Notes on exponential-logarithmic terms

by

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Abstract. The asymptotic behaviour of functions defined by exponential-logarithmic terms is studied modifying methods from [D]. This gives 1-model completeness, bounds on the last roo. and characterizations of possible limits of such functions, especially for integral exponential functiont

These notes are fruits of the efforts to generalize results in [D] and [W] to exponential-logarithmic terms. The results presented here make it possible to apply the methods in [D] also to terms with binary exponentiation. They came up in close cooperation of both authors and are a part of the thesis of the second author who has developed and simplified several ideas of the first author.

We consider these notes as a supplement to [D] and presume acquaintance with concepts and proofs from that paper so that we can confine ourselves here to the description of the necessary (and sometimes essential) modifications. The signature is extended by a unary function symbol log. The theory T is an extension of the theory of ordered exponential fields by means of the axiom

$$\text{(L)} \qquad \forall x(x \neq 0 \rightarrow e^{2\log(x)} = x^2)$$

and the Rolle- and intermediate value schema for exponential-logarithmic terms (el-terms for short). The axiom (L) means that $\log(x)$ denotes the function usually denoted by $\log(|x|)$. However this does not matter since (in the usual notation) $\log(|x|) = \frac{1}{2}\log(x^2)$ and

$$\log(x) = \log(|e^{\log(|x|)}+x|)-\log(|2|)\,.$$

We note that el-terms are continuous in substructures of models of T and that their formal derivative satisfies the usual ε-δ-definition of the derivative.

Let C_1 be a model of T and let C be a substructure of C_1. We consider el-terms in one variable x and with parameters from C. As in [D] the germs of the functions defined by these terms in some neighbourhood of $+\infty$ in C_1 form a differential field $(C_1)_\infty$ closed under e and log and ordered by the relation of eventual dominance in C_1.

In [D] it was shown first that for different models $C_1 \supseteq C$ the resulting exponential fields are isomorphic and then the major part of that paper was concerned with proving that also the order relation on $(C_1)_\infty$ is independent of C_1. This strategy must be changed for el-terms since for these terms there is an intimate relation between equality and order, e.g. $e^{\log(t)} - t = 0$ if and only if $t > 0$. That is also the reason why Theorem 2.8 of [Wi], which was used several times in [D], could not be generalized to the present setting. Consequently, the concept of an F-normal field in [D], which was adopted from [Wi], is replaced by the following

DEFINITION. A subfield K of $(C_1)_\infty$ is called *normal*, if for all $a \in K$ $a' = 0$ implies $a = c$ for some $c \in C$.

The following simple lemmas turn out to be very useful. Let $t(x)$ be an el-term. $\lim t$ denotes $\lim\limits_{x\to\infty} t(x)$.

LEMMA 1. *If* $\lim t = c \in C$ *and* $t' = 0$ *in* $(C_1)_\infty$, *then* $t = c$ *in* $(C_1)_\infty$.

LEMMA 2. *If* $\lim t = \infty$, *then* $t' \neq 0$ *in* $(C_1)_\infty$.

LEMMA 3. *Let K be a subfield of $(C_1)_\infty$ containing some $s \in K$ such that* $\lim s = \infty$. *Then for each* $a \in (C_1)_\infty$

$$a \ll 1 \bmod K \quad \textit{implies} \quad a' \ll 1 \bmod K.$$

Let $\log^{(0)}(x) = x$ and for each $i \geq 0$ $\log^{(i+1)}(x) = \log(\log^{(i)}(x))$. The definition of an outer extension is adopted from [D], p. 14 by replacing x in a) by $\log^{(i)}(x)$ for some i and giving up the divisibility of the group G. It is easy to see that Propositions 19 and 22 and the Corollaries 21 and 23 of [D] remain valid. This yields

LEMMA 4. *Let $K(e^G)$ be an outer extension of K by means of G which is dense in $K_1 \subseteq (C_1)_\infty$. Then every $f \in K_1$ has a unique representation*

$$f = ae^g(1+r)$$

where $a \in K$, $g \in G$ and r is coinitial in K_1.

PROPOSITION 5. *Let K be a normal differential subfield of $(C_1)_\infty$ containing some $s \in K$ such that* $\lim s = \infty$. *Let $K(e^G)$ be an outer extension of K by means of G. If $K(e^G)$ is dense in a field $K_1 \subseteq (C_1)_\infty$, then K_1 is normal.*

Sketch of proof. Let $a \in K_1$ be such that $a' = 0$. By Lemma 2 $|\lim a| < \infty$. Hence a has a representation $a = b + r$ where $b \in K$ and r is coinitial in K_1. $a' = 0$ yields $|b'| = |r'|$. But $r' \ll 1 \bmod K$ by Lemma 3 and $b' \in K$; hence $b' = 0$. Since K is normal, this implies $b \in K$. Now $r' = 0$ and $\lim r = 0$, hence $r = 0$ by Lemma 1. Hence $a = c$.

In a similar way it can be shown that any subfield of $(C_1)_\infty$ containing $C(x)$ dense, is normal.

DEFINITION. A subfield K_1 of $(C_1)_\infty$ is called a *simple inner extension* of K if there is some r coinitial in K such that $K_1 = K(e^r)$ or $K_1 = K(\log(1+r))$.

For each natural number $k \geq 1$ we put

$$l_k(x) = \sum_{i=1}^{k} (-1)^{i+1} \frac{x^i}{i}.$$

Then Proposition 15 of [D] can be generalized in a canonic way. Using a method similar to the proof of Proposition 5 and approximating $e(t)$, $\log(t)$ by $e_k(t)$, $l_k(t)$ respectively, it is possible to prove the important

PROPOSITION 6. *Let K be a normal differential subfield of $(C_1)_\infty$ containing some $s \in K$ such that* $\lim s = \infty$. *Let K' denote $C(x)$ or some outer extension of K and let K_1 be an inner extension of K'. Then K' is dense in K_1.*

The following concept is adopted from the definition of a ladder in [D].

DEFINITION. A field K has a *tower* $(K_i, G_i)_{i\leq n}$ of depth l and height n provided that

0) $K_0 = C$, $G_0 = Z \cdot \log^{(l+1)}(x)$, $K_{n+1} = K$,

1) for each $i = 1, \ldots, n$ there are some $j < i$, some finitely generated nontrivial Archimedean ordered subgroup $V \subseteq (K_j, +, 0, <)$ and some $g \in G_j^+$ such that $G_i = e^g V$,

2) $K_i(e^{G_i})$ is an outer extension of K_i by means of G_i $(i = 1, \ldots n)$,

3) K_{i+1} is an inner extension of $K_i(e^{G_i})$ $(i = 1, \ldots, n)$.

THEOREM 7. *If K has a tower then K is normal.*

This is proved by induction on the height of the tower using Propositions 5 and 6.

Lemmas 26–29 and Proposition 30 of [D] with their proofs carry over to the present situation without essential modifications.

Since $C(x)$ and $C(\log^{(l)}(x))$ are isomorphic in a canonic way, it is often sufficient to regard only towers of depth 0.

LEMMA 8. *Every tower of depth 0 for a field $K \subseteq (C_1)_\infty$ can be extended to a tower of depth 1 for $K(\log(x))$.*

Applying Lemmas 8 and 4 one can prove

PROPOSITION 9. *For each $f \in K \setminus \{0\}$ each tower for K can be extended to a tower for $K(\log(f))$.*

This together with the analogue of Proposition 30 of [D] suffices to prove.

THEOREM 10. *For each $f \in (C_1)_\infty$ there is a field $K \subseteq (C_1)_\infty$ such that $f \in K$ and K has a tower.*

Now let C_1, C_2 be models of T containing an exponential logarithmic field C. For each tower of subfields of $(C_1)_\infty$ there is an isomorphic tower of subfields of $(C_2)_\infty$ and the isomorphism can be chosen to be the identity on $C(x)$ and to respect $<$, e and log. As in [D], this argument makes use of the fact, that for each tower $(K_i, G_i)_{i\leq n}$ the order on K_{i+1} is uniquely determined by the order on K_i $(i \leq n)$.

Hence $T \cup D_C$ is sufficient to decide the dominance relation for el-terms. This can bestrengthened by generalizing results from [W].

Slightly modifying Wolter's proof of Theorem 6 in [W] it is possible to show

THEOREM 11. *For each el-term t with parameters from C there is a $c \in C$ such that*

$$C_1 \models \exists x \forall y \geqslant x (D_t(y) \wedge t(y) > 0) \text{ if and only if }$$
$$T \cup D_C \vdash \forall y \geqslant c (D_t(y) \wedge t(y) > 0)\,;$$
$$C_1 \models \exists x \forall y \geqslant x (D_t(y) \wedge t(y) = 0) \text{ if and only if }$$
$$T \cup D_C \vdash \forall y \geqslant c (D_t(y) \wedge t(y) = 0)$$

where $D_t(y)$ is a formula saying that $t(y)$ is defined.

Using induction on the number of iterations of e, on the number of iterations of log and on the well-known Hardy-rank, Theorem 11 can be used to obtain

THEOREM 12. *Suppose C_1, C_2 are models of T such that $C_1 \subseteq C_2$ and t is an el-term with parameters from C_1. Then the sets*

$$\{x \in C_2\colon\ t(x) \text{ is defined}\} \text{ and } \{x \in C_2\colon\ t(x) \text{ is defined and } t(x) = 0\}$$

are finite unions of intervals in C_2 with boundaries in C_1.

As in [D] Theorem 12 can be applied to prove the 1-model-completeness of T, i.e.

THEOREM 13. *Suppose C_1, C_2 are models of T such that $C_1 \subseteq C_2$ and $\varphi(x)$ is a quantifier-free formula with one variable x and with parameters from C_1. Then $C_1 \models \exists x \varphi(x)$ if and only if $C_2 \models \exists x \varphi(x)$.*

As in [D] it is possible to obtain a representation of $(C_1)_\infty$ as a field of power series of transfinite length. We confine ourselves here to the construction of the appropriate scale H_∞ such that $(C_1)_\infty$ can be embedded into $C[[H_\infty]]$.

For each natural number i, let H_0^i be the multiplicative subgroup of $(C_1)_\infty$ generated by $\log^{(i)}(x)$. Then for each $m > 0$ H_m^i and $C^{m,i}$ are defined as in [D] but such that the isomorphism E sends also $\log(a)$ to $|a|$. Then $(H_{2n}^n)_{n \in \omega}$ and $(C^{2n,n})_{n \in \omega}$ are increasing sequences. We put

$$H_\infty = \bigcup_{n \in \omega} H_{2n}^n, \qquad C^\infty = \bigcup_{n \in \omega} C^{2n,n}.$$

C^∞ can be considered as an exponential field as in [D]. It is obvious that each $a \in C$ can be uniquely represented as $a = c(\log^{(n)}(x))^m e^b (1+d)$ such that $n, m \in \omega$, $c \in C$, $d \in C^\infty$ and $\operatorname{supp}(d) < 1$, provided $a \neq 0$. With this representation we put

$$\log(a) = \log(c) + m\log^{(n+1)}(x) + b + \sum_{i=1}^{\infty} (-1)^{i+1} \frac{d^i}{i}.$$

By induction on the height of an appropriate tower we can show that there is a canonic embedding $a \mapsto \sigma_a$ of $(C_1)_\infty$ into C^∞. It is not difficult to check that for

each term t the coefficients of the series σ_t can be represented as variable-free terms build from the same parameters as t. Similar to Lemma 37 in [D] it can be shown for each el-term t and for $h = \max\operatorname{supp}(\sigma_t)$ that there is a positive $c \in C$ and an $i \in \omega$ such that $|\sigma_a - \sigma_a(h)h| < c(\log^{(i)}(x))^{-1} h$.

Hence Theorem 38 and Corollary 39 of [D] generalize to the present setting.

Now we can apply these results to the study of the binary exponential function.

Let M denote Richardson's set of integral exponential functions (cf. [R]), i.e. M is the least set of functions containing the constant function with value 1, the identity function and closed under addition, multiplication and binary exponentiation. D denotes the set of exponential constants from [R], i.e. the least set such that $1 \in D$ and such that $c, d \in D$ implies also $c+d$, cd, e^c, $c^{-1} \in D$.

A look at the power series representing integral exponential functions gives

PROPOSITION 13. *Suppose $a \in M$ and let σ_a be the transfinite power series representing a. Then*

1) $\operatorname{supp}(\sigma_a) \geqslant 1$,
2) *$\sigma_a(1)$ is a natural number,*
3) *for all $h \in \operatorname{supp}(\sigma_a)$ there are $c, d \in D \cup \{0\}$ such that $\sigma_a(h) = c - d$,*
4) $\sigma_a(\max\operatorname{supp}(\sigma_a)) \in D$.

Proposition 13 is proved by induction on the number of iterations of binary exponentiation in a.

Richardson proved (cf. Theorem 7 of [R]) that each exponential constant can be obtained as the limit of the quotient of two integral exponential functions. Now 4) of Proposition 13 gives immediately

THEOREM 14. *If $p, q \in M$ are such that p/q is bounded by some natural number, then $\lim p/q \in D \cup \{0\}$.*

Proposition 13 and Theorem 14 generalize a result of van den Dries, who proved them for terms $< 2^{2^x}$. It might seem promising to study the dominance of integral exponential functions by means of their power series. However it is not clear how to compute σ_a for a given $a \in M$ without knowing the diagram of the least exponential-logarithmic field contained in the reals. So we can only state

PROPOSITION 15. *The dominance problem for exponential-logarithmic terms is Turing-equivalent with the identity problem for constant exponential-logarithmic terms.*

References

[D] B. I. Dahn, *The limit behaviour of exponential terms*, Preprint 48 (1982) der Sektion Mathematik der Humboldt-Universität zu Berlin, Fund. Math. 124 (1984), 169–186.

[R] D. Richardson, *Solution of the identity problem for integral exponential functions*, Z. Math. Logik Grundlag. Math. 15 (1969), 333–340.

[W] H. Wolter, *On the problem of the last root for exponential terms*, Preprint 58 (1983) der Sektion Mathematik der Humboldt-Universität zu Berlin, to appear in Z. Math. Logik Grundlag. Math.
[Wi] A. Wilkie, *On exponential fields*, Preprint.

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Uniform quotients of metrizable spaces

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Abstract. The easiest possible example of a metrizable uniform space having a nonmetrizable uniform quotient is given. Using this example all metrizable spaces having metrizable uniform quotients only are fully described.

In the literature several sufficient conditions for a uniform quotient of a metric space to be metrizable are treated e.g. [1], [2], [4]. The first attempt to bring a concrete example of a nonmetrizable uniform quotient of a metric space appeared in [4], and two much simpler examples appeared later in [3].

In the sequel $f\colon X \to Y$ is a uniformly continuous onto mapping between uniform spaces. f is called a *uniform quotient mapping* if Y is endowed with the finest uniformity making f uniformly continuous. Himmelberg [2] strengthens the latter concept defining so called uniformly pseudoopen mappings (i.e. the images of uniform vicinities of the diagonal are uniform vicinities) and proves that a uniformly pseudoopen image of a metrizable space is metrizable. We start with another strengthening of uniform quotient mappings which seems to be more convenient (see Remark 1) for our problem.

DEFINITION 1. Let $f\colon X \to Y$ be a uniformly continuous mapping from X onto Y. f will be called *uniformly conservative* if for every uniform cover $\mathcal{U}$ of X the cover

$$\tilde{f}(\mathcal{U}) = \{f[\mathrm{St}(f^{-1}(y), \mathcal{U})];\ y \in Y\}$$

is uniform on Y.

It might be easily verified that every uniformly pseudoopen mapping is uniformly conservative and every uniformly conservative mapping is a uniform quotient.

PROPOSITION 1. *If $f\colon X \to Y$ is uniformly conservative (onto), X metrizable, then Y is metrizable as well.*

Proof. Take an arbitrary uniform cover $\mathcal{V}$ of Y, choose a uniform star-refinement $\mathcal{W}$ of $\mathcal{V}$ and set $\mathcal{U} = f^{-1}(\mathcal{W})$. Then for every $y \in Y$ we have

$$f[\mathrm{St}(f^{-1}(y), \mathcal{U})] \subset \mathrm{St}(y, \mathcal{W})$$