FUNDAMENTA
MATHEMATICAE
127 (1987)

On the product of a perfect paracompact space and a countable product of scattered paracompact spaces

by

K. Alster (Warszawa)

Abstract. In this note we prove that the product of a perfect paracompact space and a product of countably many scattered paracompact spaces is paracompact.

Our result improves, in a sense, the theorem of M. E. Rudin and S. Watson from [RW]. The proof of our theorem, in contrast with the proof of the theorem of Rudin and Watson is effective.

We adopt the topological terminology from [E]. A scattered space X is a space whose every subspace contains isolated points. Put $X^{(0)} = X$, $X^{(\alpha)} = \bigcap\{X^{(\beta)}\colon \beta<\alpha\}$ if α is a limit ordinal number and define $X^{(\alpha)}$ to be the set of all accumulation points of $X^{(\beta)}$ if $\alpha = \beta+1$. By ω we denote the first infinite ordinal number and by N the set of natural numbers. For $x \in X$ denote by $\alpha(x)$ the ordinal number for which $x \in X^{(\alpha(x))} \setminus X^{(\alpha(x)+1)}$.

The aim of this note is to prove (see [A_2], Problem 3)

THEOREM. *If Z is a perfect paracompact space and X_m is a scattered paracompact space, for $n \in \omega$, then the product $Z \times \mathbf{P}_{n=0}^{\infty} X_n$ is paracompact.*

Proof. Without loss of generality we may assume that $X_n = X$ for $n \in \omega$ and that there is an ordinal number λ such that $X^{(\lambda)}$ consists of a single point. Indeed, put $X = \bigoplus_{n=0}^{\infty} Y_n \cup \{a\}$, where $Y_n = \bigoplus_{i=0}^{\infty} X_i$, $a \notin Y_n$, for $n \in \omega$, X_i is a clopen subset of X, for $i \in \omega$, and the base at a is induced by the sets of the form $U(n) = \bigoplus_{j\geqslant n} Y_j \cup \{a\}$. Notice that X is a scattered paracompact space, X_n is a closed subset of X, for $n \in \omega$; hence, if $Z \times X^{\omega}$ is a paracompact space then also $Z \times \mathbf{P}_{n=0}^{\infty} X_n$ is paracompact.

Let us denote by $\mathscr{B}'$ the base of X consisting of all clopen subsets B of X for which there is an α, denoted by $\alpha(B)$, such that the cardinality of $B^{(\alpha(B))}$, abbreviated $|B^{(\alpha(B))}|$, is equal to one. The symbol p_1 stands for the projection of $Z \times X^{\omega}$ onto Z and p_2 for the projection of $Z \times X^{\omega}$ onto X^{ω}. Let us denote by $\mathscr{B}$ the base of $Z \times X^{\omega}$

consisting of all sets of the form $V\times \mathop{\mathbf{P}}\limits_{n=0}^{\infty} B_n$, where V is an open subset of Z, $\{n\in\omega:$ $B_n\neq X\}$ is finite and $B_n\in\mathscr{B}'$, for $n\in\omega$.

Let $\mathscr{U}$ be an open cover of $Z\times X^\omega$ such that $\mathscr{U}\subset\mathscr{B}$ and if $B\in\mathscr{B}$ and $B\subset U\in\mathscr{U}$ for some $U\in\mathscr{U}$ then $B\in\mathscr{U}$. In order to prove that $Z\times X^\omega$ is paracompact it is enough to define a σ-discrete open cover $G^* = \bigcup\{G_n^*\colon 1<n<\omega\}$ which refines $\mathscr{U}$. For $B = V\times \mathop{\mathbf{P}}\limits_{n=0}^{\infty} B_n$ and $(z,v)\in Z\times X^\omega$ let us put

$$n(B) = \inf\{j\in\omega\colon B_i = X \text{ for } i\geqslant j\}$$

and

$$n((z,v)) = \inf\{n(U)\colon U\in\mathscr{U} \text{ and } (z,v)\in U\}.$$

For every ordered pair (H_1,H_2), where $H_1\leqslant H_2$ and H_1 and H_2 are clopen subsets of X, denote by $V(H_1,H_2)\subset\mathscr{B}'$ a pairwise disjoint cover of $H_2\setminus H_1$ (such a cover exists because X is paracompact) and write

$$D(V(H_1,H_2)) = \{d\in X\colon \text{there is } H\in V(H_1,H_2) \text{ and } H^{(\alpha(H))} = \{d\}\}.$$

If $H_1 = H_2$ then $V(H_1,H_2) = D(V(H_1,H_2)) = \emptyset$.

We first outline the idea of the construction of a σ-discrete open refinement $G^* = \bigcup\{G_n^*\colon 1<n<\omega\}$ of $\mathscr{U}$ before presenting a formal proof. Put $p = (a,\ldots,a,\ldots)$, where $\{a\} = X^{(\alpha(X))}$. Let $G_2^*\subset\mathscr{B}$ be a σ-discrete, open in $Z\times X^\omega$, cover of $Z\times\{p\}$, which refines $\mathscr{U}$ and is such that for every $(z,p)\in Z\times\{p\}$ there exists $G = G_z\times \mathop{\mathbf{P}}\limits_{i=0}^{\infty} G_i\in G_2^*$ satisfying the following conditions: $(z,p)\in G$, $n(z,p) = n(G)$ and $G_i^{(\alpha(G_i))} = \{a\}$ for $i\in\omega$. Now let us fix $G = G_z\times \mathop{\mathbf{P}}\limits_{i=0}^{\infty} G_i\in G_2^*$ for a moment. Consider sets $V(G_i,X)$ and $Z(G) = \mathop{\mathbf{P}}\limits_{i=0}^{\infty} Z(G)_i\subset X^\omega$ such that $Z(G)_i = D(V(G_i,X))\cup\{a\}$. Notice that $V(G_i,X) = \emptyset$ if $i\geqslant n(G)$ and consequently $Z(G)$ is discrete in X^ω. Let $v = (v_0,\ldots,v_i,\ldots)$ be a point of $Z(G)$. For every $i\in\omega$ there is $R_i\in V(G_i,X)\cup\{X\}$ such that $R_i^{(\alpha(R_i))} = \{v_i\}$. Let $U(v,G)\subset\mathscr{B}$ be a σ-discrete, open in $Z\times X^\omega$, cover of $Z\times\{v\}$, which refines $\mathscr{U}$ and is such that for every $(z,v)\in Z\times\{v\}$ there exists $G' = G'_z\times \mathop{\mathbf{P}}\limits_{i=0}^{\infty} G'_i\in U(v,G)$ satisfying the following conditions: $G'\subseteq G_z\times \mathop{\mathbf{P}}\limits_{i=0}^{\infty} R_i$, $(z,v)\in G'$, if $j\geqslant n(z,v)$ then $G'_j = R_j$ and $G_j'^{(\alpha(G'_j))} = \{v_j\}$. Notice that $G_z\times \mathop{\mathbf{P}}\limits_{i=0}^{\infty} R_i$ plays the same role in the construction of $U(v,G)$ as $Z\times X^\omega$ does in the case of G_2^*. Now put

$$G_3^* = \bigcup\bigcup\{U(v,G)\colon v\in Z(G) \text{ and } G\in G_2^*\}.$$

In a similar way we construct G_n^*, for $3<n$. Before we give more details and a formal proof we shall need some more notation.

Put $\mathscr{U}_1 = \{Z\times X^{\omega}\}$, $R(Z\times X^{\omega}) = X^{\omega}$, $Z(W) = \{p\}$, where $p = (a, \ldots, a, \ldots)$ and $V_i(W) = X$, for $W \in \mathscr{W}_1$ and $i\in\omega$.

Let us assume that $\mathscr{W}_j \subset \mathscr{B}^j$, $Z(W)\subset X^{\omega}$, $V_i(W)\subset \mathscr{B}'$ and $R(W)\in p_2(\mathscr{B})$, for $W\in\mathscr{W}_j$, $i\in\omega$, $1\leqslant j<n$, where $p_2(\mathscr{B}) = \{p_2(B)\colon B\in\mathscr{B}\}$, are defined in such a way that

(1) if $1<j<n$, $W = (W_0, \ldots, W_{j-1})\in\mathscr{W}_j$ then $\{W_1, \ldots, W_{j-1}\}\subset\mathscr{U}$,

(2) if $1<j<n$ then $\mathscr{W}_{j-1} = \mathscr{W}_j|j-1$, where $\mathscr{W}_j|j-1 = \{W|j-1\colon W = (W_0, \ldots, W_{j-1})\in\mathscr{W}_j$ and $W|j-1 = (W_0, \ldots, W_{j-2}\}$,

(3) for $1\leqslant j<n$, $0\leqslant k<j$, $i\in\omega$, $W = (W_0, \ldots, W_{j-1})\in\mathscr{W}_j$, $p_2(W_k) \subset R(W|k+1)$, $p_1(W_0)\supset\ldots\supset p_1(W_{j-1})$ and $R(W|1)\supset R(W|2)\supset\ldots\supset R(W)$,

(4) for $1<j<n$, $i\in\omega$ and

$$W = (W_0, \ldots, W_{j-1})\in\mathscr{W}_j,\ V_i(W) = V(W_{j-1,i}, R_i(W))\cup\{R_i(W)\},$$

where $R(W) = \mathop{\mathbf{P}}\limits_{i=0}^{\infty} R_i(W)$, $p_2(W_{j-1}) = \mathop{\mathbf{P}}\limits_{i=0}^{\infty} W_{j-1,i}$ and $R_i(W)\in V_i(W|j-1)$ for $i\in\omega$, and $Z(W) = \mathop{\mathbf{P}}\limits_{i=0}^{\infty} Z(W)_i$, where $Z(W_i) = D(V_i(W))$.

For $W = (W_0, \ldots, W_{n-2})\in\mathscr{W}_{n-1}$ and $(z, y)\in p_1(W_{n-2})\times Z(W)$, where $y = (x_0, \ldots, x_n, \ldots)$, let $H_{(z,y)} = H_z\times \mathop{\mathbf{P}}\limits_{i=0}^{\infty} H_{(z,y),i}\in\mathscr{B}$ be such that

(5) $H_z\times \mathop{\mathbf{P}}\limits_{i=0}^{n(z,y)-1} H_{(z,y),i}\times X\times X\times\ldots\in\mathscr{U}$ and $z\in H_z\subset p_1(W_{n-2})$,

(6) $n(H_{z,y}) = r(z, y) = \max(n(z, y), n(W_{n-2}))$ and $H_{(z,y),j}^{(\alpha(H_{(z,y),j}))} = \{x_j\}$ and

(7) $H_{(z,y),j} = \begin{cases} R\in V_j(W) \text{ such that } \{x_j\} = R^{(\alpha(R))} & \text{if } n(z, y)\leqslant j<r(z, y),\\ \text{a subset of } R\in V_j(W) \text{ such that } \{x_j\} = R^{(\alpha(R))} & \text{if } j<n(z, y).\end{cases}$

Observe that R depends only on y.

For $W = (W_0, \ldots, W_{n-2})\in\mathscr{W}_{n-1}$, $y\in Z(W)$ put $O_i(y, W) = \{z\in p_1(W_{n-2})\colon n(z, y)\leqslant i\}$, for $i\in\omega$. Notice that $O_i(y, W) = \bigcup\{p_1(H_{(z,y)})\colon n(z, y)\leqslant i\}$, cf. (5), and Z is a perfect paracompact space; thus there is a σ-discrete and open in Z cover $\mathscr{V}_i(y)$ of $O_i(y, W)$, which refines $\{p_1(H_{(z,y)})\colon n(z, y)\leqslant i\}$. For every $V\in\mathscr{V}_i(y)$ there is $z(V)\in O_i(y, W)$ such that $V\subset p_1(H_{(z(V),y)})$. Put

(8) $G_{n,i}^*(y, W) = \{V\times p_2(H_{(z(V),y)})\colon \text{where } V\in\mathscr{V}_i(y)\}$,
$G_n^*(y, W) = \bigcup\{G_{n,i}^*(y, W)\colon i\in\omega\}$ and

(9) $G_n^*(W) = \bigcup\{G_n^*(y, W)\colon y\in Z(W)\}$.

Put

(10) $\mathscr{W}_n = \{W\in B^n\colon W|n-1\in\mathscr{W}_{n-1}$ and $W_{n-1}\in G_n^*(W|n-1)\}$.

If $W\in\mathscr{W}_n$ then there are unique $y = (x_0, \ldots, x_n, \ldots)\in Z(W|n-1)$, namely $y = (W_{n-1,i}^{(\alpha(W_{n-1,i}))})_{i=0}^{\infty}$, where $p_2(W_{n-1}) = \mathop{\mathbf{P}}\limits_{i=0}^{\infty} W_{n-1,i}$, and unique $R_i\in V_i(W|n-1)$, for $i\in\omega$, such that $W_{n-1}\in G_n^*(y, W|n-1)$, $W_{n-1,i}\subset R_i$ and $W_{n-1,i}^{(\alpha(W_{n-1,i}))} = R_i^{(\alpha(R_i))}$

$= \{x_i\}$, for $i \in \omega$; cf. (7), (8), (9) and (10). Put $R = \mathop{\mathbf{P}}\limits_{i=0}^{\infty} R_i = R(W)$. The sets $V_i(W)$, for $i \in \omega$, and $Z(W)$ are defined according to (4) and this concludes the induction.

Put

(11) $G_n^* = \{B \in \mathscr{B}$: there is $W = (W_0, \ldots, W_{n-1}) \in \mathscr{W}_n$ such that $W_{n-1} = B\}$, for $n \in \omega$ and $1 < n$. By (1) $G^* = \bigcup\{G_n^*\colon 1 < n < \omega\}$ is an open family, which refines $\mathscr{U}$.

In order to finish the proof of the theorem, it is enough to show that G^* is a σ-discrete cover of $Z \times X^{\omega}$.

Let $(z, y) \in Z \times X^{\omega}$, where $y = (y_n)_{n=0}^{\infty}$, $x_0 = p = (a, \ldots, a, \ldots,)$, for $\{a\} = X^{(\alpha(X))}$, $i_0 = n(z, x_0)$ and G_0 such that $(z, x_0) \in G_0 \in G_{2,i_0}^*(x_0, Z \times X^{\omega})$. Note that $K_0 = (Z \times X^{\omega}, G_0) \in \mathscr{W}_2$. Then there are $x_1 = (x_{1,0}, \ldots, x_{1,i}, \ldots) \in Z((Z \times X^{\omega}, G_0))$ such that

$$x_{1,j} = \begin{cases} a, & \text{if } n(G_0) \leqslant j, \\ R^{(\alpha(R))}, & \text{where } R \in V_j(K_0) \text{ is the smallest set, in the sense of} \\ & \text{inclusion, such that } y_j \in R \text{ if } j < n(G_0), \end{cases}$$

$$i_1 = n(z, x_1),\ (z, x_1) \in G_1 \in G_{3,i_1}^*(x_1, K_0) \quad \text{and} \quad K_1 = (Z \times X, G_0, G_1) \in \mathscr{W}_3.$$

Let us assume that $x_1, \ldots, x_n$, $K_n = (Z \times X^{\omega}, G_0, \ldots, G_n) \in \mathscr{W}_{n+2}$ and $i_1, \ldots, i_n$ are defined. Then there exist $x_{n+1} \in Z(K_n)$, G_{n+1} and i_{n+1} such that

$$x_{n+1,j} = \begin{cases} a, & \text{if } n(G_n) \leqslant j, \\ R^{(\alpha(R))}, & \text{where } R \in V_j(K_n) \text{ is the smallest set such that } y_j \in R \text{ if} \\ & j < n(G_n), \end{cases}$$

$$i_{n+1} = n(z, x_{n+1}),\ (z, x_{n+1}) \in G_{n+1} \in G_{n+3,i_{n+1}}^*(x_{n+1}, K_n) \quad \text{and}$$
$$K_{n+1} = (Z \times X^{\omega}, G_0, \ldots, G_{n+1}) \in \mathscr{W}_{n+3}.$$

Observe that

(12) if $j, n \in \omega$ and $x_{n+1,j} \neq x_{n,j}$ then $\alpha(x_{n+1,j}) < \alpha(x_n, j)$.

From (12) it follows that $D_j = \{x_{n,j}\colon n \in \omega\}$ is finite, for $j \in \omega$, so the sequence $(x_n)_{n=0}^{\infty}$ converges to some $l = (l_0, \ldots, l_n, \ldots) \in X^{\omega}$. There is $(z, l) \in B \in \mathscr{U}$. Let j be such that

(13) $x_{j,i} = l_i$ for every $i \leqslant n(B)$.

From (13) it follows that $(z, x_j) \in B$ and so $n(z, x_j) \leqslant n(B)$. We shall show that $(z, l) \in G_j$. If not then there is a k with $n(z, x_j) < k < r(z, x_j) = n(G_j)$ such that $l_k \notin G_{j,k}$, where $p_2(G_j) = \mathop{\mathbf{P}}\limits_{i=0}^{\infty} G_{j,i}$, and consequently $G_{j,k} = R_k(K_j)$. Hence by the definition of $(x_n)_{n=0}^{\infty}$, it is easy to see that for every $j \leqslant j'$, $x_{j',k} \in R_k(K_j)$ and $R_k(K_j)$ is a clopen subset of X which does not contain l_k, contradicting the fact that $(x_n)_{n=0}^{\infty}$ converges to l. We conclude that $(z, l) \in G_j$.

We now show that $(z, y) \in G_j$. If not then there is $k < n(G_j)$ such that $y_k \notin G_{j,k}$. Hence $y_k \in R \setminus G_{j,k}$, where R, which is equal to $R_k(K_j)$, was defined in connection with $x_{j,k}$. It is easy to see that $x_{t,k} \in \bigcup V(G_{j,k}, R) = R \setminus G_{j,k}$, for $j < t$, so $l_k \in R \setminus G_{j,k}$, contradicting $(z, l) \in G_j$.

In order to show that G_n^*, for $2 \leqslant n$, is σ-discrete in $Z \times X^\omega$ it is enough to prove that, for $2 \leqslant n$, there is a splitting

(14) $\mathscr{W}_n = \bigcup \{A_{n,m}: m \in \omega\}$ such that, for every $n \leqslant k$, $L_{k,m} = \{C(W): W \in A_{n,m}\}$ is discrete, where $C(W) = \bigcup \{B \in \mathscr{B}$: there is $W' = (W_0, \ldots, W'_{k-1}) \in \mathscr{W}_k$ such that $W'|n = W$ and $B = W'_{k-1}\}$.

Indeed, observe that $\bigcup \{L_{n,m}: m \in \omega\} = G_n^*$.

If $n = 2$ then (14) holds by $p_1(W_0) \supset \ldots \supset p_1(W_{j-1})$ for $W = (W_0, \ldots, W_{j-1}) \in \mathscr{W}_j$ and $j \in N$, and by the definition of $\mathscr{W}_2$. Let us assume that (14) holds for $i \leqslant n$. In order to prove (14) for $i = n+1$ it is enough to show that the desired splitting exists for $\mathscr{W}_{n+1}(W) = \{W' \in \mathscr{W}_{n+1}: W'|n = W\}$, where $W \in \mathscr{W}_n$. Let A be a subset of $\{0, 1, \ldots (n(W_{n-1})-1)\}$; notice that $Z(W)_i = X^{(\alpha(X))} = \{a\}$, for $n(W_{n-1}) \leqslant i$, and put

(15) $Z(W)(A) = \{y \in Z(W)$: $A = \{j < n(W_{n-1})$: $\{y_i\} = W_{n-1,j}^{(\alpha(W_{n-1,j}))}\}$, where $p_2(W_{n-1}) = \mathop{\mathbf{P}}\limits_{i=0}^{\infty} W_{n-1,i}$.

Note that

(16) if y and $y' \in Z(W)(A)$ and $y \neq y'$ then there is $k < n(W_n)$ such that $y_k \neq y'_k$ and if $H \in G_{n+1}^*(y, W)$, $H' \in G_{n+1}^*(y', W)$, $K = (W_0, \ldots, W_{n-1}, H)$, $K' = (W_0, \ldots, W_{n-1} H')$, then $R_k(K)$ and $R_k(K')$ are different elements of $V(W_{n-1,k}, R_k(W))$.

The family $V(W_{n-1,k}, R_k(W))$ is discrete and so, by (16), the fact that for $\{W' \in \mathscr{W}_{n+1}$: $W'|n = W$ and $W'_n \in G_{n+1}^*(y, W)\}$, for $y \in Z(W)$ (A), the desired splitting exists, we can use the same argument as in the case of $\mathscr{W}_2$, and by (3) we infer that

$$\mathscr{W}_{n+1}(W)(A) = \{W' \in \mathscr{W}_{n+1}: W'|n = W,\ W'_n \in G_{n+1}^*(y, W) \text{ and } y \in Z(W)(A)\}$$

has the splitting. From $\mathscr{W}_{n+1} = \bigcup \bigcup \{\mathscr{W}_{n+1}(W)(A)$: $W \in \mathscr{W}_n$ and

$$A \subset \{0, 1, \ldots, (n(W_{n-1})-1)\}\}$$

it follows that (14) holds for $i = n+1$.

Remark. One can prove Theorem for a little more general case; namely, it is enough to assume that X is a Lindelöf space such that each closed subset F of X contains a compact set with nonempty interior, with respect to F (cf. Theorem 2 from [A_1]).

From Theorem we derive

COROLLARY. *If Z is a hereditarily Lindelöf space and X_n, for $n \in \omega$, is a Lindelöf scattered space then* $Z \times \mathop{\mathbf{P}}\limits_{n=0}^{\infty} X_n$ is Lindelöf (see [A_1]).

Proof. Without loss of generality one can assume that $X = X_n$, for $n \in \omega$. If not, put $X = \bigoplus_{n=0}^{\infty} X_n$. Let $\mathscr{U}$ be an open cover of $Z \times X^{\omega}$. Then, by Theorem, there is an open refinement $\mathscr{H} = \bigcup\{\mathscr{H}_n: n \in \omega\}$ of $\mathscr{U}$, which covers $Z \times X^{\omega}$, where $\mathscr{H}_n$, for $n \in \omega$, is a discrete family. For $H \in \mathscr{H}$ and $k \in N$ let $H(k) = \bigcup\{U \in \mathscr{B}: n(U) \leqslant k\}$ and $\mathscr{H}_n(k) = \{H(k): H \in \mathscr{H}_n\}$, where $\mathscr{B}$ is the base defined in the proof of Theorem. Notice that $\bigcup \mathscr{H}_n = \bigcup \bigcup \{\mathscr{H}_n(k): k \in N\}$ and if p_k is the projection from $Z \times X^{\omega}$ onto $Z \times X^k$, for $k \in N$, then $p_k(\mathscr{H}_n(k)) = \{p_k(H): H \in \mathscr{H}_n(k)\}$ is discrete in $Z \times X^k$. In order to finish the proof, it is enough to show that $Z \times X^k$ is a Lindelöf space, for $k \in N$. To this purpose let us observe that if X is a Lindelöf scattered space then we may assume, without loss of generality, that X is a P-space, i.e. every G_δ-subset of X is open, because G_δ-subsets of X induce a Lindelöf topology. Now observe that if X is a P-space then X^k is a P-space, for $k \in N$, and the product of an arbitrary Lindelöf space and a Lindelöf P-space is Lindelöf.

References

[A₁] K. Alster, *A class of spaces whose Cartesian product with every hereditarily Lindelöf space is Lindelöf*, Fund. Math. 114 (1981), 173–181.

[A₂] — *On Michael's problem concerning the Lindelöf property in the Cartesian products*, Fund. Math. 121 (1984), 149–167.

[E] R. Engelking, *General topology*, Warszawa 1977.

[RW] M. E. Rudin and S. Watson, *Countable products of scattered paracompact spaces*, Proc. Amer. Math. Soc. 89 (1984), 551–552.

INSTITUTE OF MATHEMATICS
POLISH ACADEMY OF SCIENCES
Śniadeckich 8, 00-950 Warszawa, Poland

Received 24 June 1985