Construction of a Hurewicz metric space whose square is not a Hurewicz space

by

J. M. O'Farrell (Huntsville, Ala.)

Abstract. In this paper we construct a Hurewicz subset of the reals whose square admits a map onto the irrationals. Martin's Axiom is used in the construction. The method is similar to that used by Sierpiński [1] to construct a Lusin subset of the reals whose square admits a map onto the reals.

1. Terminology and notation. We denote the reals by R and the irrationals by I. If X and Y are subsets of R, then $X \oplus Y = \{x+y\colon x \in X, y \in Y\}$. A regular topological space X is a *Hurewicz space* if for every sequence $G_1, G_2, G_3, \ldots$ of open covers of X there exists a cover G of X such that $G = \bigcup \{H_i\colon i < \omega\}$ where $H_i \subseteq G_i$ and H_i is finite for each $i < \omega$. A set is a *γ-set* if for each open ω-cover, G, of X, there is a sequence $\{G_n\} \subseteq G$ with $\lim G_n = X$. If $\{A_n\colon n < \omega\}$ is a sequence of subsets of a set X, then

$$\lim A_n = \{x \in X\colon \exists n_0, \forall n \geqslant n_0, x \in A_n\}.$$

A family of subsets, $\mathscr{A}$, of a set X is said to be an *ω-cover of X* if for any finite subset F of X, there is an $a \in \mathscr{A}$ with $F \subseteq a$.

2. The irrationals are not a Hurewicz space. The square of the set to be constructed has a natural mapping onto the irrationals. The continuous image of a Hurewicz space must also be a Hurewicz space, thus the square of the constructed space cannot be a Hurewicz space by the fact that the irrationals are not a Hurewicz space. We describe here an example of a sequence of open covers of the irrationals not witnessing the Hurewicz property. We use the representation $I = N^N$ for the irrationals. The sequence of open covers of N^N are defined as follows:

$$G_n = \{(i_1, i_2, i_3, \ldots, i_n) \times N \times N \times \ldots\colon i_j \in N, \text{ for each } j,\ 1 \leqslant j \leqslant n\}.$$

Notice that this is also a basis for the irrationals not witnessing the Hurewicz property.

3. Auxillary lemma. In this section we assume Martin's Axiom. Specifically we use the fact that in a locally compact, ccc, Hausdorff space the intersection of $< 2^{\omega}$ dense open sets is nonempty.

3.1. LEMMA. *Let $S \subseteq I$ with cardinality $|S| = \varkappa < 2^\omega$ such that $S \oplus S \subseteq I$, $T \subseteq R$ such that T is the complement of the union of $< 2^\omega$ closed nowhere dense subsets of R and $p \in I$. Then there exists a point $x \in T \cap I$ such that $(p-x) \in T \cap I$ and $(\{x\} \oplus S) \cup (\{p-x\} \oplus S) \subseteq I$.*

Proof. Define $A_s = I \ominus \{s\} = \{i-s\colon i \in I\}$ for each $s \in S$. Each A_s is a dense G_δ-set in R. We note for any $x \in R$, $Y_z \subseteq R$, $z \in Z$, the following relation is true:

$$\{x\} \ominus \bigcap \{Y_z\colon z \in Z\} = \bigcap \{\{x\} \ominus Y_z\colon z \in Z\}. \tag{1}$$

Therefore for $p \in I$,

$$\{p\} \ominus \bigcap \{A_s\colon s \in S\} = \bigcap \{\{p\} \ominus A_s\colon s \in S\}.$$

By Martin's Axiom the set

$$W = (\bigcap \{A_s\colon s \in S\}) \cap (\bigcap \{\{p\} \ominus A_s\colon s \in S\}) \cap T \cap I$$

is nonempty. Let $x \in W$ and $s \in S$. Since $x \in \bigcap \{A_s\colon s \in S\}$, for each $s' \in S$ there exists an $i_{s'} \in I$ such that $x = i_{s'} - s'$. Therefore $x+s = (i_s - s)+s = i_s \in I$. Also if $x \in W$, then $x = p-y$ for some $y \in \bigcap \{A_s\colon s \in S\}$, thus $y = p-x$ and $y \in W$. It then follows as before for $s \in S$, $y+s = (p-x)+s \in I$

4. Construction. Let $\{M_\alpha\colon \alpha < 2\}^\omega$ well order the closed uncountable nowhere dense subsets of R and $\{y_\alpha\colon \alpha < 2^\omega\}$ well order I. Define $S_1 = \{y_1/2\}$ and apply the lemma with $S = S_1$, $p = y_2$, $T = R - M_1$. We obtain $S_2 = S_1 \cup \{x, y-x\}$ where S_2 has the following properties:

1) $S_2 \oplus S_2 \subseteq I$,
2) $y_2 \in S_2 \oplus S_2$,
3) $(S_2 - S_1) \subseteq (R - M_1)$.

Assume for each $\alpha < \beta < 2^\omega$ we have constructed sets S_α such that:

1) $S_\alpha \oplus S_\alpha \subseteq I$,
2) $y_\alpha \in S_\alpha \oplus S_\alpha$,
3) For $\delta < \alpha$, $S_\delta \subset S_\alpha$,
4) Cardinality $|S_\alpha| < 2^\omega$ and,
5) $(S_\alpha - \bigcup \{S_\delta\colon \delta < \alpha\}) \subseteq (R - \bigcup \{M_\delta\colon \delta < \alpha\})$.

Let $S = \bigcup \{S_\alpha\colon \alpha < \beta\}$, $p = y_\beta$, $T = (R - \bigcup \{M_\alpha\colon \alpha < \beta\})$ and let S_β denote the set resulting from application of the lemma. Define $L = \bigcup \{S_\alpha\colon \alpha < 2^\omega\}$.

Let $\{G_i\colon i < \omega\}$ be a sequence of open covers of L. Since L is separable, we choose $\{x_i\colon i < \omega\}$ a countable dense subset of L. For each x_i choose $g_i \in G_i$ such that $x_i \in g_i$. Then $v = \bigcup \{g_i\colon i < \omega\}$ is dense in L and $(L - v)$ is contained in a closed nowhere dense subset, M_α, of R. $L \cap M_\alpha$ has cardinality $< 2^\omega$ and hence by Martin's Axiom $L \cap M_\alpha$ is a γ-set [2]. It is known that γ-sets have the Hurewicz property; in fact, γ-sets satisfy property C". Therefore we may find a sequence F_i such that $F_i \subseteq G_i$, each F_i is finite and $\{F_i\colon i < \omega\}$ covers $L \cap M_\alpha$. Then for $F_i = \{g_i\} \cup F_i$, $\bigcup \{F_i\colon i < \omega\}$ is a cover of L. L is therefore a Hurewicz space.

4.1. It is known, assuming Martin's Axiom, that for Hurewicz spaces of cardinality $\varkappa < 2^\omega$, if X is a Lindelöf space then X^2 is a Hurewicz space. A. Lelek ([3], pp. 216–217) has constructed, using the continuum hypothesis, an example of a Hurewicz space X such that X^2 is not a Hurewicz space. This example used an uncountable subset of the reals possessing the Lusin property and carrying the half-open interval topology. Lelek had asked whether it was necessary to assume the continuum hypothesis in order to construct an example of a Hurewicz space whose square is not a Hurewicz space. It was also asked in the same paper whether there might exist an example of such a space that was also a metric space. The problem now is whether there is an example of a Hurewicz metric space whose square is not a Hurewicz space, which can be constructed with assumptions weaker than Martin's Axiom.

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References

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2802 Barcody Circle
HUNTSVILLE, ALABAMA 35801

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