$ is convergent).

4. Preliminary results on uniformly locally nonexpansive mappings. Let us prove the following

(4.1) Proposition. *Let f be a uniformly locally nonexpansive mapping of a metric space (M, ϱ) into itself. Then*

(a) *the set $b^f(M)$ is open,*

(b) *the induced metric ϱ_f is a metric on the set $b^f(M)$ such that ϱ_f and ϱ are locally identical on $b^f(M)$ and the restriction of f to $b^f(M)$ is a nonexpansive mapping with respect to ϱ_f,*

(c) *there exist a sequence A_n, $n = 0, 1, \ldots$, of bounded and open sets such that:*

(3) *$f(A_n)\subset A_n$ for every $n = 0, 1, \ldots$,*

(4) *$A_n\subset A_{n+1}$ for every $n = 0, 1, \ldots$, and*

(5) *$\bigcup_{n=0}^{\infty} A_n = b^f(M)$.*

First let us note the following

(4.2) Lemma. *Let f be a uniformly locally nonexpansive mapping of a metric space (M, ϱ) into itself. Then for every bounded subset A of M there exists a number $\varepsilon > 0$ such that $\varrho(f(x), f(y)) \leqslant \varrho(x, y)$ for each $z \in A$ and all $x, y \in K_\varrho(z, \varepsilon)$.*

Proof. For a fixed number $r > 0$ there is an $\varepsilon_r > 0$ such that the restriction of f to $K_\varrho(A, r) = \{y \in M: \varrho(y, A) < r\}$ is an ε_r-locally nonexpansive mapping. Hence the number $\varepsilon = \min\{r, \varepsilon_r\}$ is the required one.

Proof of Proposition (4.1). Let z be a given point of $b^f(M)$. Let $\varepsilon = \varepsilon_z/2$, where ε_z is a number defined by Lemma (4.2) for the bounded set $\{f^n(z): n \geqslant 0\}$ and let

$$V_z = K_\varrho(\{f^n(z): n \geqslant 0\}, \varepsilon).$$

Then V_z is a bounded neighbourhood of z such that $f(V_z) \subset V_z$. Moreover, $V_z \subset b^f(M)$ and $f|_{V_z}$ is an ε-locally nonexpansive mapping of V_z into itself. It follows by (d) of Remark (2.2) and (c) of Remark (2.7) that ϱ_f and ϱ are ε-locally identical on V_z. We have thus proved (a) and (b) (cf. (b) of Remark (2.7)).

To prove (c) assume that $b^f(M) \neq \emptyset$ and fix a point z_0 of $b^f(M)$. Let $A = \{f^n(z_0): n \geqslant 0\}$. Then, for every $i = 0, 1, \ldots$, we define

$$A_i = \{x \in b^f(M): \varrho_f(x, A) < i+1\}.$$

Since $f(A) \subset A$ and f is nonexpansive with respect to ϱ_f, the sets A_i, $i = 0, 1, \ldots$, satisfy condition (3) and, clearly, conditions (4) and (5). In order to finish the proof, it remains to show that for every $i = 0, 1, \ldots$, the set A_i is bounded and open. By (d) of Remark (2.7),

$$\operatorname{diam}_\varrho(A_i) \leqslant \operatorname{diam}_{\varrho_f}(A_i) \leqslant \operatorname{diam}_{\varrho_f}(A) + 2(i+1) = \operatorname{diam}_\varrho(A) + 2(i+1) < \infty,$$

i.e., A_i is bounded. Since $b^f(M)$ is open and ϱ_f and ϱ are topologically equivalent on $b^f(M)$ (cf. Remark (2.6)), the set A_i is open, which completes the proof.

Remark. Example (3.1) shows it is not generally true that the set $b^f(M)$ is closed as well as that the identity mapping of $b^f(M)$ is a uniform local isometry of $(b^f(M), \varrho)$ into $(b^f(M), \varrho_f)$.

We now consider the case of ε-locally nonexpansive mappings.

(4.3) Proposition. *Let f be an ε-locally nonexpansive mapping of a metric space (M, ϱ) into itself. Then*

(a) $M = b^f(M) \cup [M \setminus b^f(M)]$ is a decomposition of M into invariant and ε-separated sets; the induced metric ϱ_f is a metric on $b^f(M)$ such that ϱ_f and ϱ are ε-locally identical on $b^f(M)$ and the restriction of f to $b^f(M)$ is a nonexpansive mapping with respect to ϱ_f,

(b) for every $x \in M$, $f(C_\varepsilon(x)) \subset C_\varepsilon(f(x))$ and the induced metric ϱ_f is a metric on $C_\varepsilon(x)$ such that ϱ_f and ϱ are ε-locally identical on $C_\varepsilon(x)$ and the restriction of f to $C_\varepsilon(x)$ is a nonexpansive mapping of $(C_\varepsilon(x), \varrho_f)$ into $(C_\varepsilon(f(x)), \varrho_f)$.

Proof. (a) In view of (b) of Remark (2.2) and (b), (c) of Remark (2.7) and (a) of Remark (2.9), it suffices to show that $b^f(M)$ is the union of its ε-components. Let $x \in b^f(M)$ and $y \in C_\varepsilon(x)$. Then there is an ε-chain $z_0, z_1, \ldots, z_k$ from x to y. Let $d = \sum_{i=0}^{k-1} \varrho(z_i, z_{i+1})$. Since, for each $i = 0, 1, \ldots$, f^i is also ε-locally nonexpansive,

$$\varrho(f^i(y), \{f^n(x): n \geqslant 0\}) \leqslant \varrho(f^i(y), f^i(x)) \leqslant \sum_{j=0}^{k-1} \varrho(f^i(z_j), f^i(z_{j+1})) \leqslant d,$$

i.e., $f^i(y) \in \overline{K}_\varrho(\{f^n(x): n \geqslant 0\}, d)$. Since $\{f^n(x): n \geqslant 0\}$ is bounded, this shows that the sequence $\{f^n(y)\}_{n=0}^\infty$ is also bounded, i.e., $y \in b^f(M)$.

(b) Let $y \in C_\varepsilon(x)$ and let $z_0, z_1, \ldots, z_k$ be an ε-chain from x to y. Since f is ε-locally nonexpansive, $f(z_0), f(z_1), \ldots, f(z_k)$ is an ε-chain from $f(x)$ to $f(y)$, whence $f(y) \in C_\varepsilon(f(x))$. For every $n = 0, 1, \ldots$, we have

$$\varrho(f^n(x), f^n(y)) \leqslant \sum_{i=0}^{k-1} \varrho(f^n(z_i), f^n(z_{i+1})) \leqslant d,$$

where $d = \sum_{i=0}^{k-1} \varrho(z_i, z_{i+1})$. Therefore $f(C_\varepsilon(x)) \subset C_\varepsilon(f(x))$ and $\varrho_f(x, y) \leqslant d < \infty$, which, in view of (b) and (c) of Remark (2.7), completes the proof.

In the sequel we will need the following lemmas.

(4.4) Lemma. *Let f be a mapping of a metric space (M, ϱ) into a metric space (N, σ). If one of the following conditions holds:*

(a) f is an ε-locally nonexpansive mapping and (M, ϱ) is ε-convex,

(b) f is an uniformly locally nonexpansive mapping and (M, ϱ) is ε-convex for each $\varepsilon > 0$.

(c) f is a locally nonexpansive mapping and (M, ϱ) is convex and complete, then f is also a (globally) nonexpansive mapping.

Proof. Assume that (a) holds. Then for all $x, y \in M$,

$$(6) \qquad \varrho(x, y) = \inf \sum_{i=0}^{k-1} \varrho(z_i, z_{i+1}) \geqslant \inf \sum_{i=0}^{k-1} \sigma(f(z_i), f(z_{i+1})) \geqslant \sigma(f(x), f(y)),$$

where the infimum is taken over all possible ε-chains $z_0, z_1, \ldots, z_k$ from x to y.

Assume that (b) holds. Let $x, y \in M$ and let $d = \varrho(x, y)$. Then there is a number $\varepsilon > 0$ such that the restriction of f to $K_\varrho(x, d+1)$ is an ε-locally nonexpansive mapping. Since (M, ϱ) is ε-convex, it follows that the inequalites in (6) hold with the infimum taken over all possible ε-chains $z_0, z_1, \ldots, z_k \in K_\varrho(x, d+1)$ from x to y.

Assume that (c) holds. Let $x, y \in M$. Then, by a theorem of Menger (cf. [1, p. 41]), there exists a metric segment $L \subset M$ whose extremities are x and y. Since L is compact, there is an $\varepsilon > 0$ such that $f|_L$ is an ε-locally nonexpansive mapping. Since L is ε-convex (cf. the proof of Remark (2.10)), it follows from (a) above that $f|_L$ is nonexpansive. Hence, $\varrho(f(x), f(y)) \leqslant \varrho(x, y)$, which completes the proof.

(4.5) LEMMA. *Let f be a mapping of a metric space (M, ϱ) into itself. If one of the following conditions holds:*

(a) *f is nonexpansive,*

(b) *f is ε-locally nonexpansive and (M, ϱ) is ε-chainable,*

(c) *f is uniformly locally nonexpansive and (M, ϱ) is ε-convex for each $\varepsilon > 0$,*

(d) *f is locally nonexpansive and (M, ϱ) is convex and complete,*

then either $b^f(M) = \emptyset$ or $b^f(M) = M$.

Proof. Assume that $b^f(M) \neq \emptyset$. If f is nonexpansive, then for a fixed point x of $b^f(M)$ we have

$$\varrho(f^i(y), \{f^n(x): n \geqslant 0\}) \leqslant \varrho(f^i(y), f^i(x)) \leqslant \varrho(y, x),$$

i.e., $f^i(y) \in \overline{K}_\varrho(\{f^n(x): n \geqslant 0\}, \varrho(y, x))$ for every $y \in M$ and all $i = 0, 1, \ldots$ Since $\{f^n(x): n \geqslant 0\}$ is bounded, this shows that $b^f(M) = M$.

It follows from (a) of Proposition (4.3) and (b) and (c) of Lemma (4.4) that each of the conditions (b), (c) and (d) together with $b^f(M) \neq \emptyset$ implies that $b^f(M) = M$. This completes the proof.

(4.6) LEMMA. *Let f be a mapping of a metric space (M, ϱ) into itself. If one of the following conditions holds:*

(a) *f is ε-locally nonexpansive,*

(b) *f is uniformly locally nonexpansive and $b^f(M) = M$,*

then the f-closure of M, M^f, is a closed subset of M and

$$M^f = \bigcup \{w_f(x): x \in M\}. \tag{7}$$

Proof. Assume that (a) holds. Given a sequence of points $x_n \in M^f$, $n = 0, 1, \ldots$, such that $\lim x_n = x$ and a number $\delta > 0$, there exist integers $n, m \geqslant 0$ with

$$\varrho(x, x_n) < \tfrac{1}{3}\eta \quad \text{and} \quad \varrho(x_n, f^m(x_n)) < \tfrac{1}{3}\eta,$$

where $\eta = \min\{\delta, \varepsilon\}$. Hence,

$$\varrho(x, f^m(x)) \leqslant \varrho(x, x_n) + \varrho(x_n, f^m(x_n)) + \varrho(f^m(x_n), f^m(x)) \leqslant \tfrac{1}{3}\eta + \tfrac{1}{3}\eta + \varrho(x_n, x) < \eta \leqslant \delta.$$

This shows that $x \in M^f$. Therefore M^f is closed. Relation (7) is an immediate consequence of [8, Proposition 1] and our definitions (cf. (2.1)).

Assume that (b) holds. Let x_n, $n = 0, 1, \ldots$ be a sequence of points of M^f so that $\lim x_n = x$. By (c) of Proposition (4.1), there exists a bounded and invariant neighbourhood V of x. Thus, for some $\varepsilon > 0$, the restriction $f|_V$ is an ε-locally nonexpansive mapping of V into itself. From the above it follows that $x \in V^f \subset M^f$. Hence M^f is closed. In order to prove (7), it suffices to show that, for each $x \in M$, $w_f(x) \subset M^f$. Let x be a given point of M and let $A = \mathrm{cl}\{f^n(x): n \geqslant 0\}$. Thus A is bounded and invariant and $x \in A$. Hence there is an $\varepsilon > 0$ such that $f|_A$ is an ε-locally nonexpansive mapping of A into itself and it follows from the above that $w_f(x) \subset A^f \subset M^f$. This completes the proof.

Remark. It follows from Lemma (4.6) that for every uniformly locally nonexpansive mapping f of a metric space (M, ϱ) into itself we have

$$[b^f(M)]^f = \bigcup \{w_f(x): x \in b^f(M)\}.$$

Remark. Examples (3.1) and (3.4) show that the assumption $b^f(M) = M$ cannot be omitted in (b) of Lemma (4.6) even if M is finitely compact.

5. Locally nonexpansive mappings of finitely totally bounded metric spaces. In this section we shall give an answer to question A of the Introduction. For this reason we investigate the following related question: *If f is a uniformly locally nonexpansive (resp. an ε-locally nonexpansive) mapping of a finitely totally bounded metric space (M, ϱ) into itself and if some subsequence of $\{f^n(x)\}_{n=0}^\infty$, $x \in M$, is bounded, is then the sequence $\{f^n(x)\}_{n=0}^\infty$ bounded?*

Example (3.3) yields a negative answer to the above question for uniformly locally nonexpansive mappings. However, the answer to this question will be yes for ε-locally nonexpansive mappings.

We make the following

(5.1) DEFINITION. Let f be a mapping of a metric space (M, ϱ) into itself. Then for every $A \subset M$, we will denote

(a) $w(A)^f = \bigcup \{B \subset A: \mathrm{diam}_\varrho(B) < \infty$ and $f(B)$ is a dense subset of B$\}$,

(b) $b_0^f(A) = \{x \in A: \{f^n(x)\}_{n=0}^\infty$ has a bounded subsequence$\}$.

(5.2) Remark. Let f be a mapping of a metric space (M, ϱ) into itself and let $A \subset M$. Then

(a) $w(A)^f \subset b^f(A) \subset b_0^f(A)$ and $A^f \subset b_0^f(A)$,

(b) $f(w(A)^f)$ is a dense subset of $w(A)^f$,

(c) if $f(A) \subset A$, then $f(b_0^f(A)) \subset b_0^f(A)$,

(d) if $f(A) \subset A$ and if f is continuous, then $[b^f(A)]^f \subset w(A)^f$.

We prove the following

(5.3) PROPOSITION. *Let f be a uniformly locally nonexpansive mapping of a finitely totally bounded metric space (M, ϱ) into itself. Then*

(a) *$w(M)^f = [b^f(M)]^f$,*

(b) *f maps $w(M)^f$ isometrically into itself with respect to ϱ_f.*

Before proving this proposition, we state the following fact from [6]:

(5.4) LEMMA (see [6, Theorem 1]). *If f is an ε-locally nonexpansive mapping of a totally bounded metric space (M, ϱ) into itself, then*

(a) *$w(M)^f = M^f$,*

(b) *f maps $w(M)^f$ isometrically into itself with respect to ϱ_f.*

Proof of Proposition (5.3). (a) By (a) and (d) of Remark (5.2), it remains to verify that $w(M)^f \subset M^f$. Let $x \in w(M)^f$. Then there exists a bounded set $A \subset M$ such that $x \in A$, $f(A) \subset A$ and $A = w(A)^f$. Since for some $\varepsilon > 0$ the restriction $f|_A$ is

ε-locally nonexpansive, it follows by (a) of Lemma (5.4) that $A = A^f \subset M^f$. Hence $x \in M^f$.

(b) Let A_n, $n = 0, 1, \ldots$ be a sequence of bounded sets satisfying conditions (3)–(5) (cf. (c) of Proposition (4.1)). It follows by (3) and (b) of Lemma (5.4) that for every $n = 0, 1, \ldots, f$ maps $(A_n)^f$ isometrically into itself with respect to ϱ_f. From (4), $(A_n)^f \subset (A_{n+1})^f$ for each $n = 0, 1, \ldots$, and by (5) we have

$$[b^f(M)]^f = \bigcup_{n=0}^{\infty} (A_n)^f .$$

Hence, by (a) above, f maps $w(M)^f$ isometrically into itself with respect to ϱ_f. This completes the proof

(5.5) COROLLARY. *Under the assumptions of Proposition* (5.3) *and if* $b^f(M) = M$ *we have*

$$M^f = w(M)^f = \bigcup \{w_f(x) \colon x \in M\} ,$$

and M^f *is a closed subset of* M.

Proof. This follows from (a) of Proposition (5.3) and (b) of Lemma (4.6).

(5.6) COROLLARY. *Under the assumptions of Proposition* (5.3), *the restriction of f to* $w(M)^f$ *is a local isometry. If, moreover,* $b^f(M) = M$, *then the restriction of f to* M^f *is a local isometry.*

Proof. This follows from (b) of Proposition (5.3), Corollary (5.5) and the fact that ϱ_f and ϱ are locally identical on $b^f(M)$ (cf. (b) of Proposition (4.1)).

We remark that every locally nonexpansive mapping of a finitely compact metric space into itself satisfies the assumptions of Proposition (5.3) (cf. Remark (2.4)). Thus we have the following

(5.7) COROLLARY. *Let f be a locally nonexpansive mapping of a finitely compact metric space* (M, ϱ) *into itself. If* $b^f(M) = M$, *then the restriction of f to* M^f *is a local isometry of* M^f *onto itself.*

Proof. In view of Corollary (5.6) and the remark above, it remains to show that $f(M^f) = M^f$. By Corollary (5.5), M^f is closed, hence finitely compact. Let $x \in M^f$ and let $A_x = \mathrm{cl}\{f^n(x) \colon n \geq 0\}$. Thus $x \in A_x \subset M^f$, A_x is compact and $f(A_x)$ is a dense subset of A_x; hence $f(A_x) = A_x$. This completes the proof.

Remark. It is not generally true that under the assumptions of Proposition (5.3) the restriction of f to $w(M)^f$ is a uniform local isometry. Example (3.1) gives a simple illustration of this.

We now consider the case of ε-locally nonexpansive mappings. In [7] we proved the following result:

(5.8) LEMMA (see [7, Theorem (5.6)]). *Let f be a nonexpansive mapping of a finitely totally bounded metric space* (M, ϱ) *into itself. If* $b_0^f(M) \neq \emptyset$, *then* $b^f(M) = M$.

We are now in a position to prove the main result of this section.

(5.9) THEOREM. *Let f be an ε-locally nonexpansive mapping of a finitely totally bounded metric space* (M, ϱ) *into itself. Then*

(a) $b_0^f(M) = b^f(M)$,

(b) $w(M)^f = M^f$,

(c) *the restriction of f to M^f is an ε-local isometry.*

Proof. (a) In view of (a) of Remark (5.2), we only have to show that $b_0^f(M) \subset b^f(M)$. Let x be a given point of $b_0^f(M)$. Since (M, ϱ) is finitely totally bounded, there exist integers n, m such that

$$0 < n < m \quad \text{and} \quad \varrho(f^n(x), f^m(x)) < \varepsilon .$$

Let $k = m-n$ and let $h = f^k$ and, for every $i = 0, 1, \ldots, k$, let $x_i = f^{n+i}(x)$. Since f is ε-locally nonexpansive, we have $\varrho(x_i, h(x_i)) < \varepsilon$ for $i = 0, 1, \ldots, k$, and h is also an ε-locally nonexpansive mapping. Thus, by (b) of Proposition (4.3) and (a) of Remark (2.9),

$$h(C_\varepsilon(x_i)) \subset C_\varepsilon(x_i) \quad \text{for each } i = 0, 1, \ldots, k .$$

Hence, by (b) of Proposition (4.3), for each integer $i = 0, 1, \ldots, k$, the induced metric ϱ_h is a metric on $C_\varepsilon(x_i)$ such that $(C_\varepsilon(x_i), \varrho_h)$ is a finitely totally bounded metric space, and the restriction of h to $C_\varepsilon(x_i)$ is a nonexpansive mapping of $(C_\varepsilon(x_i), \varrho_h)$ into itself. By the assumption, $x \in b_0^f(M) \neq \emptyset$. Thus it follows from Lemma (5.8) that for each $i = 0, 1, \ldots, k$ the sequence $\{h^n(x_i)\}_{n=0}^{\infty}$ is bounded with respect to the metric ϱ_h; hence it is bounded with respect to the metric ϱ because $\varrho_h \geq \varrho$. Since

$$\{f^n(x) \colon n \geq 0\} = \bigcup_{i=0}^{k} \{h^n(x_i) \colon n \geq 0\} \cup \{x, f(x), \ldots, f^{n-1}(x)\} ,$$

this shows that the sequence $\{f^n(x)\}_{n=0}^{\infty}$ is bounded, i.e., $x \in b^f(M)$. Therefore $b_0^f(M) \subset b^f(M)$, as desired.

(b) It follows by (a) above and (a) of Proposition (5.3), that $w(M)^f = b_0^f(M) \cap M^f$ (cf. (a) of Remark (2.2)). Since $M^f \subset b_0^f(M)$, we have $w(M)^f = M^f$.

(c) This follows from (b) above, (b) of Proposition (5.3) and the fact that ϱ_f and ϱ are ε-locally identical on $b^f(M)$ (cf. (a) of Proposition (4.3)). Thus the proof of the theorem is complete.

Remark. It follows by (b) of Remark (2.9) and (a) of Lemma (4.4) that in the proof of (a) of the above theorem we can also use the metric ϱ_ε.

As an immediate consequence of (a) of Theorem (5.9) and (a) of Proposition (4.3) we obtain

(5.10) COROLLARY. *If f is an ε-locally nonexpansive mapping of a finitely totally bounded metric space* (M, ϱ) *into itself, then*

$$M = b^f(M) \cup [M \setminus b_0^f(M)]$$

is a decomposition of M into invariant and ε-separated sets. In particular, $M^f \subset b^f(M)$.

Consequently, we have

(5.11) COROLLARY. *If f is an ε-locally nonexpansive mapping of an ε-chainable and finitely totally bounded metric space (M, ϱ) into itself, then either $b_0^f(M) = \emptyset$ or $b^f(M) = M$.*

We now prove the following

(5.12) THEOREM. *Let f be a mapping of a metric space (M, ϱ) into itself such that $b_0^f(M) \neq \emptyset$. If one of the following holds:*

(a) *f is nonexpansive and (M, ϱ) is finitely totally bounded,*

(b) *f is ε-locally nonexpansive and (M, ϱ) is finitely totally bounded and ε-convex,*

(c) *f is uniformly locally nonexpansive and (M, ϱ) is finitely totally bounded and ε-convex for each $\varepsilon > 0$,*

(d) *f is locally nonexpansive and (M, ϱ) is finitely compact and convex,*

then $b^f(M) = M$ and f maps M^f isometrically into itself.

Proof. Assume that (a) holds. Then, by Lemma (5.8), we have $b^f(M) = M$. Since $\varrho_f = \varrho$, it follows from (b) of Proposition (5.3) and (b) of Theorem (5.9) that the restriction of f to M^f is an isometry. Therefore, by Lemma (4.4), each of the statements (b), (c) and (d) implies the assertion.

Remark. The following is a consequence of Theorem (5.9) (cf. also Theorem (6.9) of the next section): If f is an isometry of a finitely totally bounded metric space (M, ϱ) into itself and if $b_0^f(M) \neq \emptyset$, then

$$M^f = w(M)^f = b^f(M) = M .$$

In particular, f maps M onto a dense subset of itself (cf. (c) of Remark (2.2)).

The following corollary is immediate from Theorem (5.12) and the remark above.

(5.13) COROLLARY. *Under the assumptions of Theorem (5.12), f is an isometry if and only if $M^f = M$.*

Remark. Let f be a mapping of a metric space (M, ϱ) into itself. If f is a locally nonexpansive (resp. a uniformly locally nonexpansive) mapping which is defined by condition (1) with the strict inequality sign for all $x, y \in K_\varrho(z, \varepsilon)$, $x \neq y$, then f is said to be locally contractive (resp. uniformly locally contractive). It follows from Corollary (5.6) (cf. also Remark (2.4)) that if f is a locally contractive (resp. uniformly contractive) and if (M, ϱ) is finitely compact (resp. finitely totally bounded), then $w(M)^f$ is the set of all periodic points of f and it is a discrete subset of M. A similar statement can be formulated for ε-locally contractive mappings.

6. Decomposition theorems for local isometries.

We introduce the following

(6.1) DEFINITION. Let f be a mapping of a metric space (M, ϱ) into itself. Then for every $A \subset M$ we denote

$$A_i^f = A \cap M_i^f \quad \text{for } i = 0, 1, \ldots$$

where

$$M_i^f = \begin{cases} M^f & \text{if } i = 0, \\ f^{-i}(M^f) \setminus f^{-i+1}(M^f) & \text{if } i > 0. \end{cases}$$

(6.2) Remark. Let f be a mapping of a metric space (M, ϱ) into itself and let $A \subset M$. Then

(a) the sets A_i^f, $i = 0, 1, \ldots$, are disjoint,

(b) if $f(A) \subset A$, then $f(A_0^f) \subset A_0^f$ and $f(A_{i+1}^f) \subset A_i^f$ for $i = 0, 1, \ldots$

(6.3) Remark. It follows from (a) of Proposition (5.3) that if f is a uniformly locally nonexpansive mapping of a finitely totally bounded metric space (M, ϱ) into itself, then

$$[b^f(M)]_i^f = \begin{cases} w(M)^f & \text{if } i = 0, \\ f^{-i}(w(M)^f) \setminus f^{-i+1}(w(M)^f) & \text{if } i > 0. \end{cases}$$

Recall that, for a uniform local isometry f of a metric space (M, ϱ) into itself and for a bounded set $A \subset M$, ε_A denotes a positive number so that the restriction $f|_A$ is an ε_A-local isometry. The following lemma is an immediate consequence of Theorem (3.4) of [6].

(6.5) LEMMA. *Let f be a uniform local isometry of a finitely totally bounded metric space (M, ϱ) into itself and let A be a bounded subset of M such that $f(A) \subset A$. Then*

$$A = \bigcup_{i=0}^{\infty} A_i^f$$

and the sets A_i^f, $i = 0, 1, \ldots$, are ε_A-separated. Moreover, there exists an integer $i_A \geqslant 0$ such that $A_i^f = \emptyset$ for every $i \geqslant i_A$.

We are now in a position to state and prove the following decomposition theorem for uniform local isometries.

(6.6) THEOREM. *Let f be a uniform local isometry of a finitely totally bounded metric space (M, ϱ) into itself. Then*

$$b^f(M) = \bigcup_{i=0}^{\infty} [b^f(M)]_i^f ,$$

and the sets $[b^f(M)]_i^f$, $i = 0, 1, \ldots$, are open and disjoint.

Proof. Let A_n, $n = 0, 1, \ldots$, be a sequence of bounded and open sets satisfying conditions (3)–(5) (see (c) of Proposition (4.1)). It follows from Lemma (6.5) that the sets $(A_n)_i^f$, $n, i = 0, 1, \ldots$, are open. Using (5) and Definition (6.1), we have

$$(8) \qquad \bigcup_{n=0}^{\infty} (A_n)_i^f = \bigcup_{n=0}^{\infty} A_n \cap M_i^f = b^f(M) \cap M_i^f = [b^f(M)]_i^f ,$$

for every $i = 0, 1, \dots$ Hence the sets $[b^f(M)]_i^f$, $i = 0, 1, \dots$, are open and disjoint ((a) of Remark (6.2)). Using (5) and Lemma (6.5) and (8), we obtain

$$b^f(M) = \bigcup_{n=0}^{\infty} A_n = \bigcup_{n=0}^{\infty} \bigcup_{i=0}^{\infty} (A_n)_i^f = \bigcup_{i=0}^{\infty} \bigcup_{n=0}^{\infty} (A_n)_i^f = \bigcup_{i=0}^{\infty} [b^f(M)]_i^f .$$

This completes the proof.

As an immediate consequence of Theorem (6.6) we get

(6.7) COROLLARY. *Let f be a uniform local isometry of a finitely totally bounded metric space (M, ϱ) into itself. If $b^f(M) = M$, then*

$$M = \bigcup_{i=0}^{\infty} M_i^f ,$$

and the sets M_i^f, $i = 0, 1, \dots$, are open and disjoint.

We have the following immediate consequence of Corollary (6.7) and Lemma (5.8).

(6.8) COROLLARY. *Let f be a nonexpansive uniform local isometry of a finitely totally bounded metric space (M, ϱ) into itself. If $b_0^f(M) \neq \emptyset$, then*

$$M = b^f(M) = \bigcup_{i=0}^{\infty} M_i^f ,$$

and the sets M_i^f, $i = 0, 1, \dots$, are open and disjoint.

Remark. It follows from Remark (2.4) that Theorem (6.6) and Corollaries (6.7) and (6.8) remain true for local isometries of finitely compact metric spaces. Thus, these results extend and give simplified proofs for Theorems (3.1) and (3.2) in [5].

We can now prove our decomposition theorem for ε-local isometries.

(6.9) THEOREM. *Let f be an ε-local isometry of a finitely totally bounded metric space (M, ϱ) into itself. Then*

$$b_0^f(M) = b^f(M) = \bigcup_{i=0}^{\infty} M_i^f ,$$

and the sets $[M \setminus b_0^f(M)]$, $M_0^f, M_1^f, \dots$, are ε-separated.

Proof. By (a) of Theorem (5.9), $b_0^f(M) = b^f(M)$. Thus, $M_i^f \subset b^f(M)$ and hence $[b^f(M)]_i^f = M_i^f$ for every $i = 0, 1, \dots$ Thus, from Theorem (6.6), we have $b^f(M) = \bigcup_{i=0}^{\infty} M_i^f$.

To show that the sets $[M \setminus b_0^f(M)]$, $M_0^f, M_1^f, \dots$, are ε-separated, let $x, y \in M$ be such that, for some integer i, $x \in M_i^f$ and $y \notin M_i^f$. If $y \notin b_0^f(M)$, then from the above and Proposition (4.3) (or, Corollary (5.10)) we have $\varrho(x, y) \geqslant \varepsilon$. If $y \in b_0^f(M)$, then it follows by (c) of Proposition (4.1) that there exists a bounded set $A \subset M$ such that $x, y \in A$ and $f(A) \subset A$. Thus $x \in A_i^f$ and $y \notin A_i^f$. Hence, from Lemma (6.5) (with $\varepsilon_A = \varepsilon$) we infer that $\varrho(x, y) \geqslant \varepsilon$. This proves the assertion.

As an immediate consequence of Theorem (6.9) and Corollary (6.8) we get

(6.10) COROLLARY. *Let f be a nonexpansive ε-local isometry of a finitely totally bounded metric space (M, ϱ) into itself. If $b_0^f(M) \neq \emptyset$, then*

$$M = b^f(M) = \bigcup_{i=0}^{\infty} M_i^f ,$$

and the sets M_i^f, $i = 0, 1, \dots$, are ε-separated.

Remark. In [5] it is shown that if f is a local isometry of a finitely compact metric space (M, ϱ) into itself, and if $b^f(M) = M$, then there exists a unique decomposition of M into open and disjoint sets, $M = M_0 \cup M_1 \cup \dots$, such that f maps M_0 injectively into itself and $f(M_{i+1}) \subset M_i$ for $i = 0, 1, \dots$ It follows from Corollaries (6.7) and (5.6) and Remark (6.2) that in this case we have $M_i = M_i^f$ for each $i = 0, 1, \dots$ Thus, Theorem (6.9) can be reformulated as follows: Let f be an ε-local isometry of a finitely compact metric space (M, ϱ) into itself. Then there exists a unique decomposition of M into open and disjoint sets, $M = M_0 \cup M_1 \cup \cup \dots \cup M_\infty$, such that (i) f maps M_0 injectively into itself and $f(M_{i+1}) \subset M_i$ for $i = 0, 1, \dots$, and (ii) if $x \in M_i$ for some $i = 0, 1, \dots$, then the sequence $\{f^n(x)\}_{n=0}$ is bounded, while if $x \in M_\infty$, then no subsequence of $\{f^n(x)\}_{n=0}$ is bounded. Moreover, the sets M_i, $i = 0, 1, \dots$, are ε-separated.

7. Some conditions under which local isometries are isometries. In this section we first give some conditions under which decomposition theorems reduce to the equality $M = b^f(M) = M^f$. Then we apply the results of §§ 4 and 5 to obtain an answer to question B of the Introduction. As a consequence of Theorem (6.6) we have

(7.1) THEOREM. *Let f be a uniform local isometry of a connected finitely totally bounded metric space (M, ϱ) into itself. If $b^f(M) = M$, then the induced metric ϱ_f is a metric on M such that*

(a) *ϱ_f and ϱ are locally identical on M (in particular, they are topologically equivalent),*

(b) *the space (M, ϱ_f) is finitely totally bounded,*

(c) *f is an isometry with respect to ϱ_f.*

Proof. Since M is connected and $b^f(M) = M$, Corollary (6.7) implies that $M = b^f(M) = M^f$. Thus (a) and (b) follow from (b) of Proposition (4.1) and the fact that $\varrho_f \geqslant \varrho$ (cf. Remark (2.6)), while (c) is a consequence of Proposition (5.3).

Remark. It follows from Remark (2.4) that the above theorem remains true after replacing "uniform local isometry" and "finitely totally bounded" by "local isometry" and "finitely compact", respectively. However, as Example (3.2) shows, the assumptions of Theorem (7.1) do not imply that $b^{\bar{f}}(\overline{M}) = \overline{M}$ and that $\varrho_{\bar{f}}$ is finite on $\overline{M}$ (where $\bar{f}$ is the extension of f to the completion $(\overline{M}, \bar{\varrho})$ of (M, ϱ)).

As a consequence of Theorem (6.9) we get

(7.2) THEOREM. *Let f be an ε-local isometry of an ε-chainable and finitely totally bounded metric space (M, ϱ) into itself. If $b_0^f(M) \neq \emptyset$, then the induced metric ϱ_f is a metric on M such that*

(a) *ϱ_f and ϱ are ε-locally identical on M (in particular, they are topologically equivalent),*

(b) *the space (M, ϱ_f) is finitely totally bounded, and*

(c) *f is an isometry with respect to ϱ_f.*

Furthermore, then $b^f(M) = M$.

Proof. If (M, ϱ) is ε-chainable and if $b_0^f(M) \neq \emptyset$, then Theorem (6.9) implies that $M = b^f(M) = M^f$. Thus (a) and (b) follow from Proposition (4.3) (cf. Remark (2.6)), while (c) follows from Proposition (5.3) (cf. also (b) of Theorem (5.9)). This completes the proof.

Remark. Under the assumptions of Theorem (7.2) the sequences $\{f^n(x)\}_{n=0}^{\infty}$, $x \in M$, are bounded in both metrics ϱ and ϱ_f (cf. (d) of Remark (2.7)).

A metric space (M, ϱ) is said to be *connected* (resp. *convex*) *after completion*, if the completion $(\overline{M}, \bar{\varrho})$ of (M, ϱ) is connected (resp. convex). Observe that, by Remark (2.10), a finitely totally bounded metric space (M, ϱ) is convex after completion if and only if (M, ϱ) is ε-convex for each $\varepsilon > 0$. Thus, Remark (2.4) and Corollary (6.7) and Theorem (6.9) together with Theorem (5.12) imply

(7.3) THEOREM. *Let f be a mapping of a metric space (M, ϱ) into itself. If $b_0^f(M) \neq \emptyset$ and if one of the following conditions holds:*

(a) *f is a uniform local isometry and (M, ϱ) is finitely totally bounded and convex after completion,*

(a*) *f is a nonexpansive uniform local isometry and (M, ϱ) is finitely totally bounded and connected after completion,*

(b) *f is a local isometry and (M, ϱ) is finitely compact and convex,*

(b*) *f is a nonexpansive local isometry and (M, ϱ) is finitely compact,*

(c) *f is an ε-local isometry and (M, ϱ) is finitely totally bounded and ε-convex,*

(c*) *f is a nonexpansive ε-local isometry and (M, ϱ) is finitely totally bounded and ε-chainable,*

then f is an isometry and, moreover, $b^f(M) = M$.

Remark. Under the assumptions of Theorem (7.3), $M^f = M$ and $f(M)$ is a dense subset of M. Moreover, if (M, ϱ) is finitely compact, then $f(M) = M$ (cf. Corollary (5.7)).

Let us note the following special cases of Theorem (7.3).

(7.4) COROLLARY. *Let f be a uniform local isometry into itself of a metric space (M, ϱ) finitely totally bounded and convex after completion. If f has a fixed (or periodic) point, then f is an isometry.*

(7.5) COROLLARY. *Let f be a local isometry of a finitely compact and convex metric space (M, ϱ) into itself. If f has a fixed (or periodic) point, then f is an isometry.*

(7.6) COROLLARY. *Let f be an ε-local isometry of a finitely totally bounded and ε-convex metric space (M, ϱ) into itself. If f has a fixed (or, periodic) point, then f is an isometry.*

We now give an answer to question B of the Introduction. We will say that a metric space (M, ϱ) has a *transitive group of isometries* if for every two points x and y of M there exists an isometry g of M onto itself such that $g(x) = y$.

(7.7) THEOREM. *If a metric space (M, ϱ) finitely totally bounded and convex after completion has a transitive group of isometries, then every uniform local isometry of (M, ϱ) into itself is an isometry.*

Proof. Let f be a uniform local isometry of (M, ϱ) into itself and fix a point x_0 of M. Then there exists an isometry g of (M, ϱ) into itself such that $g(f(x_0)) = x_0$. Thus $g \circ f$ is a uniform local isometry with a fixed point x_0. It follows by Corollary (7.4) that $g \circ f$ is an isometry. Therefore f is also an isometry, which completes the proof.

The same argument applied to Corollary (7.5) and Corollary (7.6) gives

(7.8) THEOREM. *If a finitely compact and convex metric space (M, ϱ) has a transitive group of isometries, then every local isometry of (M, ϱ) into itself is an isometry.*

(7.9) THEOREM. *If a finitely totally bounded and ε-convex metric space (M, ϱ) has a transitive group of isometries, then every ε-local isometry of (M, ϱ) into itself is an isometry.*

Next, we consider the case of surjective local isometries. (Note that only this case was studied by Busemann [2], [3], Kirk [9]–[11] and Szenthe [12]–[14]). Given a metric space (M, ϱ), one says that $x_0 \in M$ is a *corner point* of (M, ϱ) if for every $x \in M$ there exists an isometry g of (M, ϱ) into itself such that $g(x_0) = x$.

Remark. If a metric space (M, ϱ) has a transitive group of isometries, then every point of M is a corner point of (M, ϱ). However, there exists a metric space (M, ϱ) with a corner point and such that the only isometry of (M, ϱ) onto itself is the identity mapping id_M. The space $M = \{x \in R\colon x \geqslant 0\}$ with absolute value distance gives a simple illustration of this.

Corollaries (7.4), (7.5) and (7.6) and the same argument as in the proof of Theorem (7.7) imply the following results.

(7.10) THEOREM. *If a metric space (M, ϱ) finitely totally bounded and convex after completion has a corner point, then every uniform local isometry of (M, ϱ) onto itself is an isometry.*

(7.11) THEOREM. *If a finitely compact and convex metric space (M, ϱ) has a corner point, then every local isometry of (M, ϱ) onto itself is an isometry.*

(7.12) THEOREM. *If a finitely totally bounded and ε-convex metric space (M, ϱ) has a corner point, then every ε-local isometry of (M, ϱ) onto itself is an isometry.*

Remark. It follows from Theorem (7.3) that Corollaries (7.4)–(7.6) and Theorems (7.7)–(7.12) could be stated (with weaker assumptions on the space (M, ϱ))

for nonexpansive uniform local isometries, nonexpansive local isometries and nonexpansive ε-local isometries.

8. Final comments. In this section we give some questions related to question B of the Introduction.

(8.1) QUESTION. Do Theorems (6.6) and (6.9) remain true if the assumption that the space (M, ϱ) is finitely totally bounded is replaced by the (weaker) assumption that for every point x of M the set $\{f^n(x): n \geqslant 0\}$ is finitely totally bounded?

Next, consider the following question of A. D. Aleksandrov (which appears in [12]):

(8.2) QUESTION. Under what conditions is a mapping of a metric space into itself which preserves unit distances an isometry?

We make some remarks concerning Question (8.2). Given a metric space (M, ϱ), let us say that (M, ϱ) *has property* P if every mapping of M into itself which preserves unit distances is an isometry. We will say that (M, ϱ) *has property* P* (resp. *property* P^*_ε) if every mapping of M into itself which preserves unit distances is locally nonexpansive (resp. ε-locally nonexpansive). We will say that (M, ϱ) *has property Q* if, for every $x, y \in M$ such that $0 < \varrho(x, y) < 1$, there exists a point $z \in M$ satisfying one of the following conditions:

$$\varrho(x, y)+\varrho(y, z) = \varrho(x, z) = 1 , \quad \text{or} \quad \varrho(y, x)+\varrho(x, z) = \varrho(y, z) = 1 .$$

It easily seen that if (M, ϱ) has property Q, then every nonexpansive mapping of M into itself which preserves unit distances is a $\frac{1}{2}$-local isometry. Thus, by Lemma (4.4), if (M, ϱ) is convex and complete (resp. ε-convex) and if it has property Q, then every locally nonexpansive (resp. ε-locally nonexpansive) mapping of M into itself which preserves unit distances is a $\frac{1}{2}$-local isometry. Hence, Theorems (7.8) and (7.9) have the following consequences:

(8.3) THEOREM. *Let (M, ϱ) be a finitely compact and convex metric space which has a transitive group of isometries. If (M, ϱ) has properties* Q *and* P*, *then (M, ϱ) has property* P.

(8.4) THEOREM. *Let (M, ϱ) be a finitely totally bounded metric space which has a transitive group of isometries. If (M, ϱ) has property* Q *and if for some ε, $0 < \varepsilon \leqslant \frac{1}{2}$, it is ε-convex and has property P^*_ε, then (M, ϱ) has property* P.

In view of this, it seems appropriate to pose the following question:

(8.5) QUESTION. When does a metric space (M, ϱ) have property P* (resp. property P^*_ε)?

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