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Uniform quotients of metrizable spaces

by

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Abstract. The easiest possible example of a metrizable uniform space having a nonmetrizable uniform quotient is given. Using this example all metrizable spaces having metrizable uniform quotients only are fully described.

In the literature several sufficient conditions for a uniform quotient of a metric space to be metrizable are treated e.g. [1], [2], [4]. The first attempt to bring a concrete example of a nonmetrizable uniform quotient of a metric space appeared in [4], and two much simpler examples appeared later in [3].

In the sequel $f\colon X\to Y$ is a uniformly continuous onto mapping between uniform spaces. f is called a *uniform quotient mapping* if Y is endowed with the finest uniformity making f uniformly continuous. Himmelberg [2] strengthens the latter concept defining so called uniformly pseudoopen mappings (i.e. the images of uniform vicinities of the diagonal are uniform vicinities) and proves that a uniformly pseudoopen image of a metrizable space is metrizable. We start with another strengthening of uniform quotient mappings which seems to be more convenient (see Remark 1) for our problem.

DEFINITION 1. Let $f\colon X\to Y$ be a uniformly continuous mapping from X onto Y. f will be called *uniformly conservative* if for every uniform cover $\mathcal{U}$ of X the cover

$$\tilde{f}(\mathcal{U}) = \{f[\mathrm{St}(f^{-1}(y), \mathcal{U})];\ y\in Y\}$$

is uniform on Y.

It might be easily verified that every uniformly pseudoopen mapping is uniformly conservative and every uniformly conservative mapping is a uniform quotient.

PROPOSITION 1. *If $f\colon X\to Y$ is uniformly conservative (onto), X metrizable, then Y is metrizable as well.*

Proof. Take an arbitrary uniform cover $\mathcal{V}$ of Y, choose a uniform star-refinement $\mathcal{W}$ of $\mathcal{V}$ and set $\mathcal{U} = f^{-1}(\mathcal{W})$. Then for every $y\in Y$ we have

$$f[\mathrm{St}(f^{-1}(y), \mathcal{U})] \subset \mathrm{St}(y, \mathcal{W})$$

Therefore every uniform cover of Y may be refined by a cover of the form $\bar{f}(\mathscr{U})$ for some uniform cover $\mathscr{U}$ of X. f is uniformly conservative, X has a countable basis for uniform covers, so Y has a countable basis for covers as well.

Now we present two results which will show that under some conditions quotients are uniformly conservative.

PROPOSITION 2. *Suppose $f\colon X\to Y$ is continuous and onto, X compact, (hence f is a uniform quotient), then f is uniformly conservative.*

Proof (standart). Take $y\in Y$ and an open neighborhood U of $f^{-1}(y)$. For every z distinct from y take $V(z)$, $W(z)$ disjoint open in Y such that $z\in V(z)$, $y\in W(z)$. The cover

$$\{U\}\cup\{f^{-1}[V(z)];\ z\in Y\setminus\{y\}\}$$

is an open cover of X, we may choose a finite subcover

$$\{U\}\cup\{f^{-1}[V(z_i)];\ i=1,\dots,k\}.$$

The set $W=\bigcap_{i=1}^{k}W(z_i)$ is an open neighborhood of y.

If $x\in W$ then $f^{-1}(x)\cap f^{-1}[V(z_i)]=\emptyset$ for all i; hence $f^{-1}(x)\subset U$, and hence $W\subset f[U]$, therefore $f[U]$ is a neighborhood of y. Take any open cover $\mathscr{U}$ of X, the set $f[\mathrm{St}(f^{-1}(y),\mathscr{U})]$ is a neighborhood of y for all $y\in Y$, so $\bar{f}(\mathscr{U})$ is a uniform cover of Y.

COROLLARY 1. *Every uniform quotient of a compact metrizable space is metrizable.*

Remark 1. Proposition 2 remains no more true if we write uniformly pseudo-open instead of uniformly conservative as shows the following easy example:

Put $X=[0,3]$, $Y=[0,2]$ compact intervals, $f(x)=x$ for $x\in[0,1]$, $f(x)=x-1$ for $x\in[2,3]$, $f(x)=1$ otherwise. $\bar{f}$ is obviously uniformly conservative but not uniformly pseudoopen.

PROPOSITION 3. *Let $f\colon X\to Y$ be a uniform quotient,*

$$Y_1=\{y\in Y;\ |f^{-1}(y)|\geqslant 2\}.$$

If the family $\{f^{-1}(y);\ y\in Y_1\}$ is uniformly discrete in X then f is uniformly conservative.

Proof. Take a uniform cover $\mathscr{U}$ of X. We may suppose that for every $y,z\in Y_1$ the sets $\mathrm{St}(f^{-1}(y),\mathscr{U})$ and $\mathrm{St}(f^{-1}(z),\mathscr{U})$ are disjoint. Suppose $\mathscr{V}$ is a uniform star-refinement of $\mathscr{U}$, $z\in Y$.

a) If $z\in Y_1$, then easily $\mathrm{St}(z,\bar{f}(\mathscr{V}))\subset f[\mathrm{St}(f^{-1}(z),\mathscr{U})]$.

b) If $z\notin Y_1$ and $\mathrm{St}(f^{-1}(z),\mathscr{V})$ does not intersect any $f^{-1}(y)$ for $y\in Y_1$, then evidently $\mathrm{St}(z,\bar{f}(\mathscr{V}))\subset f[\mathrm{St}(f^{-1}(z),\mathscr{U})]$.

c) If $z\notin Y_1$ and $\mathrm{St}(f^{-1}(z),\mathscr{V})$ intersects some $f^{-1}(y)$ with $y\in Y_1$, then this y is unique and we may find $U\in\mathscr{U}$ containing $\mathrm{St}(f^{-1}(z),\mathscr{V})$ and intersecting $f^{-1}(y)$, hence $\mathrm{St}(z,\bar{f}(\mathscr{V}))\subset f[\mathrm{St}(f^{-1}(y),\mathscr{U})]$.

So $\bar{f}(\mathscr{U})$ is star-refined by $\bar{f}(\mathscr{V})$, hence f is uniformly conservative.

COROLLARY 2. *If X is metrizable, f is a quotient mapping such that the family of all nontrivial fibres is uniformly discrete, then the quotient space is metrizable.*

PROPOSITION 4. *Suppose X is metrizable and every quotient of it is also metrizable, Y is a subspace of X. Then every quotient of Y is metrizable.*

Proof. Suppose $f\colon Y\to Q$ is uniform quotient, let us embed Q as a uniform subspace into some injective space Z and extend f to a uniformly continuous $\bar{f}\colon X\to Z$. If we take the quotient uniformity on $\bar{f}[Z]$, then this uniform space has to be metrizable and contains Q as a uniform subspace. So Q is metrizable as well.

COROLLARY 3. *Every precompact metrizable space has metrizable quotients only.*

Let us turn our attention to nonmetrizable quotients now. Let us denote by D_2 the adjacent sequence (see [6], where also the strange notation is justified), that is the set $N\times 2$ with the uniformity metrized by the metric $d(x,x)=0$, $d(\langle n,0\rangle,\langle n,1\rangle)=1/n$, $d(x,y)=1$ otherwise. It is proved in [5] that every metrizable space which is not uniformly discrete contains as a uniform subspace either a cauchy sequence or a copy of D_2. Every quotient of a cauchy sequence must be metrizable due to its precompactness, so if we prove (and we shall prove it) that D_2 has a nonmetrizable quotient, it will be the easiest possible example.

EXAMPLE 1. Take $f\colon D_2\to\omega$ defined as $f(1,1)=0$, $f(n,0)=f(n+1,1)=n$. Then the corresponding quotient uniformity Q on ω is not metrizable.

Proof. We start with the following notation: if $f\colon X\to Y$ is a uniform quotient and if X is metrized by a metric d, we shall denote by d_f the following pseudometric on Y:

$$d_f(x,y)=\inf\sum_{0}^{n-1}d\big(f^{-1}(x_i),f^{-1}(x_{i+1})\big)$$

where $x_0=x$, $x_n=y$ and the infimum is taken over all such chains. Obviously f is uniformly continuous into the pseudometric space (Y,d_f).

The following observation is due to Marxen:

LEMMA ([4]). *Suppose Y is metrized by σ, then there is a metric d' uniformly equivalent to d such that d'_f is uniformly equivalent to σ.*

Using this idea we come back to our example. Suppose Q is metrizable, we have some σ metrizing D_2 such that σ_f metrizes Q. σ may be characterized by some sequence $\{a_n\}$ of real numbers such that $1\geqslant a_n\searrow 0$, $\sigma(\langle n,0\rangle,\langle n,1\rangle)=a_n$. We may and shall suppose (Q has a quotient uniformity) that $\sum a_n=\infty$. Now define a new metric $\sigma^1(\langle n,0\rangle,\langle n,1\rangle)=b_n$ on D_2 (equivalent to σ) as follows:

For every $k\in N$ denote $n(1)=1$, otherwise $n(k)$ is the first natural number such that

(i) $a_{n(k)}+\dots+a_{n(k)+k-1}<\dfrac{1}{k}$,

(ii) $a_{n(k)}\geqslant a_{n(k-1)+k-1}$.

For $n(k) \leqslant n \leqslant n(k+1)$ we define $b_n = \frac{1}{k}$.

For all natural n we have $b_n \geqslant a_n$; hence $\sigma^1 \geqslant \sigma$, hence $\sigma_f^1 \geqslant \sigma_f$.

On the other hand, for any $k \geqslant 2$ there are points $v = n(k)-1$, $w = n(k)+k-1$ in Q such that

$$\sigma_f(v, w) = \sum_{i=1}^{k-1} a_{n(k)+i} < \frac{1}{k}$$

$$\sigma_f^1(v, w) = \sum_{n=n(k)}^{n(k)+k-1} b_n = 1$$

Therefore σ_f^1 is not uniformly equivalent to σ_f, and hence σ_f does not metrize Q, which contradicts the assumption.

Remark 2. Recall that the uniform weight of a space is the smallest cardinality of a base of uniform covers. Hušek and Pelant [3] proved that if Y is a uniform quotient of a metrizable space, then its uniform weight is less or equal to $cof\,({}^{\omega}\omega)$. (The latter symbol stands for the smallest cardinality of a cofinal set in the set of all mappings on ω into ω with the pointwise order.) If one looks carefully how the uniform covers in the space Q from Example 1 look like, one can easily see that the uniform weight of Q is just $cof\,({}^{\omega}\omega)$, so our example is as "far from being metrizable" as possible.

Now we are prepared to prove the main result:

THEOREM. *The following properties of a metrizable uniform space X are equivalent*:

(1) *Every uniform quotient of X is metrizable,*

(2) *X does not contain D_2 as a uniform subspace,*

(3) *Every uniform subspace of X is uniformly discrete or contains a Cauchy sequence as a uniform subspace.*

(4) *The completion $\hat{X}$ of X is of the form $L \cup (X \setminus L)$, where L is compact and for every $\varepsilon > 0$ the subspace $\hat{X} \setminus \mathcal{O}_\varepsilon(L)$ is uniformly discrete ($\mathcal{O}_\varepsilon(L)$ stands for the ε-neighborhood of L).*

Proof. (1)$\Rightarrow$(2) follows from Proposition 4 and Example 1.

(2)$\Leftrightarrow$(3) is contained in the remark foregoing Example 1.

(3)$\Rightarrow$(4). It might be easily observed that the condition (2) is closed under completions, so if X fulfils (3), so does $\hat{X}$. Take all copies of a cauchy sequence in $\hat{X}$ and denote by L the set of all their limit points. (3) implies that L is compact. If $\hat{X} \setminus \mathcal{O}_\varepsilon(L)$ is not uniformly discrete for some $\varepsilon > 0$, it must contain a cauchy sequence not converging to a point in L, which contradicts the definition of L.

(4)$\Rightarrow$(1). Using Proposition 4 we may suppose that X is complete. Let ϱ_1 be any metric metrizing the uniformity of X. If f: $X \to Y$ is a quotient mapping, using Proposition 2 we may find a metric σ metrizing L such that the corresponding $\sigma_{f|L}$ metrizes the quotient uniformity on $f[L]$, the latter being a subspace of Y. Take ϱ_2 any uniformly continuous extension of σ over all of X. Set $\varrho = \max(\varrho_1, \varrho_2)$. ϱ is generating the uniformity of X. For every $x \in X$ take

$$L_x = \{y \in L;\ \varrho(x, y) = \varrho(x, L)\} \neq \emptyset .$$

Now define for $x, y \in X$:

$$d(x, x) = 0 ,$$

$$d(x, y) = \inf_{\substack{u \in L_x \\ v \in L_y}} \big(\varrho(x, u) + \varrho(u, v) + \varrho(v, y)\big) \text{ if } x \neq y .$$

It is easy to see that d is a metric on X, $d \geqslant \varrho$ and $d(x, y) = \varrho(x, y)$ for x, y in L.

For every $\varepsilon > 0$ there is $\eta(\varepsilon) > 0$ such that $\varrho(x, y) \geqslant \eta(\varepsilon)$ for $x, y \notin \mathcal{O}_\varepsilon^{\varrho}(L)$. Put $\delta = \min\left(\frac{\varepsilon}{5}, \eta\left(\frac{\varepsilon}{5}\right)\right)$. Then for $\varrho(x, y) < \delta$ we have $\varrho(x, L) < \frac{\varepsilon}{5}$, $\varrho(y, L) < \frac{\varepsilon}{5}$.

For arbitrary $\mu > 0$ we may find $u \in L_x$, $v \in L_y$ so that

$$d(x, y) < \varrho(x, u) + \varrho(u, v) + \varrho(v, y) + \mu ,$$

$$\varrho(u, v < \varrho(u, x) + \varrho(x, y) + \varrho(v, y) < \tfrac{3}{5}\varepsilon .$$

So we have $d(x, y) \leqslant \varepsilon$, and hence d is uniformly equivalent to ϱ. Now it is easy to check that the corresponding metric d_f metrizes the quotient uniformity on Y.

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