A note on a paper of J. A. Guthrie and M. Henry

by

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Abstract. In a recent paper, J. A. Guthrie and M. Henry assert that if $\mathcal{F}$ is a family of continuous functions from a space $X$ into $[0, 1]$ whose cozero sets form a hereditarily closure-preserving family, then $\mathcal{F}$ is relatively complete. Here we show that this result is not correct in general, but it is true when every point in the space is a $G_δ$ set.

In this paper we suppose that regular spaces are $T_1$, and $\mathbb{R}$ will denote the set of real numbers with the usual topology.

In [1], Burke, Engelking and Lutzer prove the following generalization of the Nagata–Smirnov’s theorem:

**Theorem A.** A regular space is metrizable if, and only if, it has a $σ$-hereditarily closure-preserving ($σ$-HCP) base.

In [2], Guthrie and Henry deduce Theorem A from these results:

**Theorem B.** A topological space is pseudometrizable if, and only if, it has the weak topology induced by a $σ$-relatively complete collection.

**Theorem C.** Let $\mathcal{F}$ be a family of continuous functions from a space $X$ into $[0, 1]$. If the cozero sets of the functions in $\mathcal{F}$ form a HCP collection, then $\mathcal{F}$ is relatively complete.

The example given by Burke, Engelking and Lutzer in [1], Example 8, proves that Theorem C is not correct because if for every $H_δ \in \mathcal{H}$ we define

$$f_δ(x) = \begin{cases} 1 & \text{when } x \in H_δ, \\ 0 & \text{when } x \in X - H_δ \end{cases}$$

then the family $\mathcal{F} = \{f_δ, 0 ≤ δ < ω_1\}$ verifies the hypothesis of Theorem C but, obviously, it is not relatively complete.

However, we have the following result:

**Proposition 1.** Let $\mathcal{F} = \{f_i, i \in I\}$ be a family of continuous functions from a space $X$ into $[0, 1]$, whose cozero sets form a HCP collection. Then $\mathcal{F}$ is relatively complete if each point in $X$ is a $G_δ$ set.
Proof. Let $J \subseteq I$. Then $\sup \{f_j, j \in J\}$ is continuous (see the first part of [2] Theorem 6).

Let $f = \inf \{f_j, j \in J\}$ and let $U_j$ be the cozero set of $f_j$. Clearly, $f(x)$ is defined for each $x \in X$. If $f(x) = 0$, it is easily verified that $f$ is continuous at $x$. If $f(x) \neq 0$ then $x \in \bigcap \{U_j, j \in J\}$. We can put $\{x\} = \bigcap_{n=1}^{\infty} V_n$ where every $V_n$ is open. Well-order $J$ and let

$$H_j = \begin{cases} U_j \cap V_j & \text{when } j < \omega_0 \\ U_j & \text{when } j \geq \omega_0 \end{cases}$$

By using the technique of [1], Lemma 4, we prove that $\bigcap \{H_j, j \in J\} = \{x\}$ is open. Consequently, $f$ is continuous at the isolated point $x$. Then $f$ is continuous in $X$ and the proof is complete.

Now, Theorem A is an immediate consequence of Theorem B and Proposition 1. The authors are indebted to Dr. M. López-Pellicer for his many valuable suggestions.

References
