Generalized Lusin sets with the Blackwell property

by

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Abstract. A separable space $(X, \mathscr{B})$ has the Blackwell property if whenever $\mathscr{C}$ is a countably generated sub-σ-algebra of $\mathscr{B}$ separating points of X, then $\mathscr{C} = \mathscr{B}$. We characterize which Lusin and Sierpiński sets have the Blackwell property and consider when the property is preserved under the taking of products.

§ 0. Introduction. At the 1983 Oberwolfach Conference on Measure Theory, the first author posed the problem of whether the construction of universally null Blackwell spaces is possible within ZFC. At present, the matter does not seem to have found resolution. However, Jakub Jasiński [7] in Gdańsk has very recently shown that under the assumption of CH, there are some Lusin sets with the Blackwell property and some without. As he points out, his arguments apply also to their category analogue, the Sierpiński sets.

The present paper is an effort to unify these results with other theorems about Blackwell spaces in a rather general framework. Many examples of Blackwell spaces may then be seen to arise as generalized Lusin sets with respect to a σ-ideal of Borel sets. Vary the σ-ideal, and different species of Blackwell spaces emerge.

Our point of departure remains the observation that every set is a generalized Lusin set with respect to some Borel σ-ideal. For each fixed σ-ideal, only the "larger" Lusin sets enjoy the Blackwell property, and the strong Blackwell property is reserved for those still even larger. Further gradations of size may be observed for sets various products of which are also Blackwell.

Section 1 defines Blackwell properties and introduces the concept of a Lusin set with respect to a σ-ideal, as well as the notion of size ($\mathscr{I}$-density) mentioned above. Section 2 goes on to characterize $\mathscr{I}$-Lusin sets with Blackwell and strong Blackwell properties for an arbitrary σ-ideal (Proposition 2). It is seen that if $\mathscr{I}$ is a uniformisable σ-ideal, then for $\mathscr{I}$-Lusin sets, these properties actually coincide.

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Propositions 3 and 4 take up the matter of Blackwell properties for spaces of the form $X^n \times S$, where X is an $\mathscr{I}$-Lusin set and S is a standard space.

Section 3 presents proofs (assuming CH) that there are $\mathscr{I}$-Lusin sets with and without Blackwell properties (Propositions 6 and 7). Proposition 5 guarantees the existence of $\mathscr{I}$-Lusin sets of any prescribed order of density. Section 4 takes on the σ-ideal of countable sets. This σ-ideal turns out to be uniformisable, with the consequence that for spaces with totally imperfect complement, Blackwell and strong Blackwell properties coincide. This result was obtained earlier by the first author [18], but here it is viewed as part of a general theory.

Section 5 defines the notion of a uniformisable set. Like the Blackwell spaces, such sets turn out to be a generalization of analytic spaces (Proposition 10), and they have a close relationship with the Blackwell problem (Proposition 13).

The rest of the paper specializes to the σ-ideal of measure zero sets. In this case, the $\mathscr{I}$-Lusin sets are precisely the classical Sierpiński sets. This σ-ideal is not uniformisable (Lemma 13), and indeed, Blackwell and strong Blackwell properties part company for such spaces, at least under CH (Proposition 18). Interestingly enough, the construction also solves an open problem of probabilistic measure theory posed by Ramachandran [14]: the notions of independent random variable due to Steinhaus and Kolmogoroff do not necessarily agree on Blackwell domains.

Section 7 takes up the question of Blackwell properties for various products of Sierpiński sets. The structure of the σ-ideal of measure zero sets is sufficiently rich so as to allow a fuller analysis than that given in Section 2, e.g. Proposition 19. Some work of Kellerer on marginal problems puts in an unexpected guest appearance at this juncture.

Section 8 obtains a pair of measurable selection theorems (Propositions 23 and 25) using disintegration of measures, hyperspaces of compact sets and Strassen's Theorem from probability in concert. With these results, it becomes possible to establish another characterization of Sierpiński sets with Blackwell properties (Proposition 26). This theorem finds employment in Section 9, where it is used to prove Proposition 27 on Blackwell properties for countable unions of such sets.

Section 10 considers the question of whether $X \times S$ is Blackwell whenever S is standard and X is a strongly Blackwell Sierpiński set. The issue remains unresolved, but the question is shown to be equivalent to a conjecture about measurable selections and doubly-stochastic measures (Proposition 29).

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§ 1. Preliminaries. For the elements of descriptive set theory and basic results about Borel structures, the reader is referred to [1] and [9]. The notation and terminology of this paper generally follow [1].

A *separable space* is a measurable (Borel) space $(X, \mathscr{B})$ whose σ-algebra $\mathscr{B}$ is countably generated (c.g.) and contains all singletons drawn from the set X. If $\mathscr{C}$ is a sub-σ-algebra of $\mathscr{B}$, and $A \subset X$, then the notation $\mathscr{C}(A) = \{C \cap A: C \in \mathscr{C}\}$ will be used to denote the relative $\mathscr{C}$-structure on A. A separable space $(S, \mathscr{B})$ is *standard* (resp. *analytic*) if there is a complete (resp. analytic) separable metric topology on S for which $\mathscr{B}$ is the corresponding Borel structure.

A separable space $(X, \mathscr{B})$ has the *Blackwell property* if whenever $\mathscr{C}$ is a c.g. sub-σ-algebra of $\mathscr{B}$ separating points of X, then necessarily $\mathscr{C} = \mathscr{B}$. Say that $(X, \mathscr{B})$ has the *strong Blackwell property* if whenever $\mathscr{C} \subset \mathscr{D}$ are c.g. sub-σ-algebras of $\mathscr{B}$ with the same atoms, the necessarily $\mathscr{C} = \mathscr{D}$. For equivalent definitions, consult [1].

Throughout this paper, $(S, \mathscr{B})$ will denote a fixed uncountable standard space. Often, the space will be denoted simply as S. A σ-ideal $\mathscr{I}$ of sets in $\mathscr{B}(S)$ is *continuous* if it contains all singleton subsets of S. Let $\mathscr{I}$ be any σ-ideal in $\mathscr{B}(S)$. (The set S is never a member of $\mathscr{I}$.) A subset X of S is *$\mathscr{I}$-Lusin* if X is uncountable and the intersection of X with every member of $\mathscr{I}$ is countable.

We shall often consider the n-fold product of S with itself as a standard space, denoted by S^n. Here n may be any positive integer. If $N_1, \ldots, N_n$ are subsets of S, then $\langle N_1, \ldots, N_n \rangle$ is the set of all $(s_1, \ldots, s_n)$ in S^n such that for some k, one has $s_k \in N_k$. Thus $\langle N_1, \ldots, N_n \rangle$ is the complement in S^n of $N_1^c \times N_2^c \times \ldots \times N_n^c$. A subset R of S^n is *$\mathscr{I}$-reticulate in S^n* if these are sets $N_1, \ldots, N_n$ in $\mathscr{I}$ with $R \subset \langle N_1, \ldots, N_n \rangle$. A subset T of $S \times S$ is an *$\mathscr{I}$-thread* if

1) T is the graph of a Borel-isomorphism between sets A and B in $\mathscr{B}(S)$;
2) T is not $\mathscr{I}$-reticulate.

A subset $\mathscr{G}$ of $S \times S$ is an *$\mathscr{I}$-graph* if

1) $\mathscr{G}$ is the graph of some Borel function $g: A \to S$, where $A \in \mathscr{B}(S)$;
2) $\mathscr{G}$ is not $\mathscr{I}$-reticulate.

Note. The set $\mathscr{G}$ is allowed to be of either form $\{(s, t): t = g(s)\}$ or $\{(s, t): s = g(t)\}$. A σ-ideal $\mathscr{I}$ is *uniformisable* if every set R in $\mathscr{B}(S \times S)$ which is not $\mathscr{I}$-reticulate contains an $\mathscr{I}$-thread.

A subset X of S is *$\mathscr{I}$-dense (of order 1) in S* if every $B \in \mathscr{B}(S)$ disjoint from X is a member of $\mathscr{I}$. Say that X is *$\mathscr{I}$-dense of order n in S* if X^n intersects every R in $\mathscr{B}(S^n)$ which is not $\mathscr{I}$-reticulate. It is easy to see that for each $n \geqslant 1$, $\mathscr{I}$-density of order $n+1$ implies $\mathscr{I}$-density of order n.

One particular type of σ-ideal in $\mathscr{B}(S)$ will prove to be of special interest. Let X be a fixed uncountable subset of S. Define $\mathscr{I}(X)$ to be the σ-ideal of all B in $\mathscr{B}(S)$ with $B \cap X$ countable. Then $\mathscr{I}(X)$ is a continuous σ-ideal, and X is $\mathscr{I}(X)$-dense in S. In fact, one may easily prove.

LEMMA 1. *Let $\mathscr{I}$ be a σ-ideal in $\mathscr{B}(S)$ and let X be an uncountable subset of S.*

1) *The set X is $\mathscr{I}$-Lusin in S if and only if $\mathscr{I} \subset \mathscr{I}(X)$.*

2) *If $\mathscr{I}(X) \subset \mathscr{I}$, then X is $\mathscr{I}$-dense in S. The converse holds if $\mathscr{I}$ is assumed to be continuous.*

Thus, $\mathscr{I}(X)$ is the largest σ-ideal $\mathscr{I}$ in $\mathscr{B}(S)$ for which X is $\mathscr{I}$-Lusin, whilst $\mathscr{I}(X)$ is the smallest continuous σ-ideal $\mathscr{I}$ for which X is $\mathscr{I}$-dense.

Suppose $\mathscr{I}$ is a given σ-ideal in $\mathscr{B}(S)$ and that $\mathscr{C}$ and $\mathscr{D}$ are c.g. sub-σ-algebras of $\mathscr{B}(S)$. Say that $\mathscr{C}$ is *$\mathscr{I}$-proper* in $\mathscr{D}$ if

1) $\mathscr{C} \subset \mathscr{D}$;

2) if $N \in \mathscr{I}$, then $\mathscr{C}(N^c) \neq \mathscr{D}(N^c)$.

Given Marczewski functions f and g for countable generators of σ-algebras $\mathscr{C}$ and $\mathscr{D}$, we define the set

$$T(\mathscr{C}, \mathscr{D}) = \{(s,t) \in S \times S : f(s) = f(t) \quad \text{and} \quad g(s) \neq g(t)\}.$$

The definition of $T(\mathscr{C}, \mathscr{D})$ does not depend on the particular choice of the Marczewski functions f and g.

LEMMA 2. *Let $\mathscr{C} \subset \mathscr{D}$ be c.g. sub-σ-algebras of $\mathscr{B}(S)$. Then the following are equivalent*:

1) *$\mathscr{C}$ is $\mathscr{I}$-proper in $\mathscr{D}$;*

2) *there is some set D in $\mathscr{D}$ not equivalent to any $\mathscr{C}$-set modulo $\mathscr{I}$: in other words, $D \triangle C \notin \mathscr{I}$ for any C in $\mathscr{C}$;*

3) *$T(\mathscr{C}, \mathscr{D})$ is not $\mathscr{I}$-reticulate in $S \times S$.*

Proof. 1)$\Rightarrow$2): We proceed to prove the contrapositive and assume that no set D as in 2) exists. Let $D_1, D_2, \ldots$ be a countable generator for $\mathscr{D}$. Then there is a sequence $C_1, C_2, \ldots$ in $\mathscr{C}$ with $C_n \sim D_n \bmod(\mathscr{I})$. Put

$$N = \bigcup (D_n \triangle C_n),$$

a set in $\mathscr{I}$. Then $\mathscr{C}(N^c) = \mathscr{D}(N^c)$, since for each n, $D_n \cap N^c = C_n \cap N^c$. Thus $\mathscr{C}$ is not $\mathscr{I}$-proper in $\mathscr{D}$.

2) $\Rightarrow$1): Suppose D in $\mathscr{D}$ is as specified and that $N \in \mathscr{I}$. Then for no set C in $\mathscr{C}$ does it happen that $D \cap N^c = C \cap N^c$. If this were true, then $C \triangle D \subset N$, a contradiction.

3) $\Rightarrow$1): Suppose that $N \in \mathscr{I}$ is such that $\mathscr{C}(N^c) = \mathscr{D}(N^c)$. Then one sees that $T(C, \mathscr{D}) \subset \langle N, N \rangle$.

1) $\Rightarrow$3): Suppose that these are sets N_1 and N_2 in $\mathscr{I}$ with $T(\mathscr{C}, \mathscr{D}) \subset \langle N_1, N_2 \rangle$. Put $N = N_1 \cup N_2$. Then $\mathscr{C}(N^c) \subset \mathscr{D}(N^c)$ have the same atoms. Since every standard set is strongly Blackwell [1; p. 21], we have $\mathscr{C}(N_s^c) = \mathscr{D}(N^c)$, as desired. ■

Say that a subset X of S satisfies condition (J) or (J+) for the σ-ideal $\mathscr{I}$ if

(J) whenever $\mathscr{C}$ is a c.g. sub-σ-algebra of $\mathscr{B}(S)$ which is $\mathscr{I}$-proper in $\mathscr{B}(S)$, then there is some $\mathscr{C}$-atom C such that $C \cap X$ contains at least two points (i.e. $\mathscr{C}$ does not separate points of X);

(J+) whenever $\mathscr{C}$ and $\mathscr{D}$ are c.g. sub-σ-algebras of $\mathscr{B}(S)$ with $\mathscr{C}$ $\mathscr{I}$-proper in $\mathscr{D}$, then there is some $\mathscr{C}$-atom C such that $C \cap X$ contains two points separated by $\mathscr{D}$.

Compare the definition of (*)-Blackwell conditions in [18], as well as their use for a special case in [7] ("*Jasiński conditions*"). Condition (J+) guarantees that whenever $\mathscr{C}$ is very much smaller than $\mathscr{D}$ (with respect to $\mathscr{I}$), then the set X contains witnesses to this fact.

Another sort of σ-ideal on which our attention will focus consists of sets which are null with respect to a given probability measure. Let m be a Borel probability measure on S. Define $\mathscr{I}(m$ to be the collection of all m-null sets in $\mathscr{B}(S)$. The σ-ideal $\mathscr{I}(m)$ is continuous if and only if m is a continuous measure. Most often, we shall take m to be continuous. A subset X of S is $\mathscr{I}(m)$-dense in S if and only if $m^*(X) = 1$. If m is continuous, then the $\mathscr{I}(m)$-Lusin sets are the classical sets with property (S) studied by Sierpiński [2], [20]. We shall often use the terms $\mathscr{I}(m)$-Lusin and m-Sierpiński interchangeably.

The following result will be quite handy in the sequel:

LEMMA 3. *Let m be an atomless probability on S. If B is a set in $\mathscr{B}(S^2)$ of positive $(m \otimes m)$-area, then B contains an $\mathscr{I}(m)$-thread.*

Note. The result also holds whenever m is purely atomic, but not if m is of mixed type.

Proof. By the usual isomorphism theorems, it is no loss of generality to take m to be Lebesgue measure on the interval $S =]0, 1[$. For each real c, define $B(c) = \{(s,t) \in B : t = s+c\}$. Then by Fubini's Theorem (at a 45° angle), at least one of the sets $B(c)$ has positive linear measure. Such a set $B(c)$ is an $\mathscr{I}(m)$-thread. ■

Consider the following restriction on a σ-ideal $\mathscr{I}$:

(R) Whenever B_1 and B_2 are two members of $\mathscr{B}(S) \setminus \mathscr{I}$, then the rectangle $B_1 \times B_2$ contains an $\mathscr{I}$-thread.

LEMMA 4. *Let $\mathscr{I}$ be a continuous σ-ideal in $\mathscr{B}(S)$ satisfying restriction* (R). *If $X \subset S$ satisfies* (J), *then X is $\mathscr{I}$-dense. In particular, this implication holds when $\mathscr{I}$ is uniformisable.*

Proof. If X is not dense in S, then there is some set $B \subset S \setminus X$ in $\mathscr{B} \setminus \mathscr{I}$. Since $\mathscr{I}$ is continuous, B decomposes into two (necessarily uncountable) disjoint sets B_1 and B_2 in $\mathscr{B} \setminus \mathscr{I}$. Applying restriction (R), we find an $\mathscr{I}$-thread G inside $B_1 \times B_2$. Now G is the graph of a Borel-isomorphism g between sets C_1 and C_2 in $\mathscr{B}(S)$. Define f: $S \to S$ by the rule

$$f(s) = \begin{cases} g(s), & s \in C_1, \\ s, & s \notin C_1. \end{cases}$$

Define $\mathscr{C} = \mathscr{B}(f) = \{f^{-1}(B) : B \in \mathscr{B}(S)\}$. We claim that $\mathscr{C}$ is $\mathscr{I}$-proper in $\mathscr{B}(S)$. For each $N \in \mathscr{I}$, there is in G a point $(s, g(s))$ not in $(N \times S) \cup (S \times N)$. Then s and $g(s)$ are points of N^c not separated by $\mathscr{C}$. So $\mathscr{C}(N^c) \neq \mathscr{B}(N^c)$.

However, $\mathscr{C}$ separates points of X. The Lemma follows by contraposition. ■

We shall see later that a σ-ideal may satisfy restriction (R) but not be uniformisable. This will be the case with the σ-ideal of null sets for a continuous measure (see Section 6). Also, it is not true that every continuous σ-ideal $\mathscr{I}$ satisfies restriction (R). An example to illustrate the point will be given in Section 6.

§ 2. Results for general σ-ideals. As usual, $\mathscr{I}$ is taken to be a σ-ideal in the standard structure $\mathscr{B}(S)$.

PROPOSITION 1. *Suppose X is $\mathscr{I}$-dense in S.*

A) *If X is $\mathscr{I}$-dense of order 2 in S, then X satisfies condition* (J+).

B) *If X satisfies condition* (J+), *then $X\times X$ intersects every $\mathscr{I}$-graph in $S\times S$.*

C) *If X satisfies condition* (J), *then $X\times X$ intersects every $\mathscr{I}$-thread in $S\times S$.*

Proof. Part A. As noted in Lemma 2, $\mathscr{C}$ proper in $\mathscr{D}$ means that $T(\mathscr{C},\mathscr{D})$ is not $\mathscr{I}$-reticulate. If X is dense of order 2, then $(X\times X)\cap T(\mathscr{C},\mathscr{D})\neq\emptyset$. So there are points x and x' in X not separated by $\mathscr{C}$ but separated by $\mathscr{D}$. This establishes condition (J+).

Part B. We assume that $S=]0,1[$ under the usual linear order and Borel structure. The contrapositive will be proved.

Begin by taking $\mathscr{G}$ to be an $\mathscr{I}$-graph with $\mathscr{G}\cap(X\times X)=\emptyset$. We shall assume that there is some A in $\mathscr{B}(S)$ and a Borel function $k\colon A\to S$ such that $\mathscr{G}=\operatorname{graph}(k)=\{(s,k(s))\colon s\in A\}$. This assumption is made without loss of generality. Define

$$\Delta=\{(s,s)\colon s\in S\},$$
$$\Delta_+=\{(s,t)\colon s<t\},$$
$$\Delta_-=\{(s,t)\colon s>t\}$$

as subsets of $S\times S$. Since X is $\mathscr{I}$-dense in S, it follows that $\mathscr{G}\cap\Delta$ is $\mathscr{I}$-reticulate. So one of the sets $\mathscr{G}\cap\Delta_-$ or $\mathscr{G}\cap\Delta_+$ is an $\mathscr{I}$-graph. We shall assume that $\mathscr{G}\cap\Delta_-$ is an $\mathscr{I}$-graph. The other case, where it is $\mathscr{G}\cap\Delta_+$ that is an $\mathscr{I}$-graph involves an argument entirely parallel to the one we now offer.

With these assumptions in place, there is some $\varepsilon>0$ such that $\mathscr{G}\cap\Delta_-(\varepsilon)$ is an $\mathscr{I}$-graph, where

$$\Delta_-(\varepsilon)=\{(s,t)\in S\times S\colon s-\varepsilon>t\}.$$

Also, there is some open interval $\mathscr{O}$ of length ε such that $\mathscr{G}\cap\Delta_-(\varepsilon)\cap(\mathscr{O}\times S)$ is an $\mathscr{I}$-graph. This set is the graph of a Borel function h defined on a set D in $\mathscr{B}(S)$:

$$\mathscr{G}\cap\Delta_-(\varepsilon)\cap(\mathscr{O}\times S)=\{(s,h(s))\colon s\in D\}.$$

Now, whenever s and t are elements of D, then $h(s)<s-\varepsilon<t$, so that $h(D)\cap D=\emptyset$. Define functions f and g on S by the formulae

$$f(s)=\begin{cases}h(s), & s\in D,\\ s, & s\in D^c;\end{cases}$$
$$g(s)=\begin{cases}(h(s),0), & s\in D,\\ (s,s), & s\in D^c.\end{cases}$$

Here, g takes values in $\boldsymbol{R}^2$. Let $\mathscr{C}$ and $\mathscr{D}$ be the sub-σ-algebras generated by f and g, respectively.

We claim that $\mathscr{C}$ is $\mathscr{I}$-proper in $\mathscr{D}$. This will follow from a proof that $T(\mathscr{C},\mathscr{D})$ is not $\mathscr{I}$-reticulate. For this, we need only note that

$$T(\mathscr{C},\mathscr{D})=\{(s,h(s))\colon s\in D\}\cup\{(h(t),t)\colon t\in D\}.$$

On the other hand, $T(\mathscr{C},\mathscr{D})\cap(X\times X)=\emptyset$, so that $\mathscr{C}(X)$ and $\mathscr{D}(X)$ have the same atoms. Thus condition (J+) fails for X.

Part C. Essentially the same proof as for part B, taking $\mathscr{G}$ to be an $\mathscr{I}$-thread and noting that, in this case, $\mathscr{D}$ as defined above equals $\mathscr{B}(S)$. ■

PROPOSITION 2. *Let X be an $\mathscr{I}$-Lusin subset of S.*

1) *If X satisfies* (J+), *then X is strongly Blackwell. The converse obtains if X is $\mathscr{I}$-dense in S.*

2) *If X satisfies* (J), *then X is Blackwell. The converse obtains if X is $\mathscr{I}$-dense in S.*

Proof. We show that if X satisfies condition (J+), then X is strongly Blackwell. The implication (J)→Blackwell runs parallel and its proof is omitted.

So suppose that $\mathscr{C}(X)\subset\mathscr{D}(X)$ are c.g. sub-σ-algebras of $\mathscr{B}(X)$ with the same atoms. Then there are c.g. σ-algebras $\mathscr{C}\subset\mathscr{D}\subset\mathscr{B}(S)$ whose relativizations to X are $\mathscr{C}(X)$ and $\mathscr{D}(X)$. Condition (J+) implies that $\mathscr{C}$ is not proper in $\mathscr{D}$. Therefore, there is some set N in $\mathscr{I}$ with $\mathscr{C}(N^c)=\mathscr{D}(N^c)$. Since X is an $\mathscr{I}$-Lusin set, it follows that $X\cap N$ is countable.

Let A be the union of all $\mathscr{C}$-atoms C with the property that $C\cap X\cap N\neq\emptyset$. There are only countably many such $\mathscr{C}$-atoms, so that $A\in\mathscr{C}(S)$. Since $\mathscr{C}(X\cap A)$ and $\mathscr{D}(X\cap A)$ have the same countable set of atoms, one has $\mathscr{C}(X\cap A)=\mathscr{D}(X\cap A)$. Now $X\cap A^c\subset N^c$, so that $\mathscr{C}(X\cap A^c)=\mathscr{D}(X\cap A^c)$. Given D in $\mathscr{D}$, there are $\mathscr{C}$-sets C_1 and C_2 such that

$$\begin{aligned}D\cap X&=(D\cap X\cap A)\cup(D\cap X\cap A^c)\\&=(C_1\cap X\cap A)\cup(C_2\cap X\cap A^c)\end{aligned}$$

is a member of $\mathscr{C}(X)$. So $\mathscr{C}(X)=\mathscr{D}(X)$, as desired.

To conclude the proof, we show that if X is $\mathscr{I}$-dense and strongly Blackwell, then X satisfies condition (J+). Again, the implication Blackwell→(J) is similar and so its proof is omitted.

So suppose that $\mathscr{C}$ and $\mathscr{D}$ are c.g. sub-σ-algebras of $\mathscr{B}(S)$ with $\mathscr{C}$ proper in $\mathscr{D}$. We shall assume that (J+) fails and derive a contradiction. If (J+) fails, then $\mathscr{C}(X)\subset\mathscr{D}(X)$ have the same atoms. From the strong Blackwell property, $\mathscr{C}(X)=\mathscr{D}(X)$. Let g be a Marczewski function for $\mathscr{D}(S)$. Then when restricted to X, g generates $\mathscr{D}(X)=\mathscr{C}(X)$. Call this restricted function g_0. Since it is $\mathscr{C}(X)$-measurable, it extends to a $\mathscr{C}$-measurable function f on S. Put

$$B=\{s\in S\colon f(s)=g(s)\}.$$

Then $B\in\mathscr{B}(S)$; in fact, $B\in\mathscr{D}$.

Since $X\subset B$, we know that $S\setminus B\in\mathscr{I}$. We claim that $\mathscr{C}(B)=\mathscr{D}(B)$. For, suppose,

that b and b' are points in B separated by $\mathscr{D}$. Then $g(b) \neq g(b')$. Thus $f(b) \neq f(b')$ and so b and b' are separated by $\mathscr{C}$. This proves the claim.

A contradiction has been established: $\mathscr{C}$ is proper in $\mathscr{D}$, but $\mathscr{C}(B) = \mathscr{D}(B)$ with $S \setminus B \in \mathscr{I}$. ■

Remark. It should be noted that in the proof of the converse, the hypothesis that X is $\mathscr{I}$-Lusin was not used.

COROLLARY. *Suppose that $\mathscr{I}$ is a uniformisable σ-ideal. Let X be an $\mathscr{I}$-Lusin set $\mathscr{I}$-dense in S. Then the following are equivalent*:

1) *X is dense of order 2;*
2) *X is strongly Blackwell;*
3) *X is Blackwell.*

Note. The implication 1)⇒2) does not require $\mathscr{I}$ to be uniformisable or X to be $\mathscr{I}$-dense; 3) ⇒1) does not require $\mathscr{I}$ to be continuous or X to be $\mathscr{I}$-Lusin; 2)⇒3) needs no conditions on $\mathscr{I}$ or X.

DEFINITION. A subset X of S is *analytically $\mathscr{I}$-dense (of order 1) in S* if for each analytic subset A of $S \setminus X$, there is some set N in $\mathscr{I}$ with $A \subset N$.

Clearly, this notion is stronger than that of simple $\mathscr{I}$-density. The following is immediate:

LEMMA 5. *A subset X of S is analytically $\mathscr{I}(X)$-dense if and only if Borel sets separate X from every analytic A with $A \cap X = \emptyset$.*

EXAMPLE 1. Let A be an analytic subset of S. Then A is analytically $\mathscr{I}(A)$-dense in S. This is Lusin's first separation principle in another costume.

EXAMPLE 2. Let X be a co-analytic subset of S. Then X is analytically $\mathscr{I}(X)$ dense in S if and only if X is a Borel set.

Say that a subset X of S is *analytically $\mathscr{I}$-dense of order n in S* if X^n intersects every analytic set $A \subset S^n$ which is not $\mathscr{I}$-reticulate in S^n. In other words, if $A \subset S^n \setminus X^n$, then there are sets $N_1, \ldots, N_n$ in $\mathscr{I}$ with $A \subset \langle N_1, \ldots, N_n \rangle$.

It is easy to show that analytic $\mathscr{I}$-density of order $n+1$ implies that of order n for $n = 1, 2, \ldots$

LEMMA 6. *Let X be an $\mathscr{I}$-Lusin subset of S. The following statements are equivalent*:

1) *X is analytically $\mathscr{I}$-dense of order n;*

2) *whenever A is an analytic subset of $S^n \setminus X^n$, then there are $\mathscr{I}$-sets $N_1, \ldots, N_n$ contained in $S \setminus X$ such that $A \subset \langle N_1, \ldots, N_n \rangle$.*

Proof. The implication 2⇒1 is immediate The proof of 1⇒2 proceeds *via* induction on n. For $n = 1$, we see that if $A \subset S \setminus X$ is analytic, then there is some N in $\mathscr{I}$ with $A \subset N$. But since X is $\mathscr{I}$-Lusin, we know that $N \cap X$ is countable. So $A \subset N \setminus X \subset S \setminus X$ fills the bill.

Now we assume the result for all dimensions less than n ($n \geqslant 2$) and suppose

that X is analytically $\mathscr{I}$-dense of order n. Suppose that $A \subset S^n \setminus X^n$ is an analytic set. There are sets $N_1, \ldots, N_n$ in $\mathscr{I}$ with $A \subset \langle N_1, \ldots, N_n \rangle$. Since X is $\mathscr{I}$-Lusin, the set $N_1 \cap X$ may be listed as $x_1, x_2, \ldots$ For each $i \geqslant 1$, the section $A(x_i)$ of A over x_i is an analytic subset of $S^{n-1} \setminus X^{n-1}$. By the induction hypothesis, there are $\mathscr{I}$-sets $N_2^{(i)}, \ldots, N_n^{(i)}$ contained in $S \setminus X$ with $A(x_i) \subset \langle N_2^{(i)}, \ldots, N_n^{(i)} \rangle$. For $2 \leqslant k \leqslant n$, put $N_k' = N_k \cup N_k^{(1)} \cup N_k^{(2)} \cup \ldots$ Then $A \subset \langle N_1 \setminus X, N_2', N_3', \ldots, N_n' \rangle$.

Repeat the procedure for each of the other $n-1$ co-ordinates in S^n. This will produce $\mathscr{I}$-sets $M_1, \ldots, M_n$ contained in $S \setminus X$ such that $A \subset \langle M_1, \ldots, M_n \rangle$, as desired. ■

PROPOSITION 3. *Let $\mathscr{I}$ be a σ-ideal in the standard structure $\mathscr{B}(S)$ and suppose that $X \subset S$ is an $\mathscr{I}$-Lusin set analytically $\mathscr{I}$-dense of order $2n$ ($n \geqslant 1$). Then $X^n \times S$ is a strongly Blackwell space.*

Proof. It suffices, by Propositions 1 and 2, to prove that $X^n \times S$ is $\mathscr{I}(X^n \times S)$-dense of order 2 in $S^n \times S$. So suppose R is a member of $\mathscr{B}(S^n \times S \times S^n \times S)$ contained in $(S^n \times S \times S^n \times S)$ $(X^n \times S \times X^n \times S)$. Let f: $S^n \times S \times S^n \times S \to S^n \times S^n$ be defined as the projection

$$f(s_1, \ldots, s_n, s_{n+1}, t_1, \ldots, t_n, t_{n+1}) = (s_1, \ldots, s_n, t_1, \ldots, t_n).$$

Then $f(R)$ is an analytic subset of $(S^n \times S^n) \setminus (X^n \times X^n)$. Since X is analytically $\mathscr{I}$-dense of order $2n$, there are $\mathscr{I}$-sets $A_1, \ldots, A_i$ and $B_1, \ldots, B_n$ contained in $S \setminus X$ (Lemma 6) with

$$f(R) \subset \langle A_1, \ldots, A_n, B_1, \ldots, B_n \rangle,$$

and so

$$R \subset \langle \langle A_1, \ldots, A_n, \emptyset \rangle, \langle B_1, \ldots, B_n, \emptyset \rangle \rangle.$$

Since $\langle A_1, \ldots, A_n, \emptyset \rangle$ and $\langle B_1, \ldots, B_n, \emptyset \rangle$ do not intersect $X^n \times S$, we see that R is $\mathscr{I}(X^n \times S)$-reticulate, as desired. ■

PROPOSITION 4. *Let $\mathscr{I}$ be a σ-ideal in the standard Borel structure $\mathscr{B}(S)$ and suppose that $X \subset S$ is analytically $\mathscr{I}$-dense (order 1) in S. For $n \geqslant 1$, if $X^n \times S$ is Blackwell, then X is analytically $\mathscr{I}$-dense of order $2n$ in S.*

Proof. Suppose that A is an analytic subset of $S^{2n} \setminus X^{2n}$. There is some Borel mapping h of S onto A. We define a Borel subset T of $(S^n \times S) \times (S^n \times S)$ as

$$T = \{(s_1, \ldots, s_n, s_{n+1}, t_1, \ldots, t_n, t_{n+1}) : s_{n+1} = t_{n+1} \quad \text{and}$$
$$h(s_{n+1}) = (s_1, \ldots, s_n, t_1, \ldots, t_n)\}.$$

The set T is disjoint from $(X^n \times S) \times (X^n \times S)$. Furthermore, each section of T (either direction) over a point in $S^n \times S$ is at most singleton.

Now since $X^n \times S$ is Blackwell, it follows (Propositions 1 and 2) that $(X^n \times S) \times (X^n \times S)$ meets every $\mathscr{I}(X^n \times S)$-thread in $(S^n \times S) \times (S^n \times S)$. Since the one-one graph T is not met, it must be that there is some N in $\mathscr{I}(X^n \times S)$ with $T \subset \langle N, N \rangle$.

Let f be the natural projection map from $S^n \times S$ onto S^n. Then $M = f(N)$ is an analytic subset of S^n such that

a) $M \cap X^n$ is countable;

b) $A \subset \langle M, M \rangle$.

We have proved that to each analytic $A \subset S^{2n} \setminus X^{2n}$ there corresponds at least one analytic $M \subset S^n$ with properties a) and b).

The proof now proceeds by induction on n. For $n = 1$, take M corresponding to A as above. Since X is analytically $\mathscr{I}$-dense in S, there is some $\mathscr{I}$-set N with $M \subset N$. Then $A \subset \langle N, N \rangle$ as desired. Now we assume that the result has been established for dimensions $k \leqslant n$ and consider the case for $n+1$. Since $X^{n+1} \times S$ is Blackwell, so too is $X^n \times S$, so that (by the induction hypothesis) X is analytically $\mathscr{I}$-dense of order $2n$ and therefore also of order $n+1$. Again, given $A \subset S^{2(n+1)} \setminus X^{2(n+1)}$, take $M \subset S^{n+1}$ corresponding to A as before. There are $\mathscr{I}$-sets $N_1, \ldots, N_{n+1}$ with $M \subset \langle N_1, \ldots, N_{n+1} \rangle$. So

$$A \subset \langle N_1, \ldots, N_{n+1}, N_1, \ldots, N_{n+1} \rangle$$

as desired.

COROLLARY. *Let $X \subset S$ be an $\mathscr{I}$-Lusin set analytically $\mathscr{I}$-dense in S and suppose that A is an uncountable analytic subset of S. Then for each $n \geqslant 1$, the following are equivalent:*

1) *$X^n \times S$ is strongly Blackwell;*
2) *$X^n \times A$ is strongly Blackwell;*
3) *$X^n \times A$ is Blackwell;*
4) *$X^n \times S$ is Blackwell;*
5) *X is analytically $\mathscr{I}$-dense of order $2n$ in S.*

Proof. The implications $1 \Rightarrow 2$ and $3 \Rightarrow 4$ follow from the fact that A is a measurable image of S, and that S is isomorphic to a Borel subset of A. The implication $2 \Rightarrow 3$ is obvious, $4 \Rightarrow 5$ is Proposition 4, whilst $5 \Rightarrow 1$ is Proposition 3. ■

§ 3. Existence theorems. In this section, we make use of the continuum hypothesis (CH) to prove the existence of $\mathscr{I}$-Lusin sets with and without Blackwell properties. Later, through the use of Proposition 5 (regarding the existence of $\mathscr{I}$-Lusin sets of any prescribed density) we shall demonstrate the existence of sets $X \subset S$ such that $X^n \times S$ is strong Blackwell, but $X^{n+1} \times S$ is not Blackwell.

LEMMA 7 (CH). *Let $\mathscr{I}$ be a continuous σ-ideal in $\mathscr{B}(S)$. Let n be a positive integer and suppose that $\mathscr{A}$ is a collection of subsets of S^n such that*

1) *no set in $\mathscr{A}$ is $\mathscr{I}$-reticulate;*

2) *for each A in $\mathscr{A}$ and N in $\mathscr{I}$, the set $A \setminus \langle N, \ldots, N \rangle$ is a member of $\mathscr{A}$.*

Suppose that $Y \subset S$ is such that Y^n intersects every set in $\mathscr{A}$. Then there is an $\mathscr{I}$-Lusin $X \subset Y$ with the same property.

Proof. Well-order the members of $\mathscr{A}$ and of $\mathscr{I}$ in transfinite series $A_0 A_1 A_2 \ldots A_\alpha \ldots$ and $N_0 N_1 N_2 \ldots N_\alpha \ldots$, $\alpha < \aleph_1$. Select points $y_1^{(0)}, \ldots, y_n^{(0)}$ in Y with

$(y_1^{(0)}, \ldots, y_n^{(0)}) \in A_0 \setminus \langle N_0, \ldots, N_0 \rangle$. For each α, put $M_\alpha = \bigcup \{N_\beta : \beta \leqslant \alpha\} \cup \bigcup \{y_i^{(\beta)} : 1 \leqslant i \leqslant n \text{ and } \beta < \alpha\}$ and choose points $y_1^{(\alpha)}, \ldots, y_n^{(\alpha)}$ in Y with

$$(y_1^{(\alpha)}, \ldots, y_n^{(\alpha)}) \in A_\alpha \setminus \langle M_\alpha, \ldots, M_\alpha \rangle .$$

Properties of $\mathscr{A}$ ensure the possibility of such selections. Define

$$X = \{y_i^{(\alpha)} : 1 \leqslant i \leqslant n \text{ and } \alpha < \aleph_1\} .$$

This set has the desired properties. ■

PROPOSITION 5 (CH). *Let $\mathscr{I}$ be a continuous σ-ideal in $\mathscr{B}(S)$ and let n be a positive integer. Suppose that $Y \subset S$ has one of the following properties:*

1) *$Y \times Y$ intersects every $\mathscr{I}$-thread in $S \times S$;*
2) *$Y \times Y$ intersects every $\mathscr{I}$-graph in $S \times S$;*
3) *Y is $\mathscr{I}$-dense of order n in S;*
4) *Y is analytically $\mathscr{I}$-dense of order n in S.*

Then there is an $\mathscr{I}$-Lusin set $X \subset Y$ with the same property.

Indication. This is a simple application of Lemma 7: let $\mathscr{A}$ be either the collection of $\mathscr{I}$-threads, $\mathscr{I}$-graphs, or Borel or analytic subsets of S^n which are not $\mathscr{I}$-reticulate.

PROPOSITION 6 (CH). *Let $\mathscr{I}$ be a continuous σ-ideal in $\mathscr{B}(S)$. Then there is an $\mathscr{I}$-Lusin subset X of S with the strong Blackwell property.*

Proof. From Proposition 5 with $Y = S$, we find an $\mathscr{I}$-Lusin set X which is $\mathscr{I}$-dense of order 2 in S. The result follows from Propositions 1 and 2. ■

PROPOSITION 7 (CH). *Let $\mathscr{I}$ be a continuous σ-ideal in $\mathscr{B}(S)$ satisfying restriction* (R). *There is an $\mathscr{I}$-Lusin subset X of S which is dense but lacks the Blackwell property.*

Proof. Since the σ-ideal $\mathscr{I}$ is continuous, there are disjoint sets B_1 and B_2 in $\mathscr{B}(S) \setminus \mathscr{I}$. Restriction (R) implies that $B_1 \times B_2$ contains a thread G. There are sets C_1 and C_2 in $\mathscr{B}(S) \setminus \mathscr{I}$ so that G is the graph of a Borel-isomorphism g of C_1 onto C_2. Define $f: S \to S$ by the rule

$$f(s) = \begin{cases} g(s), & s \in C_1 , \\ g^{-1}(s), & s \in C_2 , \\ s, & s \in S \setminus (C_1 \cup C_2) . \end{cases}$$

Note that $g(s) \neq s$ for any s in C_1 and that $f = f^{-1}$.

Let $B_0 B_1 B_2 \ldots B_\alpha \ldots$ $(\alpha < c)$ be a listing in transfinite series of all members of $\mathscr{B}(S) \setminus \mathscr{I}$ and let $N_0 N_1 N_2 \ldots N_\alpha \ldots$ $(\alpha < c)$ be a similar listing of $\mathscr{I}$. For each countable ordinal α, put $M_\alpha = \bigcup \{N_\beta : \beta \leqslant \alpha\}$.

Choose x_0 from $B_0 \setminus M_0$ and for each $\alpha < c$ select x_α from

$$B_\alpha \setminus (M_\alpha \cup \{f(x_\beta) : \beta < \alpha\}) .$$

Put $X = \{x_\alpha : \alpha < c\}$. Then X is $\mathscr{I}$-Lusin and dense in S. But $X \times X$ does not intersect G, so that from Proposition 1, condition (J) fails for X. Proposition 2 shows that X lacks the Blackwell property. ■

§ 4. **Borel-density.** Let $\mathscr{I}$ be the σ-ideal of all countable subsets of the uncountable standard space S. Then $\mathscr{I}$ is continuous, and the $\mathscr{I}$-Lusin sets are precisely the uncountable subsets of S. The notions of $\mathscr{I}$-reticulate sets, $\mathscr{I}$-thread, and $\mathscr{I}$-density for this σ-ideal coincide with those of "reticulate set", "thread", and "Borel-density" as used in [17], [18], and [19].

LEMMA 8. *The σ-ideal $\mathscr{I}$ is uniformisable.*

Indication. This follows from results in [5], as well as an argument in Sarbadhikari's note [15].

The following was obtained in [18]. It is now seen to be a consequence of our general theory, in particular the corollary to Proposition 2.

PROPOSITION 8. *Let X be a subset of an uncountable Polish space S such that $S \setminus X$ is totally imperfect. Then the following are equivalent:*

1) *X is Blackwell;*
2) *X is strongly Blackwell;*
3) *X is $\mathscr{I}$-dense ("Borel-dense" in [17]) of order 2.*

COROLLARY. *If X is a Blackwell subset of S with totally imperfect complement, then any Y such that $X \subset Y \subset S$ is strongly Blackwell.*

§ 5. **Uniformisable sets.** Let X be a fixed uncountable subset of S. Define $\mathscr{I}(X)$ to be the σ-ideal consisting of all B in $\mathscr{B}(S)$ with $B \cap X$ countable. Then $\mathscr{I}(X)$ is continuous, and X is $\mathscr{I}(X)$-dense of order 1 in S.

LEMMA 9. *Suppose that X is $\mathscr{I}(X)$-dense of order 2 in S. Then the sets R in $\mathscr{B}(S \times S)$ which are not $\mathscr{I}(X)$-reticulate are precisely those for which $R \cap (X \times X)$ is not contained in a countable union of horizontal and vertical sections.*

Proof. Clearly, any R with this property is not $\mathscr{I}(X)$-reticulate. Conversely, if $R \cap (X \times X)$ is contained in a countable union of sections, then by removing these sections from R, we obtain a Borel set R_0 disjoint from $X \times X$. Second-order density implies that R_0 is $\mathscr{I}(X)$-reticulate. So also is R. ■

PROPOSITION 9. *Let A be an analytic subset of S. Then A is $\mathscr{I}(A)$-dense of order 2 in S.*

Proof. Let R be a member of $\mathscr{B}(S \times S)$ disjoint from $A \times A$. Then $R_1 = R \cap (S \times A)$ is an analytic set whose projection onto the first factor is disjoint from A. Lusin's first separation principle implies that there is some $N_1 \in \mathscr{B}(S)$ such that $N_1 \cap A = \emptyset$ and $R_1 \subset (N_1 \times S)$ Now $R_2 = R \setminus (N_1 \times S)$ is a member of $\mathscr{B}(S \times S)$ whose projection onto the second factor does not meet A. Thus there is some $N_2 \in \mathscr{B}(S)$ such that $N_2 \cap A = \emptyset$ and $R_2 \subset (S \times N_2)$. Then $N = N_1 \cup N_2 \in \mathscr{I}(A)$, and $R \subset (N \times S) \cup (S \times N)$. ■

PROPOSITION 10. *If A is analytic, then the σ-ideal $\mathscr{I}(A)$ is uniformisable.*

Proof. Suppose that R is a member of $\mathscr{B}(S \times S)$ not $\mathscr{I}(A)$-reticulate. From Proposition 9 and Lemma 9, we know that $R \cap (A \times A)$ is not contained in a countable union of sections. It follows from [5; Theorem 4.4] or [15] that $R \cap (A \times A)$ contains an uncountable standard set T each of whose horizontal and vertical sections is at most a singleton. See also the discussion in [19]. Thus T is an $\mathscr{I}(A)$-thread. ■

If X is any uncountable subset of S, then certainly X is an $\mathscr{I}(X)$-Lusin set. However, there may be a great many other $\mathscr{I}(X)$-Lusin sets essentially larger than X.

LEMMA 10. *Let A be an analytic subset of S. Then the $\mathscr{I}(A)$-Lusin sets are those uncountable subsets Y of S whose intersection with each constituent of the co-analytic set $S \setminus A$ is countable.*

Proof. If X is an $\mathscr{I}(A)$-Lusin set, and C is a constituent of $S \setminus A$, then $C \in \mathscr{I}(A)$, so that $X \cap C$ is countable. On the other hand, suppose X intersects each constituent of $S \setminus A$ in a countable set. Given N in $\mathscr{I}(A)$, we know that $N \cap A$ is countable, so that $N \setminus A$ is Borel and contained in a countable union of constituents (boundedness theorem). This forces $X \cap N$ to be countable. ■

PROPOSITION 11. *Let A be an analytic subset of S and suppose that Y is a set whose intersection with each constituent of $S \setminus A$ is countable. Then $A \cup Y$ is strongly Blackwell.*

Remark. Jakub Jasiński has recently [7] used Martin's Axiom to obtain a partial converse to this result.

Proof. From Proposition 9, the sets A and $A \cup Y$ are $\mathscr{I}(A)$-dense of order 2. Lemma 10 says that $A \cup Y$ is $\mathscr{I}(A)$-Lusin. The result now follows from the corollary to Proposition 2. ■

The following result has been obtained independently by Jasiński [7]. It follows easily from the theory developed here, and we include it for the sake of completeness.

PROPOSITION 12. *Let X be an uncountable Blackwell (resp. strongly Blackwell) subset of S. If Y is an $\mathscr{I}(X)$-Lusin set, then $X \cup Y$ is Blackwell (resp. strongly Blackwell).*

Proof. Clearly, $X \cup Y$ is $\mathscr{I}(X)$-Lusin and $\mathscr{I}(X)$-dense in S. If X is Blackwell (resp. strongly Blackwell), then from Proposition 2, X and $X \cup Y$ satisfy condition (J), (resp. condition (J+)). Another application of Proposition 2 does the job. ■

Say that a subset X of an uncountable standard space S is a *uniformisable set* if $\mathscr{I}(X)$ is uniformisable. We have shown that every analytic set is uniformisable. Thus, the notion of uniformisable space generalizes and properly contains that of analytic space. To see this, note that if $S \setminus X$ is totally imperfect (X is Borel-dense), then $\mathscr{I}(X)$ is the σ-ideal of countable subsets of S. So from Lemma 8, X is uniformisable.

PROPOSITION 13. *Let X be a subset of S which is Blackwell, but not strongly Blackwell. Then X not a uniformisable set.*

Proof. As mentioned, X is certainly an $\mathscr{I}(X)$-Lusin set $\mathscr{I}(X)$-dense of order 1. The result is immediate from Propositions 1 and 2. ■

Blackwell spaces without the strong Blackwell property have been constructed under the axioms CH and MA+ not -CH. We shall give another construction (CH) is section 6. The situation in ZFC is still not known. See [4] and the discussion in [1].

§ 6. Sierpiński sets. Throughout this section we let m denote a continuous probability on the Borel structure $\mathscr{B}(S)$. Also, $\mathscr{I} = \mathscr{I}(m)$ will be the σ-ideal of all m-null sets in $\mathscr{B}(S)$. Then $\mathscr{I}$ is continuous, and the $\mathscr{I}$-Lusin subsets of S are the classical Sierpiński sets [2]. A subset X of S is $\mathscr{I}$-dense if and only if $m^*(X) = 1$.

We shall prove that $\mathscr{I}$ satisfies restriction (R) but is not uniformisable. Additionally, $\mathscr{I}$ will be used to construct a continuous σ-ideal violating restriction (R). To facilitate discussion, it becomes convenient to introduce the following set function, defined on subsets R of $S\times S$:

$$\mu R = \inf\{mA+mB:\ R\subset(A\times S)\cup(S\times B);\ A, B\in\mathscr{B}(S)\}\,.$$

Then μ is a Carathéodory outer measure with very few μ-measurable sets.

LEMMA 11. *A set $R\subset S\times S$ is $\mathscr{I}$-reticulate if and only if $\mu R = 0$.*

Proof. One direction (only if) is obvious. To prove the converse, suppose that $\mu R = 0$. Choose sets A_n and B_n in $\mathscr{B}(S)$ with $R\subset(A_n\times S)\cup(S\times B_n)$ and $m(A_n)+m(B_n)<2^{-n}$. Put $A = \limsup A_n$ and $B = \limsup B_n$. Then $R\subset(A\times S)\cup(S\times B)$, and from the Borel–Cantelli Lemma, $mA = mB = 0$. So R is reticulate. ■

PROPOSITION 14. *Suppose that $(S\times S)\setminus R$ is a countable union of Borel rectangles (e.g. if S is metric and R is closed). Then $\mu R\geqslant\varepsilon$ if and only if there is a measure ν on $\mathscr{B}(S\times S)$ both of whose marginals equal m such that $\mu R\geqslant\varepsilon$.*

Proof. This is essentially a result of Strassen [21], discussed and extended for the measurable setting in [16]. ■

LEMMA 12. *The σ-ideal $\mathscr{I}$ satisfies restriction* (R).

Proof. Suppose that B_1 and B_2 are sets in $\mathscr{B}(S)$ with $m(B_1)m(B_2)$ positive. Then there are sets $C_1\subset B_1$ and $C_2\subset B_2$ in $\mathscr{B}(S)$ with $m(C_1) = m(C_2) > 0$. Also there is a Borel-isomorphism f of C_1 onto C_2 preserving m-measure: for all Borel $B\subset C_1$, one has $mf(B) = m(B)$. The graph of f is a thread inside $B_1\times B_2$. ■

LEMMA 13. *The σ-ideal $\mathscr{I}(m)$ is not uniformisable.*

Proof. Let f be a Borel-isomorphism of S onto $S\times S$ mapping m onto the product measure $m\otimes m$. Let $p\colon S\times S\to S$ be projection onto the first co-ordinate and define $g\colon S\to S$ to be the composition $g = p\circ f$. Then $G = \mathrm{graph}(g)$ is non-reticulate in $S\times S$ but contains no threads.

To see this, suppose that $G\subset(N_1\times S)\cup(S\times N_2)$ with $m(N_1) = m(N_2) = 0$. Then $S\subset N_1\cup g^{-1}(N_2)$, but $mg^{-1}(N_2) = mf^{-1}p^{-1}(N_2) = (m\otimes m)p^{-1}(N_2) = m(N_2) = 0$, a contradiction. So G is not reticulate.

On the other hand, suppose that T is a thread in G. Then T is the graph of the restriction of g to some C in $\mathscr{B}(S)$. On C, the function g is one-one. So $f(C)$ is a subset of $S\times S$ each of whose vertical sections is at most singleton. So $m(C) = (m\otimes m)f(C) = 0$, a contradiction. ■

Let $\mathscr{J}$ be an arbitrary σ-ideal in $\mathscr{B}(S)$. Say that a subset R of $S\times S$ is $(\mathscr{I},\mathscr{J})$-*reticulate* if R can be covered by a set of the form $(N_1\times S)\cup(S\times N_2)$ with $m(N_1) = 0$ and $N_2\in\mathscr{J}$.

LEMMA 14. *A subset R of $S\times S$ is not $(\mathscr{I},\mathscr{J})$-reticulate if and only if there is some $\varepsilon>0$ such that for all B in $\mathscr{B}(S)$ with $m(B)\geqslant 1-\varepsilon$, the set $R\cap(B\times S)$ is not $(\mathscr{I},\mathscr{J})$-reticulate.*

Proof. One direction (if) is obvious. To prove the converse, we suppose that for each n, there is some B_n in $\mathscr{B}(S)$ with $m(B_n)\geqslant 1-2^{-n}$ such that $R\cap(B_n\times S)$ is $(\mathscr{I},\mathscr{J})$-reticulate. Thus there are sets N_n and M_n with $R\cap(B_n\times S)\subset(N_n\times S)\cup(S\times M_n)$ and $m(N_n) = 0$ and $M_n\in\mathscr{J}$. Put $A_n = S\setminus B_n$ and note that

$$R\subset\big((A_n\cup N_n)\times S\big)\cup(S\times M_n)\,.$$

Define $A = \limsup(A_n\cup N_n)$ and $M = \bigcup M_n$. Then $R\subset(A\times S)\cup(S\times M)$ and one has $M\in\mathscr{J}$ and, by the Borel–Cantelli lemma, $m(A) = 0$. Thus R is $(\mathscr{I},\mathscr{J})$-reticulate. ■

EXAMPLE. Not every continuous σ-ideal satisfies restriction (R).

Construction. Realize S as the union of the intervals $I_1 = [0,1]$ and $I_2 = [2,3]$ as a compact metric space under the usual structure. Define a σ-ideal $\mathscr{N}$ in $\mathscr{B}(S)$ to be the collection of all Borel sets of the form $N_1\cup N_2$, where N_1 is a subset of I_1 of Lebesgue measure $m(N_1) = 0$ and N_2 is a subset of I_2 of first Baire category.

We shall prove that $I_1\times I_2$ contains no $\mathscr{N}$-threads. Suppose for the sake of contradiction that T were such a thread. Then T is the graph of some Borel-isomorphism f. Let ε be as specified in Lemma 14 for the set T any apply Lusin's Theorem to produce a compact $K\subset I_1$ with $m(K)\geqslant 1-\varepsilon$ and such that f is continuous on K (in fact, a homeomorphism). Now from Lemma 14, $T\cap(K\times I_2)$ is a thread; it is the graph of the restriction of f to K, which function we call g.

The set K decomposes into a union $K = N\cup F$, where $m(N) = 0$ and F is of first category in K. Since g is a homeomorphism, $g(F)$ is of first category in $g(K)$ and hence also in I_2. Thus $T\cap(K\times I_2) = \mathrm{graph}(g)\subset(N\times S)\cup(S\times g(F))$ is $\mathscr{N}$-reticulate. This gives the desired contradiction. ■

Jasiński [7] has, under the assumption of CH, constructed a Sierpiński set X_1 such that if $X_2\supset X_1$ is another Sierpiński set, then automatically X_2 is strong Blackwell. In his example, $m^*(X_1) = 1$. We shall see that such behavior is actually typical.

PROPOSITION 15 (CH). *There are Sierpiński subsets of S with the strong Blackwell property.*

Proof. Apply Proposition 6. ■

PROPOSITION 16 (CH). *There are Sierpiński subsets of S without the Blackwell property.*

Proof. Apply Proposition 7 and Lemma 12. ■

Remark. Jasiński shows [7] without CH that if there is a Sierpiński set, then there is some Sierpiński set without the Blackwell property.

PROPOSITION 17. *Suppose that X_1 and X_2 are $\mathscr{I}(m)$-Lusin sets ("m-Sierpiński sets") with $X_1 \subset X_2$ and $m^*(X_1) = m^*(X_2)$.*

1) *If X_1 is Blackwell, then X_2 is Blackwell;*

2) *If X_1 is strongly Blackwell, then X_2 is strongly Blackwell.*

Proof. Choose $B \supset X_2$ with $B \in \mathscr{B}(S)$ and $m(B) = m^*(X_2)$. We know that $m^*(X_2) > 0$, so that $n = m/m(B)$ is a continuous probability on the standard space $(B, \mathscr{B}(B))$. Since X_1 is $\mathscr{I}(n)$-dense in B, Proposition 2 will apply to prove the results. ∎

However, for the σ-ideal $\mathscr{I} = \mathscr{I}(m)$, the conditions in Proposition 1 are not equivalent. The nonuniformisability of $\mathscr{I}$ exerts great influence in the following.

PROPOSITION 18 (CH). *Let m be a continuous probability on a standard space S. Then S contains an $\mathscr{I}(m)$-Lusin set X which is Blackwell, but not strongly Blackwell.*

Proof. Any two standard, continuous probability spaces (S_1, m_1), (S_2, m_2) are isomorphic, so it suffices to work with a particular instance of such a space. Put $S = [0, 1[\times]0, 1[$ under the usual Borel structure. Put $A_1 = \{0\} \times]0, 1[$ and $A_2 = S \setminus A_1 =]0, 1[\times]0, 1[$. Define a Borel probability m on S as follows:

1) on subsets of A_1, m agrees with one half the usual linear Lebesgue measure;

2) on subsets of A_2, m agrees with one half the usual planar area measure.

We define functions f and g on S as follows:

a) $f(x, y) = y$;

b) $g(x, y) = y + \operatorname{sgn}(x)$.

Let $\mathscr{C}$ and $\mathscr{D}$ be the σ-algebras on S generated by f and g, respectively. Clearly $\mathscr{C} \subset \mathscr{D}$.

We claim that $\mathscr{C}$ is $\mathscr{I}(m)$-proper in $\mathscr{D}$. To see this, suppose that N is a null set for m. Then $f(A_1 \cap N)$ is a linear null set in $]0, 1[$, and from Fubini's Theorem, almost all horizontal sections of $A_2 \cap N$ have linear measure zero. It follows that there is some y in $]0, 1[$ such, that

1) $(0, y) \in N$;

2) $(x, y) \in N$ for some $x > 0$.

The points $(0, y)$ and (x, y) are not separated by $\mathscr{C}$, but are separated by $\mathscr{D}$. Thus $\mathscr{C}(N^c) \neq \mathscr{D}(N^c)$, and $\mathscr{C}$ is proper in $\mathscr{D}$.

It follows that the set $T(\mathscr{C}, \mathscr{D})$ is not $\mathscr{I}(m)$-reticulate. A pair (s, t) belongs to $T(\mathscr{C}, \mathscr{D})$ if and only if s and t are members of the same horizontal y-section of S, and one of the two points is of the form $(0, y)$. Thus for any three points s_1, s_2, s_3 in S, some pair (s_i, s_j) does not belong to $T(\mathscr{C}, \mathscr{D})$.

List in transfinite series $\mathscr{E}_0 \mathscr{E}_1 \dots \mathscr{E}_\alpha \dots \alpha < \aleph_1$ all c.g. sub-σ-algebras of $\mathscr{B}(S)$ proper in $\mathscr{B}(S)$. List also as $N_0 N_1 \dots N_\alpha \dots \alpha < \aleph_1$ all m-null members of $\mathscr{B}(S)$. We shall choose points s_α and t_α in S *via* transfinite induction.

Assume that points s_β, t_β have been chosen for $\beta < \alpha$. Remove a Borel subset of S that is the union of the horizontal sections $[0, 1[\times \{f(s_\beta)\}$ and $[0, 1[\times \{f(t_\beta)\}$ all $\beta < \alpha$) as well as the union $N_0 \cup N_1 \cup \dots \cup N_\alpha$. This Borel set has m-measure zero. Denote what remains of S by R.

Since $\mathscr{E}_\alpha$ is proper, we know that $\mathscr{E}_\alpha(R)$ is not separated. We choose points s_α, t_α from R not separated by $\mathscr{E}_\alpha$, but which are such, that the pair (s_α, t_α) does not belong to $T(\mathscr{C}, \mathscr{D})$. We show presently that such a choice is possible:

1) If $\mathscr{E}_\alpha(R)$ contains an atom with at least three elements, then as mentioned above, two of these elements form a pair (s, t) not in $T(\mathscr{C}, \mathscr{D})$. Put $s_\alpha = s$ and $t_\alpha = t$.

2) In this case, we assume that all $\mathscr{E}_\alpha(R)$-atoms are of cardinality $\leqslant 2$ and suppose that if $\{s, t\}$ is any $\mathscr{E}_\alpha(R)$-atom, then $(s, t) \in T(\mathscr{C}, \mathscr{D})$. Let W be the union of all $\mathscr{E}_\alpha(R)$-atoms of cardinality two (it follows from [6; p. 119] that W is analytic). Then $A = W \cap A_2$ is an analytic subset of $]0, 1[\times]0, 1[$, each of whose horizontal sections is either singleton or empty. Thus there is a Borel set $B \supset A$ such, that $m(B) = 0$. But since $\mathscr{E}_\alpha$ is proper in $\mathscr{B}$, we know that $\mathscr{E}_\alpha(R \setminus B) \neq \mathscr{B}(R \setminus B)$. This is the desired contradiction.

Having defined the points s_α, t_α for each $\alpha < \aleph_1$, put $X = \{s_\alpha, t_\alpha : \alpha < \aleph_1\}$. Then X is a Sierpiński subset of S satisfying condition (J) but for which condition (J+) fails at $(\mathscr{C}, \mathscr{D})$.

Lemmas 4 and 12 imply that $m^*(X) = 1$. It follows from Proposition 2 that X is Blackwell, but not strongly Blackwell. ∎

The space constructed in Proposition 18 yields an answer to an unsolved problem of D. Ramachandran: P930 in [14].

COROLLARY (CH). *There is a probability space $(X, \mathscr{B}(X), P)$ whose Borel structure $\mathscr{B}(X)$ is Blackwell, but for which the notions of Kolmogoroff-and Steinhaus-independence are not equivalent.*

Remark. In [14] it is shown that the two definitions of independence are equivalent whenever $\mathscr{B}(X)$ is strongly Blackwell.

Proof. Let X be the space constructed in the proof of Proposition 18 and put $P = m^*$. Let h and f be projection onto the first and second factor as functions defined on X. Then h and f are Steinhaus-independent random variables, meaning that

$$\mathscr{B}(h) = \{h^{-1}(A) : A \text{ Borel}\},$$

$$\mathscr{B}(f) = \{f^{-1}(A) : A \text{ Borel}\}$$

are P-independent σ-algebras. This follows from the fact that m is a product measure on S and from the equation $m^*(B \cap X) = m(B)$ for B in $\mathscr{B}(S)$.

However, h and f are not Kolmogoroff-independent, meaning that

$$\mathscr{A}(h) = \{h^{-1}(A) : A \text{ linear set}, h^{-1}(A) \in \mathscr{B}(X)\},$$

$$\mathscr{A}(f) = \{f^{-1}(A) : A \text{ linear set}, f^{-1}(A) \in \mathscr{B}(X)\}$$

are not P-independent. To see this, put $A = f(X \cap A_1)$. Then A is a linear set with $f^{-1}(A) = X \cap A_1$ a member of $\mathscr{A}(f)$. Now $f^{-1}(A) = h^{-1}(0)$ and $Pf^{-1}(A)$

$= Ph^{-1}(0) = 1/2$. This event is not independent of itself, so $\mathscr{A}(h)$ and $\mathscr{A}(f)$ are not independent. ■

§ 7. Products of Sierpiński sets. This section takes up the matter of when certain products of Sierpiński sets have the Blackwell property. In order to apply the techniques of Section 2, we begin by showing that for the σ-ideal $\mathscr{I} = \mathscr{I}(m)$, the notions of analytic and simple $\mathscr{I}$-density coincide.

Let m be a Borel probability measure on a compact metric space S. Let $L(S)$ be the set of all Borel functions from S to the interval $[0, 1]$. For each $n \geqslant 1$, we define a set function ι_n on subsets of the product S^n. For any $R \subset S^n$, define

$$\iota_n(R) = \inf\{\int h_1\,dm + \ldots + \int h_n\,dm\colon\ h_1, \ldots, h_n \in L(S) \text{ and } 1_R(s_1, \ldots, s_n) \leqslant h_1(s_1) + \ldots + h_n(s_n), \text{ all } s_i \in S\}.$$

Here, 1_R is the indicator (characteristic) function of the set R. Hans Kellerer has considered this set function in his paper on the marginal problem [8]. (See 1.3 and 1.8 therein for an equivalent definition).

LEMMA 15. *Let m, S, and ι_n be as described above.*

1) *The set function ι_n is a regular (Choquet) capacity on S^n.*

2) *A subset R of S^n is $\mathscr{I}(m)$-reticulate if and only if $\iota_n(R) = 0$.*

Indications. Part 1 follows from 1.28 and 1.30 in [8]. Part 2 is established in [8] as part (b) in the proof of 1.15.

LEMMA 16. *Let m be a Borel probability on a standard space S. For each $n \geqslant 1$, a subset X of S is analytically $\mathscr{I}$-dense of order n if and only if X is $\mathscr{I}$-dense of order n.*

Proof. One direction (only if) is immediate. For the other, we note that on analytic set $A \subset S^n$ is not $\mathscr{I}(m)$-reticulate if and only if there is some Borel $B \subset A$ which is not $\mathscr{I}(m)$-reticulate. This follows from Lemma 15 and the capacity theorem of Choquet. (If S is topologized as a compact metric space, then B may be chosen compact.) ■

PROPOSITION 19. *Let m be a Borel probability on a standard space and let X be an m-Sierpiński subset of S with $m^*(X) = 1$. Let A be an uncountable analytic subset of S. The following are equivalent:*

1) *$X^n \times S$ is strongly Blackwell;*
2) *$X^n \times A$ is strongly Blackwell;*
3) *$X^n \times A$ is Blackwell;*
4) *$X^n \times S$ is Blackwell;*
5) *X is $\mathscr{I}(m)$-dense of order $2n$ in S.*

Indication. This re-expresses the corollary to Proposition 4, taking Lemma 16 into account.

The next set of results show that not all Sierpiński sets are created equal: some are "larger" than others. The larger ones enjoy more Blackwell properties.

PROPOSITION 20. *Let m be a continuous Borel probability on a standard space S.*

For each $n = 1, 2, \ldots$, there is a subset Y of S such that Y is $\mathscr{I}(m)$-dense of order n, but not order $n+1$ in S.

Proof. We take S to be the interval $]0, 1[$ under its usual Borel structure. Let $f_1, f_2, \ldots$ be the Borel automorphisms of S defined by $f_k(s) = s^k$. Let $\mathscr{I}$ be the σ-ideal of all countable subsets of S. The proof of Proposition 13 in [17] establishes the existence of a $Y \subset S$ such that

1) *Y is $\mathscr{I}$-dense of order n in S;*
2) *Y^{n+1} does not meet the subset*

$$G = \{(s_1, \ldots, s_n, s_{n+1})\colon f_1(s_1) = \ldots = f_n(s_n) = f_{n+1}(s_{n+1})\} \quad \text{of} \quad S^{n+1}.$$

Property 1 implies that Y is also $\mathscr{I}(m)$-dense of order n. It remains only to note that G is not $\mathscr{I}(m)$-reticulate in S^{n+1}. ■

COROLLARY 1 (CH). *Let m be a continuous Borel probability on a standard space S. For each $n = 1, 2, \ldots$, there is an m-Sierpiński subset X of S such that X is $\mathscr{I}(m)$-dense fof order n, but not order $n+1$ in S.*

Proof. Let Y be as in Proposition 20 and apply Proposition 5. Note that if Y is not dense of order $n+1$, then neither is any $X \subset Y$ ■

COROLLARY 2 (CH) *Let m be a continuous probability on a standard space S. For each $n = 1, 2, \ldots$, there is an m-Sierpiński subset X of S such that*

1) *$X^n \times S$ is strongly Blackwell;*
2) *$X^{n+1} \times S$ is not Blackwell.*

Proof. This follows from the preceeding corollary, combined with Proposition 19: take S to be $\mathscr{I}(m)$-dense of order $2n$, but not of any higher order. ■

§ 8. Selection theorems for m-threads and m-graphs. This section is devoted to establishing converses to parts B and C of Proposition 1 for the σ-ideal $\mathscr{I}(m)$. This will yield a complete characterization of Sierpiński sets with Blackwell properties and will facilitate a discussion of the combinatorial behavior of these sets.

Let m be a Borel probability on a compact metric space (S, d). Let $\mathscr{K}(S)$ denote space of all compact subsets of S under the exponential topology of Hausdorff and Vietoris [9; p. 160]. The space $\mathscr{K}(S)$ is compact and is metrized by the Hausdorff metric [9; p. 214]. Let $C(S)$ denote the space of all continuous functions from S to itself as a complete separable metric space under the uniform (supremum) norm. Finally, let $\mathscr{P}(S)$ be the space of all Borel probabilities on S. This space, under the weak topology induced by continuous real functions on S, is compact metric [12].

LEMMA 17. *The real function $(K, \nu) \to \nu(K)$ is measurable (in fact, upper semi-continuous) on $\mathscr{K}(S) \times \mathscr{P}(S)$.*

Proof. Given $\varepsilon \geqslant 0$ we prove that $\{(K, \nu)\colon \nu(K) \geqslant \varepsilon\}$ is a closed set. Suppose that $(K_n, \nu_n) \to (K, \nu)$ as $n \to \infty$ with $\nu_n(K_n) \geqslant \varepsilon$ for each n. Fix $\delta > 0$ and put

$$K^\delta = \{x\colon d(x, u) \leqslant \delta \text{ for some } u \text{ in } K\}$$

as the closed δ-neighbourhood of K. Choose n large so that $K_n \subset K^\delta$ and so that $\nu(K^\delta) \geq \nu_n(K^\delta) - \delta \geq \nu_n(K_n) - \delta \geq \varepsilon - \delta$. Letting $\delta \to 0$ gives $\nu(K) \geq \varepsilon$. ■

Let f be a continuous real function on S. Let $m(p, \cdot)$ be a Markov transition kernel (disintegration) indexed by p in $f(S)$ such that:

1) For each p in $f(S)$, the set function $B \to m(p, B)$ is a Borel probability on S;

2) For each B in $\mathscr{B}(S)$, the real function $p \to m(p, B)$ is Borel measurable on $f(S)$;

3) For each B in $\mathscr{B}(S)$, one may write

$$m(B) = \int_{f(S)} m(p, B)\, df(m)(p) ;$$

4) For each p in $f(S)$, one has $m(p, f^{-1}(p)) = 1$.

For the existence of such kernels, see [13].

LEMMA 18. *The mapping $p \to m(p, \cdot)$ is measurable from $f(S)$ into $\mathscr{P}(S)$.*

Proof. This follows immediately from the fact that the Borel structure on $\mathscr{P}(S)$ is generated by the evaluation functionals $\nu \to \nu(B)$ for B ranging over $\mathscr{B}(S)$. ■

LEMMA 19. *The mapping $p \to f^{-1}(p)$ is Borel measurable from $f(S)$ to $\mathscr{K}(S)$.*

Proof. See [9; p. 165]. ■

LEMMA 20. *The set $\{(K, L): K \subset L\}$ is closed in $\mathscr{K}(S) \times \mathscr{K}(S)$.*

Proof. See [9; p. 167]. ■

LEMMA 21. *The mapping $(K, L) \to K \times L$ is continuous from $\mathscr{K}(S) \times \mathscr{K}(S)$ to $\mathscr{K}(S \times S)$.*

Proof. This follows from Remark 1 on p. 49 of [10] combined with the relation $\lim(K_n \times L_n) = (\lim K_n) \times (\lim L_n)$ in [9; p. 339]. ■

LEMMA 22. *Let R be an F_σ (σ-compact) subset of $S \times S$. Then the set*

$$\mathscr{A} = \{p: R \text{ is not } m(p, \cdot)\text{-reticulate}\}$$

is a Borel subset of $f(S)$.

Proof. Write R as the union of compact sets $R_1 \cup R_2 \cup \ldots = R$. Then we see that

$$\mathscr{A} = \bigcup_{n=1}^{\infty} \{p: R_n \text{ is not } m(p, \cdot)\text{-reticulate}\},$$

so it suffices to work with a particular compact R_n. From Lemma 11 it follows that R_n is $m(p, \cdot)$-reticulate if and only if R_n^c contains rectangles with arbitrarily large sides for $m(p, \cdot)$. For each $k \geq 1$, define the following subset of $f(S) \times \mathscr{K}(S)$:

$$\mathscr{L}_k = \{(p, L): L \times L \subset R_n^c \text{ and } m(p, L) \geq 1 - 1/k\}.$$

Lemma 17, 18 and 21 imply that each $\mathscr{L}_k$ is Borel. Let $P(\mathscr{L}_k)$ be the (analytic) projection of $\mathscr{L}_k$ to $f(S)$. Then we may write

$$\{p: R_n \text{ is } m(p, \cdot)\text{-reticulate}\}$$

as

$$\bigcap_{k=1}^{\infty} P(\mathscr{L}_k).$$

We have shown that the set $\mathscr{A}$ is co-analytic.

Proposition 14 and Lemma 11 say that R_n is not $m(p, \cdot)$-reticulate if and only if there is some probability ν on $S \times S$ with marginals $\nu_1 = \nu_2 = m(p, \cdot)$ such that $\nu(R_n) > 0$. Thus we define the following subset of $f(S) \times \mathscr{P}(S \times S)$:

$$\mathscr{L} = \{(p, \nu): \nu_1 = \nu_2 = m(p, \cdot) \text{ and } \nu(R_n) > 0\}.$$

Lemma 17 and 18 combined with the fact that the operations $\nu \to \nu_1$ and $\nu \to \nu_2$ are continuous show that $\mathscr{L}$ is a Borel set. Now $\{p: R_n \text{ is not } m(p, \cdot)\text{-reticulate}\}$ is the projection of $\mathscr{L}$ to $f(S)$. Thus, $\mathscr{A}$ is an analytic set.

The set $\mathscr{A}$ is both co-analytic and analytic, and therefore Borel. ■

Define a subset T_0 of $S \times S$ by

$$T_0 = \{(s, t): f(s) = f(t)\}.$$

Then T_0 is compact.

PROPOSITION 21. *For any σ-compact (F_σ) subset R of T_0, the following conditions are equivalent:*

1) *R is m-reticulate;*

2) *$f(m)\{p: R \text{ is not } m(p, \cdot)\text{-reticulate}\} = 0$.*

Proof. Suppose that R is m-reticulate, so that $R \subset (N \times S) \cup (S \times N)$ with $m(N) = 0$. Then

$$\int_{f(S)} m(p, N)\, df(m)(p) = m(N) = 0,$$

so that for p outside some Borel subset M of $f(S)$ with $f(m)(M) = 0$, one has $m(p, N) = 0$. For such p, clearly R is $m(p, \cdot)$-reticulate. Condition 2 follows.

Now assume that Condition 2 holds. Put $Q = \{p: R \text{ is } m(p, \cdot)\text{-reticulate}\}$. We know that $f(m)(Q) = 1$. Fix $\varepsilon > 0$. For each p in Q, there is a compact $K \subset f^{-1}(p)$ with $K \times K \subset R^\varepsilon$ and $m(p, K) \geq 1 - \varepsilon$

Consider now the subset of $Q \times \mathscr{K}(S)$ defined by

$$\mathscr{V} = \{(p, K): K \subset f^{-1}(p), K \times K \subset R^\varepsilon, m(p, K) \geq 1 - \varepsilon\}.$$

Then from Lemmas 17–21 it follows that $\mathscr{V}$ is a Borel subset of $Q \times \mathscr{K}(S)$, and so has a continuous selector on most of Q: by the Jankov–von Neumann Theorem, there is a continuous mapping $p \to K(p)$ defined on a compact subset C of Q such that

1) $m(C) \geq 1 - \varepsilon$;

2) for each p in C, the pair $(p, K(p))$ belongs to $\mathscr{V}$.

The range of the mapping $p \to K(p)$ is compact, and so therefore is the union of all sets in its range, i.e.

$$L = \bigcup_{p \in C} K(p).$$

Also,

$$m(L) = \int_{f(S)} m(p, L)\, df(m)(p) \geqslant \int_C m(p, L)\, df(m)(p)\,,$$

and since $K(p) \subset L$, one has $m(p, L) \geqslant m(p, K(p)) \geqslant 1-\varepsilon$. Therefore

$$m(L) \geqslant (1-\varepsilon)(1-\varepsilon)\,.$$

Now we claim that $L \times L \subset R^c$. Suppose there is some point (x, y) in $(L \times L) \cap R$. Then, $(x, y) \in T_0$, so that $f(x) = f(y)$. Now there are points p, q in C with $x \in K(p) \subset f^{-1}(p)$ and $y \in K(q) \subset f^{-1}(q)$. This means that $p = q$. So $(x, y) \in \in (K(p) \times K(p)) \cap R$, a contradiction from the definition of $\mathscr{V}$.

We have shown that for each $\varepsilon > 0$, there is a Borel set L with $m(L) \geqslant (1-\varepsilon)^2$ and such, that $L \times L \subset R^c$. It follows that R is m-reticulate (Lemma 11). ■

We shall require the following technical

LEMMA 23. *Let $f\colon K_1 \to K_2$ be a continuous mapping between compact metric spaces (K_1, d_1) and (K_2, d_2). Then f is one-one if and only if there is some countable dense subset R of K_1 and a sequence of positive numbers $\varepsilon_1\ \varepsilon_2\ \varepsilon_3 \ldots$ such that the following condition holds:*

(*) *whenever r and s are members of R with $d_1(r, s) \geqslant 1/n$ $(n = 1, 2, \ldots)$, then $d_2(f(r), f(s)) > \varepsilon_n$.*

Remark. The proof shows that if f is one-one, then (*) holds for $R = K_1$.

Proof. Suppose that f is indeed one-one on K_1. Then the inverse function $g = f^{-1}$ is uniformly continuous from $f(K_1)$ onto K_1. Thus for each $n \geqslant 1$ there is some positive ε_n such that $d_2(u, v) \leqslant \varepsilon_n$ implies $d_1(g(u), g(v)) \leqslant 1/n$ for all u, v in $f(K_1)$. Rephrased, this means that whenever $d_1(x, y) \geqslant 1/n$, then also $d_2(f(x), f(y)) > \varepsilon_n$ for all x, y in K_1.

Suppose that f, R, and ε_n satisfy (*) and suppose for the sake of contradiction that there are points $x \neq y$ in K_1 with $f(x) = f(y)$. There is some $n \geqslant 1$ such that $d_1(x, y) > 1/n$. Also, there is some $\delta > 0$ such that $d_1(x, u) < \delta$ implies $d_2(f(x), f(u)) < \varepsilon_n/2$ for all u in K_1 and such that $d_1(y, v) < \delta$ implies $d_2(f(y), f(v)) < \varepsilon_n/2$ for all v in K_1. Choose points r and s in R such that

1) $d_1(x, r) < \delta$,
2) $d_1(y, s) < \delta$,
3) $d_1(r, s) > 1/n$.

Then

$$d_2(f(r), f(s)) \leqslant d_2(f(r), f(x)) + d_2(f(y), f(s)) < \varepsilon_n\,,$$

a contradiction. ■

We now are in a position to prove our first major selection theorem. The insistence on 0-dimensionality in the following result is for purely technical reasons (it enables one to extend continuous functions), and will not appear in the final product, which deals with the measurable case.

PROPOSITION 22. *Let m be a Borel probability on a 0-dimensional compact metric space S. Suppose that f is a continuous real function on S generating the sub-σ-algebra $\mathscr{C}$ of $\mathscr{B}(S)$. Suppose that $\mathscr{C}$ is $\mathscr{I}(m)$-proper in $\mathscr{B}(S)$. Then there is a homeomorphism g between compact subsets of S whose graph K satisfies:*

1) $K \subset \{(s, t)\colon f(s) = f(t)$ *and* $s \neq t\} = T(\mathscr{C}, \mathscr{B})$.
2) *K is a thread (not m-reticulate).*

Proof. We shall define four analytic subsets $\mathscr{A}_1$, $\mathscr{A}_2$, $\mathscr{A}_3$, $\mathscr{A}_4$ of $f(S) \times \mathscr{K}(S \times S)$. Then a selection theorem will be applied to their intersection much as in the proof of Proposition 21.

Define the set $\mathscr{A}_1$ by the rule

$$\mathscr{A}_1 = \{(p, K)\colon K \subset f^{-1}(p) \times f^{-1}(p)\}\,.$$

Lemmas 19, 20, and 21 show that $\mathscr{A}_1$ is closed. Likewise define

$$\mathscr{A}_2 = \{(p, K)\colon K \cap \Delta = \emptyset\}\,,$$

where Δ is the diagonal in $S \times S$. The set $\mathscr{A}_2$ is open.

Define $\mathscr{A}_3$ to be the collection of all (p, K) such that K is not $m(p, \cdot)$-reticulate. To see that $\mathscr{A}_3$ is analytic, we define a subset $\mathscr{E}_1$ of $f(S) \times \mathscr{K}(S \times S) \times \mathscr{P}(S \times S)$ by

$$\mathscr{E}_1 = \{(p, K, \nu)\colon \nu(K) > 0 \text{ and } \nu_1 = \nu_2 = m(p, \cdot)\}\,,$$

where ν_1 and ν_2 are the univariate marginals of ν. Then $\mathscr{E}_1$ is Borel and from Proposition 11 we see that $\mathscr{A}_3$ is the projection of $\mathscr{E}_1$ to $f(S) \times \mathscr{K}(S \times S)$.

The definition of $\mathscr{A}_4$ takes a few steps. When they are complete, $\mathscr{A}_4$ will be defined as the collection of all (p, K) such that K is the graph of a homeomorphism between compact subsets of S.

Consider the subset $\mathscr{E}_2$ of $\mathscr{K}(S) \times S^\infty$ consisting of all $(L, s_1, s_2, s_3, \ldots)$ such that $s_1 s_2 s_3 \ldots$ is a sequence of points drawn from L which is dense in L. Then E_2 is a G_δ set, since it may be written as the intersection of

$$\bigcap_{n=1}^{\infty} \{(L, s_1, s_2, s_3, \ldots)\colon s_n \in L\}\,,$$

which is closed, with

$$\bigcap_{n=1}^{\infty} [\{(L, s_1, s_2, \ldots)\colon U_n \cap L = \emptyset\} \cup \bigcup_{k=1}^{\infty} \{(L, s_1, s_2, \ldots)\colon s_k \in U_n\}]\,,$$

where $U_1 U_2 \ldots$ is a fixed countable open base for the topology of S.

Define a subset E_3 of $\mathscr{C}(S) \times \mathscr{K}(S) \times S^\infty \times (\boldsymbol{R}^+)^\infty$ to be the intersection of the sets

$$\{(f, L, s_1, s_2, \ldots; \varepsilon_1, \varepsilon_2, \ldots)\colon d(s_i, s_j) \leqslant 1/n \text{ or } d(f(s_i), f(s_j)) > \varepsilon_n\}$$

as n, i, j range over all positive integers. Then $\mathscr{E}_3$ is a G_δ set, as is $\mathscr{E}_4$, the intersection of $\mathscr{E}_3$ with $C(S) \times \mathscr{E}_2 \times (\boldsymbol{R}^+)^\infty$.

Define $\mathscr{E}_5$ to be the (analytic) projection of $\mathscr{E}_4$ to $C(S)\times\mathscr{K}(S)$. Then from Lemma 23, $\mathscr{E}_4$ consists of all pairs (f, L), where f is a continuous function from S to S which is one-one on the compact set L.

Consider the mapping from $C(S)\times\mathscr{K}(S)$ to $\mathscr{K}(S\times S)$ sending (f, L) to Graph $(f)\cap(L\times S)$. Following [9; p. 180] and [10; p. 70], as well as [3; p. 72], we see that the intersection operation $(K, L)\to K\cap L$ is measurable. From [9; p. 223], the same may be said of the operation $f\to\text{Graph}(f)$. Thus the map in question is measurable. Define $\mathscr{E}_6$ to be the image of $\mathscr{E}_5$ under this map; clearly, $\mathscr{E}_6$ is an analytic set.

Finally, we may define $\mathscr{E}_4$ to be the product $f(S)\times\mathscr{E}_6$. As promised, $\mathscr{E}_4$ consists of all pairs (p, K), where K is the graph of some one-one continuous function (homeomorphism) defined on a compact subset S.

We have tacitly assumed that every such function may be extended continuously to all of S. Since S is 0-dimensional, this is guaranteed [9; p. 281]. Define $\mathscr{A}$ to be the intersection $\mathscr{A}=\mathscr{A}_1\cap\mathscr{A}_2\cap\mathscr{A}_3\cap\mathscr{A}_4$.

Consider now the set T defined by

$$T = T(\mathscr{C}, \mathscr{B}) = \{(s, t): f(s) = f(t) \text{ and } s \neq t\}.$$

For each p in $f(S)$, define the "cell" $T(p)$ as

$$T(p) = \{(s, t): f(s) = f(t) = p \text{ and } s \neq t\}.$$

Since $\mathscr{C}=\mathscr{B}(f)$ is proper in $\mathscr{B}(S)$, the set T is not m-reticulate. Now T is a σ-compact (open) subset of

$$T_0 = \{(s, t): f(s) = f(t)\},$$

so that Proposition 21 applies with $R=T$. Put

$$Q = \{p: T \text{ is not } m(p, \cdot)\text{-reticulate}\}.$$

Then

1) $m(Q)>0$;

2) for each p in Q, the set $T(p)$ is not $m(p, \cdot)$-reticulate.

We shall now demonstrate that for each p in Q, there is some K in $\mathscr{K}(S\times S)$ such that $(p, K)\in\mathscr{A}$. So fix $p\in Q$.

Case 1. The measure $m(p, \cdot)$ is purely discrete. We see that $m(p, \cdot)$ cannot be a Dirac mass concentrated at a single point x in $f^{-1}(p)$, since $T(p)$ is not $m(p, \cdot)$-reticulate. So there must be two points x_1 and x_2 in $f^{-1}(p)$ with $m(p, x_1)$ and $m(p, x_2)$ positive. Then put $K=\{(x_1, x_2)\}$.

Clearly, (p, K) belongs to $\mathscr{A}$, since

1) $K\subset f^{-1}(p)\times f^{-1}(p)$;

2) $K\cap\Delta=\emptyset$;

3) K is not $m(p, \cdot)$-reticulate;

4) K is the graph of a one-one continuous function between compact (singleton) subsets of S.

Case 2. The measure $m(p, \cdot)$ is not discrete, so there exists some Borel set $A\subset f^{-1}(p)$ with $m(p, A)>0$ and on which $m(p, \cdot)$ is continuous. Let n denote the measure $m(p, \cdot)$ restricted to subsets of A. By the usual isomorphism tricks, there is a Borel-isomorphism k of A onto a real interval J taking n to Lebesgue measure $k(n)$ on J. Then $(s, t)\to(k(s), k(t))$ takes $A\times A$ isomorphically onto $J\times J$ and sends $(A\times A)\cap\Delta$ to the diagonal Δ_0 of $J\times J$.

Now $(J\times J)\setminus\Delta_0$ contains the graph of a Borel automorphism h of J onto itself which preserves the linear measure $k(n)$. Then the composition $g=k^{-1}\circ h\circ k$ is an automorphism of A onto itself preserving the measure n such that graph$(g)\subset(A\times A)\setminus\Delta$. By Lusin's theorem, there is a compact $L\subset A$ with $n(L)>0$ on which g is continuous (a homeomorphism).

Define K to be the graph of the restriction of g to the set L. Then (p, K) belongs to $\mathscr{A}$, since

1) $K\subset f^{-1}(p)\times f^{-1}(p)$;

2) $K\cap\Delta=\emptyset$;

3) K is not $m(p, \cdot)$-reticulate;

4) K is the graph of a one-one continuous mapping between compact subsets of S.

Remark. The above reasoning shows that $Q=\{p: m(p, \cdot) \text{ is not a point measure}\}$.

We have established that for each p in Q, there is some K in $\mathscr{K}(S\times S)$ with $(p, K)\in\mathscr{A}$. An application of the Jankov–von Neumann selection theorem, combined with Lusin's theorem produces a continuous selector for $\mathscr{A}$ over most of Q: there is a continuous mapping $p\to K(p)$ defined on a compact subset C of Q such that

1) $m(C)>0$;

2) for each p in C, the pair $(p, K(p))$ belongs to $\mathscr{A}$.

The range of the mapping $p\to K(p)$ is compact, and so therefore is the union of all sets in that range, i.e.

$$K = \bigcup_{p\in C} K(p).$$

Now for $p\in C$, the set $K(p)$ is not $m(p, \cdot)$-reticulate. By Proposition 21, it follows that K is not m-reticulate. Clearly, K is the graph of a one-one function. Since K is compact, this function is a homeomorphism g. Lastly, we note that $K\subset T$. ■

The measurable case easily reduces to the continuous one described in Proposition 22.

PROPOSITION 23. *Let m be a Borel probability on a standard space S. Suppose that f is a measurable real function on S generating the sub-σ-algebra $\mathscr{C}$ of $\mathscr{B}(S)$. Suppose that $\mathscr{C}$ is proper in $\mathscr{B}(S)$. Then there is a thread K contained in $T(\mathscr{C}, \mathscr{B})=\{(s, t): f(s)=f(t)$ and $s\neq t\}$.*

Proof. We assume that S has been metrized as a 0-dimensional compact metric space. The condition that $\mathscr{C}$ be proper in $\mathscr{B}(S)$ is equivalent to the statement that $T = T(\mathscr{C}, \mathscr{B})$ is not reticulate. From Lemma 11, there is some $\varepsilon > 0$ such that for any B in $\mathscr{B}(S)$ with $m(B) \geqslant 1-\varepsilon$, the set $(B\times B)\cap T$ is not reticulate.

Keeping ε as specified and applying Lusin's theorem to f, we obtain a compact $S_0 \subset S$ such that

1) $m(S_0) \geqslant 1-\varepsilon$;

2) f is continuous on S_0.

Then define a probability m_0 on S_0 by the rule

$$m_0(B) = m(B|S_0) = \frac{m(B\cap S_0)}{m(S_0)}\,.$$

Let f_0 be the restriction of f to S_0. Then Proposition 22 may be applied to m_0, S_0, and f_0, noting that $\mathscr{B}(S_0)$ is m_0-proper in $\mathscr{B}(S_0)$ and that if a set $K \subset S_0 \times S_0$ is not m_0-reticulate, then R is certainly not m-reticulate. ■

The techniques we have developed enable us to prove the following selection theorem. The argument is similar to the one for Proposition 22, with a couple of subtle differences. These differences turn out to mark the distinction between the Blackwell and strong Blackwell properties for Sierpiński sets.

Proposition 24. *Let m be a Borel probability on a 0-dimensional compact metric space S. Suppose that f and g are continuous real functions on S generating sub-σ-algebras $\mathscr{C} \subset \mathscr{D}$ in $\mathscr{B}(S)$ with $\mathscr{C}$ m-proper in $\mathscr{D}$.*

Then there is a continuous mapping k between compact subsets of S whose graph K satisfies:

1) $K \subset \{(s,t)\colon f(s) = f(t) \text{ and } g(s) \neq g(t)\} = T(\mathscr{C}, \mathscr{D})$;

2) *K is not m-reticulate (K is an $\mathscr{I}(m)$-graph).*

Proof. Let $m(p,\cdot)$ be a transition kernel indexed by p in $f(S)$ just as above. Note that $\mathscr{C}$ proper in $\mathscr{D}$ means that $T(\mathscr{C}, \mathscr{D})$ is not m-reticulate.

Following the proof of Proposition 22, we define four analytic subsets $\mathscr{A}_1$, $\mathscr{A}_2$, $\mathscr{A}_3$, $\mathscr{A}_4$ of $f(S)\times\mathscr{K}(S\times S)$. The set $\mathscr{A}_1$ is defined as

$$\mathscr{A}_1 = \{(p,K)\colon K \subset f^{-1}(p)\times f^{-1}(p)\}\,.$$

Let D be the closed set

$$D = \{(s,t)\colon g(s) = g(t)\}$$

and define

$$\mathscr{A}_2 = \{(p,K)\colon K\cap D = \emptyset\}\,.$$

Then $\mathscr{A}_1$ is closed, and $\mathscr{A}_2$ is open.

Let $\mathscr{A}_3$ be the collection of all (p,K) such that K is not $m(p,\cdot)$-reticulate. Lastly, $\mathscr{A}_4$ is defined as the collection of all (p,K) such that K is the graph of a continuous mapping between compact subsets of S. As in the proof of Proposition 22,

the sets $\mathscr{A}_3$ and $\mathscr{A}_4$ are analytic. The set $\mathscr{A}$ is defined to be the intersection $\mathscr{A} = \mathscr{A}_1\cap\mathscr{A}_2\cap\mathscr{A}_3\cap\mathscr{A}_4$.

For each p in $f(S)$, define the "cell" $T(p)$ as

$$T(p) = \{(s,t)\colon f(s) = f(t) = p \text{ and } g(s) \neq g(t)\}\,.$$

Applying Lemma 22 to $T(\mathscr{C}, \mathscr{D})$, we produce the Borel set

$$Q = \{p\colon T(\mathscr{C}, \mathscr{D}) \text{ is not } m(p,\cdot)\text{-reticulate}\}$$

with the properties

1) $m(Q) > 0$;

2) for each $p \in Q$, the set $T(p)$ is not $m(p,\cdot)$-reticulate.

We now demonstrate that for each p in Q, there is some K in $\mathscr{K}(S\times S)$ with $(p,K) \in \mathscr{A}$. So fix $p \in Q$.

Fact. The measure $m(p,\cdot)$ cannot be entirely concentrated on a single $\mathscr{D}$-atom $g^{-1}(q)$. Were it so, then $g^{-1}(q)\times g^{-1}(q) \subset (S\times S)\setminus T(p)$ would contradict property 2 *supra*.

Case 1. There is some point y in $f^{-1}(p)$ with $m(p,y) > 0$. Define $q = g(y)$ and let L be a compact subset of $f^{-1}(p)\setminus g^{-1}(q)$ with $m(p,L) > 0$. (We have used the Fact.) Put $K = L\times\{y\}$. Then $(p,K) \in \mathscr{A}$.

Case 2. The measure $m(p,\cdot)$ is continuous. Define g_0 to be the restriction of g to the set $f^{-1}(p)$. Let n be the image of the measure $m(p,\cdot)$ under the map g_0. By the Fact, n is not a simple point mass. Thus there is some linear Borel set A with $0 < nA < 1$. Put $B = g_0^{-1}(A)$ and $B' = f^{-1}(p)\setminus B$. Then $m(p,B)m(p,G') > 0$, so that $B\times B'$ is a Borel subset of $T(p)$ containing a compact $\mathscr{I}(m(p,\cdot))$-thread K, as desired.

Remark. It is instructive to carry out the decomposition of $T(\mathscr{C}, \mathscr{D})$ into the sets $T(p)$ as well as the corresponding disintegration of m for the example constructed in Proposition 18. The essential difference between conditions (J) and (J+) then becomes quite vivid.

We have established that for each p in Q, there is some K in $\mathscr{K}(S\times S)$ with $(p,K) \in \mathscr{A}$. An application of the Jankov–von Neumann selection theorem, combined with Lusin's theorem, produces a continuous selector for $\mathscr{A}$ over most of Q: there is a continuous mapping $p \to K(p)$ defined on a compact subset C of Q such that

1) $m(C) > 0$;

2) for each p in C, the pair $(p, K(p))$ belongs to $\mathscr{A}$.

The range of the mapping $p \to K(p)$ is compact, and so therefore is the union of all sets in its range, i.e.

$$K = \bigcup_{p\in C} K(p)\,.$$

Now for $p \in C$, the set $K(p)$ is not $m(p,\cdot)$-reticulate. By Proposition 21, it follows that K is not m-reticulate. Clearly, K is the graph of some function k. Since K is compact, k is continuous. Finally, we note that $K \subset T(\mathscr{C}, \mathscr{D})$. ■

As before, we pass to the measurable case. The proof of the following theorem is entirely analogous to that of Proposition 23.

PROPOSITION 25. *Let m be a Borel probability on a standard space S. Suppose that f and g are measurable real functions on S generating the sub-σ-algebras $\mathscr{C} \subset \mathscr{D}$ in $\mathscr{B}(S)$ with $\mathscr{C}$ m-proper in $\mathscr{D}$. Then there is an $\mathscr{I}(m)$-graph K contained in*

$$T(\mathscr{C}, \mathscr{D}) = \{(s, t): f(s) = f(t) \text{ and } g(s) \neq g(t)\}.$$

It is now possible to obtain a complete characterization of Sierpiński sets with Blackwell and strong Blackwell properties.

PROPOSITION 26. *Let m be a Borel probability on a standard space S. Suppose that X is $\mathscr{I}(m)$-Lusin in S with $m^*(X) = 1$. Then the following are equivalent:*

1) *X is a Blackwell set;*
2) *X satisfies condition* (J);
3) *$X \times X$ meets every $\mathscr{I}(m)$-thread in $S \times S$.*

Also, the following are equivalent:

4) *X is strongly Blackwell;*
5) *X satisfies condition* (J+);
6) *$X \times X$ meets every $\mathscr{I}(m)$-graph in $S \times S$.*

Proof. The equivalences $1 \Leftrightarrow 2$ and $4 \Leftrightarrow 5$ follow from Proposition 1. The implications $2 \Leftrightarrow 3$ and $5 \Leftrightarrow 6$ follow from Proposition 2. Lastly, the implications $3 \Leftrightarrow 2$ and $6 \Leftrightarrow 5$ are easily deduced from Propositions 23 and 25, respectively.

§ 9. Direct sums of Sierpiński sets. Suppose X is a subset of the standard space S and that $\mathscr{I}$ is a σ-ideal in $\mathscr{B}(S)$. Let N be an arbitrary, nonvoid, countable set (finite or infinite). If N is given the discrete Borel structure, then $X \times N$ is a direct union of copies of X contained in the standard space $S \times N$. Define a σ-ideal $\mathscr{J}$ in $\mathscr{B}(S \times N)$ to be the collection of all countable unions of sets of the form $M \times \{n\}$, where $M \in \mathscr{I}$ and $n \in N$.

LEMMA 24. *If X is $\mathscr{I}$-dense in S, then $X \times N$ is $\mathscr{J}$-dense in $S \times N$. If X is an $\mathscr{I}$-Lusin set, then $X \times N$ is a $\mathscr{J}$-Lusin set in $S \times N$.*

Proof. Immediate. ■

PROPOSITION 27. *Let m be a Borel probability on a standard space S and suppose that X is an m-Sierpiński subset of S.*

1) *If X is Blackwell, then so is $S \times N$.*
2) *If X is strong Blackwell, then so is $S \times N$.*

Proof. Without loss of generality, one may assume that $m^*(X) = 1$. We establish 2) then indicate the (similar) proof of 1).

Let d be any probability on N with $d(n) > 0$ for each n in N. Put $m' = m \otimes d$ on $S \times N$. Then the σ-ideal $\mathscr{J}$ discussed above consists of the null sets for m'. Lemma 24 implies that $X \times N$ is an m'-Sierpiński set. From Proposition 1, $X \times X$ meets every $\mathscr{I}(m)$-graph in $S \times S$. It is easily seen that $(X \times N) \times (X \times N)$ intersects every m'-graph in $(S \times N) \times (S \times N)$. Proposition 26 implies that $X \times N$ is strong Blackwell.

The same proof applies to Condition 1, with "graph" replaced by "thread". ■

A forthcoming paper by the first author will contain a proof that whenever X is a Blackwell space, then so too is $X \times N$. The situation for strong Blackwell spaces is still unresolved.

§ 10. A conjecture about m-graphs. The preceeding analysis has not answered the question of whether $X \times S$ is Blackwell whenever X is a strong Blackwell Sierpiński set. However, it will be shown that this matter reduces to a conjecture about measurable selections, one that is of some independent interest.

Specifically, we have not been able to decide the following statements:

HYPOTHESIS (G). *Let m be a continuous Borel probability on a standard space S. If a set B in $\mathscr{B}(S^2)$ is not $\mathscr{I}(m)$-reticulate in S^2, then B contains an $\mathscr{I}(m)$-graph.*

HYPOTHESIS (G′). *Let m be a continuous Borel probability on a standard space S. If Y is a subset of S such that $Y \times Y$ meets every $\mathscr{I}(m)$-graph, then Y is $\mathscr{I}(m)$-dense of order 2.*

PROPOSITION 28. *Hypothesis* (G) *implies hypothesis* (G′). *Under* CH, *the converse is also true.*

Proof. The first sentence is obvious. Assuming CH, we prove that if hypothesis (G) fails, then so does hypothesis (G′). Suppose that $B \in \mathscr{B}(S^2)$ is not $\mathscr{I}(m)$-reticulate and contains no $\mathscr{I}(m)$-graphs. From Lemma 3, $(m \otimes m)(B) = 0$. By Fubini's Theorem, we may assume that for each $s \in S$, the sections

$$B_s = \{t \in S: (s, t) \in B\},$$
$$B^s = \{t \in S: (t, s) \in B\}$$

are of m-measure zero. (We may remove the other sections and preserve the character of B.) We also assume that B is symmetric about the line $y = x$. (We may replace B with its symmetrization $B \cup B^1$, where $B^1 = \{(t, s): (s, t) \in B\}$.)

Well-order the collection of all $\mathscr{I}(m)$-graphs as $G_0 G_1 G_2 \ldots G_\alpha \ldots$, $\alpha < \aleph_1$, and choose points $y_1^{(\alpha)}$, $y_2^{(\alpha)}$, $\alpha < \aleph_1$, according to the following scheme: $(y_1^{(0)}, y_2^{(0)}) \in G_0 \setminus B$; for each α, define Z_α to be the set of all $s \in S$ such that either $(s, y_1^{(\alpha)}) \in B$ of $(s, y_2^{(\alpha)}) \in B$. Put $Y_\alpha = \bigcup \{Z_\beta: \beta < \alpha\}$ and note that $m(Y_\alpha) = 0$. In general, we select $(y_1^{(\alpha)}, y_2^{(\alpha)}) \in G_\alpha \setminus (B \cup \langle Y_\alpha, Y_\alpha \rangle)$.

Put $Y = \{y_1^{(\alpha)}, y_2^{(\alpha)}: \alpha < \aleph_1\}$. The set $Y \times Y$ meets each of the $\mathscr{I}(m)$-graphs G_α, but $(Y \times Y) \cap B$ is void.

The importance of hypothesis (G) for the present analysis lies in the following equivalence. It provides a partial answer to problem (P8) in [1], at least for Sierpiński sets.

PROPOSITION 29. *Let S be an uncountable standard space and consider the following statements:*

1) *Hypothesis* (G);

2) *Suppose m is a continuous probability on $\mathcal{B}(S)$ and that $X \subset S$ is an m-Sierpiński set. If X is strongly Blackwell, then so also is $X \times S$.*

Statement 1 *implies statement* 2. *Under* CH, *they are equivalent.*

Proof. 1$\Rightarrow$2. Assume that $X \subset S$ is a strongly Blackwell m-Sierpiński set. It is no loss of generality to assume that $m^*(X) = 1$. By Proposition 2, $X \times X$ intersects every $\mathcal{I}(m)$-graph in $S \times S$. Hypothesis (G) implies hypothesis (G'), so that X is $\mathcal{I}(m)$-dense of order 2. Proposition 1 then implies that $X \times S$ is strongly Blackwell.

2$\Rightarrow$(CH). Suppose hypothesis (G) fails. Then by Proposition 28, so does hypothesis (G'). Thus there is some $Y \subset S$ such that $Y \times Y$ meets every $\mathcal{I}(m)$-graph and such that Y is not $\mathcal{I}(m)$-dense of order 2. From Proposition 5, we see that there is an m-Sierpiński set $X \subset Y$ with the same properties. Then X is strongly Blackwell, but $X \times S$ is not a Blackwell space (Proposition 19). ■

Hypothesis (G) seems to be related to the analysis of the so-called "doubly stochastic measures" (*vide* [11]). Not-(G) implies that there is an extreme doubly-stochastic measure not concentrated on any countable union of measurable graphs. An example of such a measure is known to exist [11], so this implication yields nothing new, and its proof (using the Choquet–Bishop–de Leeuw theorem on integral representations) is therefore omitted. It may be taken as very mild evidence that (G) is false.

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