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A note on a paper of J. A. Guthrie and M. Henry

by

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Abstract. In a recent paper, J. A. Guthrie and M. Henry assert that if $\mathscr{F}$ is a family of continuous functions from a space X into $[0, 1]$ whose cozero sets form a hereditarily closure-preserving family, then $\mathscr{F}$ is relatively complete. Here we show that this result is not correct in general, but it is true when every point in the space is a G_δ set.

In this paper we suppose that regular spaces are T_1, and R will denote the set of real numbers with the usual topology.

In [1], Burke, Engelking and Lutzer prove the following generalization of the Nagata–Smirnov's theorem:

THEOREM A. *A regular space is metrizable if, and only if, it has a σ-hereditarily closure-preserving* (σ-HCP) *base.*

In [2], Guthrie and Henry deduce Theorem A from these results:

THEOREM B. *A topological space is pseudometrizable if, and only if, it has the weak topology induced by a σ-relatively complete collection.*

THEOREM C. *Let $\mathscr{F}$ be a family of continuous functions from a space X into* $[0, 1]$. *If the cozero sets of the functions in $\mathscr{F}$ form a* HCP *collection, then $\mathscr{F}$ is relatively complete.*

The example given by Burke, Engelking and Lutzer in [1], Example 8, proves that Theorem C is not correct because if for every $H_\alpha \in \mathscr{H}$ we define

$$f_\alpha(x) = \begin{cases} 1 & \text{when } x \in H_\alpha, \\ 0 & \text{when } x \in X - H_\alpha \end{cases}$$

then the family $\mathscr{F} = \{f_\alpha, 0 \leqslant \alpha < \omega_1\}$ verifies the hypothesis of Theorem C but, obviously, it is not relatively complete.

However, we have the following result:

PROPOSITION 1. *Let $\mathscr{F} = \{f_i, i \in I\}$ be a family of continuous functions from a space X into* $[0, 1]$, *whose cozero sets form a* HCP *collection. Then $\mathscr{F}$ is relatively complete if each point in X is a G_δ set.*

Proof. Let $J \subset I$. Then $\sup\{f_j,\ j \in J\}$ is continuous (see the first part of [2] Theorem 6).

Let $f = \inf\{f_j, j \in J\}$ and let U_j be the cozero set of f_j. Clearly, $f(x)$ is defined for each $x \in X$. If $f(x) = 0$, it is easily verified that f is continuous at x. If $f(x) \neq 0$ then $x \in \bigcap \{U_j, j \in J\}$. We can put $\{x\} = \bigcap_{n=1}^{\infty} V_n$ where every V_n is open. Well-order J and let

$$H_j = \begin{cases} U_j \cap V_j & \text{when } j < \omega_0 \\ U_j & \text{when } j \geqslant \omega_0 \end{cases}$$

By using the technique of [1], Lemma 4, we prove that $\bigcap \{H_j, j \in J\} = \{x\}$ is open. Consequently, f is continuous at the isolated point x. Then f is continuous in X and the proof is complete.

Now, Theorem A is an immediate consequence of Theorem B and Proposition 1.

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References

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