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Combinatorial aspects of measure and category

by

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Abstract. In this paper we study set-theoretical properties of the ideal of meager sets. We prove that the real line is not the union of less than 2^ω meager sets iff for every family of reals of cardinality less than 2^ω there exists an "infinitely equal" real. We also find a characterization of uniformity of the ideal of meager sets.

0. Preface. The purpose of this paper is to give combinatorial description of some elementary properties of the ideal of meager sets and the ideal of null sets. In fact, we deal only with the ideal of meager sets. We find a characterization of basic set-theoretical properties of this ideal. For a more complete picture we also formulate, in the same language, the already known characterization of the analoguous properties of the ideal of null sets.

Let us start with the following definition.

DEFINITION. For any ideal $I \subseteq P(R)$ let $c(I)$ denote the smallest 2^ω-complete ideal containing I.

We define the following sentences.

$$A(I) \equiv c(I) \subseteq I,$$

$$B(I) \equiv R \notin c(I),$$

$$U(I) \equiv \mathop{\forall}_{|X|<2^\omega} X \subseteq R \;\; X \in I,$$

$$C(I) \equiv \mathop{\forall}_{|\mathscr{F}|<2^\omega} \mathscr{F} \subseteq I \; \exists H \in I \; \forall F \in \mathscr{F} \;\; H - F \neq \emptyset .$$

Let I_c and I_m denote the ideal of meager subsets of R and the ideal of Lebesgue measure zero sets, respectively. Let I_k denote the σ-ideal generated by compact subsets of ω^ω. We are interested in properties A, B, U and C for those ideals. For simplicity let $A(c)$ abbreviate $A(I_c)$, $B(k)$ stand for $B(I_k)$ and so on. It is well known that the properties A, B, U and C are equivalent when stated for the real line R, the Baire space ω^ω or the Cantor set 2^ω.

Throughout the paper we use the standard terminology. For any set X we write $[X]^{<\omega} = \{Z \subseteq X\colon |Z| < \omega\}$. $[X]^{\omega} = \{Z \subseteq X\colon |Z| = \omega\}$ and $X^{<\omega} = \bigcup_n X^n$. For any $s, t \in X^{<\omega}$ we denote by $s\hat{\ }t$ the concatenation of sequences s and t. For $s \in \omega^{<\omega}$ we denote $[s] = \{x \in \omega^{\omega}\colon s \subseteq x\}$. The family $\{[s]\colon s \in \omega^{\omega}\}$ is a standard base of ω^{ω}. For $n, m < \omega$ let $[n, m) = \{i < \omega\colon n \leqslant i < m\}$. Denote by $\mathbf{1}$ the function given by $\mathbf{1}(n) = 1$ for $n < \omega$.

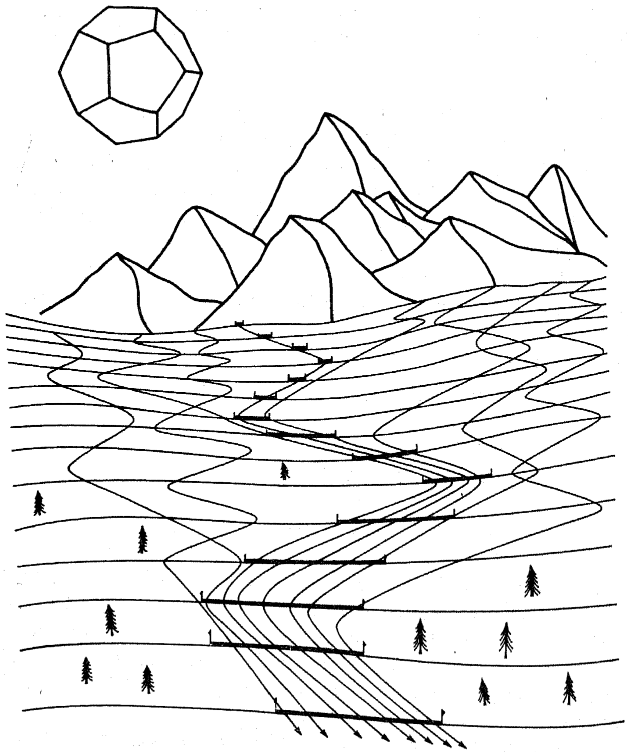
Symbols "$\forall^{\infty}$" and "$\exists^{\infty}$" abbreviate "for all except finitely many" and "there exist infinitely many", respectively.

We now define some combinatorial properties.

Definition. Let H be any countable set. Every element

$$\varphi \in \prod_n [H]^{<\omega} \stackrel{\mathrm{df}}{=} \mathrm{ST}^H$$

is called a *slalom*.



For any function $g \in \omega^{\omega}$ let

$$\mathrm{ST}^H_g = \prod_n [H]^{\leqslant g(n)}.$$

The set H will be irrelevant for our purposes, only its power will be of importance. Therefore we define

$$\mathrm{ST} = \mathrm{ST}^{\omega} \quad \text{and} \quad \mathrm{ST}_g = \mathrm{ST}^{\omega}_g.$$

Let

$$Q_0 = P(\omega) - \{\emptyset\} = \{X \subseteq \omega\colon \exists n\ n \in X\}$$
$$Q_1 = \{X \subseteq \omega\colon |X| = \omega\} = \{X \subseteq \omega\colon \exists^{\infty} n\ n \in X\}.$$
$$Q_2 = \{X \subseteq \omega\colon |\omega - X| < \omega\} = \{X \subseteq \omega\colon \forall^{\infty} n\ n \in X\}.$$
$$Q_3 = \{\omega\} = \{X \subseteq \omega\colon \forall n\ n \in X\}.$$

For any family $F \subseteq \omega^{\omega}$ and any function $h \in \omega^{\omega}$ we define

$$\mathrm{In}_i(F, h) \equiv \exists \varphi \in \mathrm{ST}_h\ \forall f \in F\ \{n\colon f(n) \in \varphi(n)\} \in Q_i \quad \text{for } i = 0, 1, 2, 3;$$

and for any family $\Phi \subseteq \mathrm{ST}$ and any function $f \in \omega^{\omega}$ let

$$\mathrm{Out}_i(\Phi, f) \equiv \forall \varphi \in \Phi\ \{n\colon f(n) \notin \varphi(n)\} \in Q_i \quad \text{for } i = 0, 1, 2, 3.$$

In this paper we study the following combinatorial principles.

Definition. For $i = 0, 1, 2, 3$ let

$$\mathrm{In}_i \equiv \forall_{|F| < 2^{\omega}} F \subseteq \omega^{\omega}\ \exists h \in \omega^{\omega}\ \mathrm{In}_i(F, h),$$
$$\mathrm{In}^i \equiv \exists h \in \omega^{\omega}\ \forall_{|F| < 2^{\omega}} F \subseteq \omega^{\omega}\ \mathrm{In}_i(F, h),$$
$$\mathrm{Out}^i \equiv \forall_{|\Phi| < 2^{\omega}} \Phi \subseteq \mathrm{ST}\ \exists g \in \omega^{\omega}\ \mathrm{Out}_i(\Phi, g),$$
$$\mathrm{Out}_i \equiv \forall h \in \omega^{\omega}\ \forall_{|\Phi| < 2^{\omega}} \Phi \subseteq \mathrm{ST}_h\ \exists g \in \omega^{\omega}\ \mathrm{Out}_i(\Phi, g).$$

The lemmas below state the basic properties of the sentences defined above.

Lemma 0.1. (1) $\mathrm{In}^0 \equiv \mathrm{In}^1$,

(2) $\mathrm{In}_0 \equiv \mathrm{In}_1$,

(3) $\mathrm{Out}_0 \equiv \mathrm{Out}_1$,

(4) $\mathrm{Out}^0 \equiv \mathrm{Out}^1$. ∎

An easy proof is left to the reader.

The lemma shows that the case $i = 0$ can be eliminated. The next lemma eliminates the case of $i = 3$.

Lemma 0.2. *The sentences* In_3, In^3, Out_3, Out^3 *are false*. ∎

Easy computation shows that

Lemma 0.3. (1) $\mathrm{In}_2 \equiv \mathrm{Out}^2$,

(2) $\mathrm{In}_1 \equiv \mathrm{Out}^1$. ∎

For $i = 1, 2$ and for a function $h \in \omega^\omega$ we can define

$$\mathrm{In}_h^i \equiv \forall F \subseteq_{|F| < 2^\omega} \omega^\omega \ \mathrm{In}_i(F, h).$$

Then

$$\mathrm{In}^i \equiv \exists h \in \omega^\omega \ \mathrm{In}_h^i$$

and we have another easy lemma:

LEMMA 0.4. $\mathrm{In}_h^2 \to \lim_{n\to\infty} h(n) = \infty$. ■

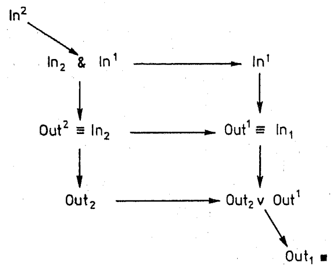
The next lemma shows that if we restrict ourselves to functions h converging to infinity then the sentences In_h^2 are equivalent.

LEMMA 0.5. *If* $\lim_{n\to\infty} g(n) = \infty$ *then* $\mathrm{In}_f^2 \to \mathrm{In}_g^2$. ■

An analoguous lemma for the case $i = 1$ will be proved in the next section.

We have six different sentences of type In and Out for $i = 1, 2$. The relations between them are described in the following obvious theorem.

THEOREM 0.6.



It turns out that each sentence from this diagram is equivalent to one of the properties A, B, U and C for the ideals I_c, I_m or I_k. Let us now recall some known facts.

THEOREM 0.7. (1) $\mathrm{Out}^2 \equiv \mathrm{In}_2 \equiv A(k) \equiv U(k)$,
(2) $\mathrm{Out}^1 \equiv \mathrm{In}_1 \equiv B(k) \equiv C(k)$. ■

THEOREM 0.8 (Miller, Truss). $A(c) \equiv \mathrm{In}_2 \,\&\, \mathrm{In}^1$.
(For the proof see [Mi1].) ■

THEOREM 0.9. $A(m) \equiv \mathrm{In}^2$.
(For the proof see [Ba2].) ■

THEOREM 0.10. $C(m) \equiv \mathrm{Out}_1$. ■

This theorem was independently proved by A. Miller, J. Cichoń, J. Raisonnier and J. Stern. (For the proof see [R–S] or [Fr]). The rest of this paper is devoted to the remaining sentences In^1, Out_2 and $\mathrm{Out}_2 \vee \mathrm{Out}^1$. We will prove that they are equivalent to $B(c)$, $U(c)$ and $C(c)$, respectively.

1. Baire category theorem. In this section we show that $B(c) \equiv \mathrm{In}^1$. In his paper [Mi 1] A. Miller showed that

THEOREM 1.1 (Miller).

$$B(c) \equiv \forall F \subseteq_{|F| < 2^\omega} \omega^\omega \ \exists g \in \omega^\omega \ \forall f \in F \ \exists^\infty n (f(n) = g(n) \,\&\, \forall i < n \ g(i) < n). \ ■$$

and in the paper [Mi2] he proved

THEOREM 1.2 (Miller).

$$B(c) \equiv \forall F \subseteq_{|F| < 2^\omega} \omega^\omega \ \forall G \subseteq_{|G| < 2^\omega} [\omega]^\omega \ \exists g \in \omega^\omega \ \forall f \in F \ \forall X \in G \ \exists^\infty n (f(n) = g(n) \,\&\, n \in X). \ ■$$

In these paper he asked whether the conditions $\forall i < n \ g(i) < n$ in Theorem 1.1 and the quantifier $\forall G \subseteq [\omega]^\omega$ in Theorem 1.2 are necessary. We will show that indeed, these conditions can be dropped out. We start with some definitions.

DEFINITION.

$$\omega^{=\omega} \stackrel{\mathrm{df}}{=} \{\omega^X\colon X \in [\omega]^\omega\};$$

the elements of the space $\omega^{=\omega}$ will be called *partial functions*. For any functions $f, g \in \omega^{=\omega}$ and $n < \omega$ the sentence $f(n) = g(n)$ means that the values of $f(n)$ and $g(n)$ do exist and are equal. Now we define, for any function $h \in \omega^\omega$,

$$\widetilde{\mathrm{In}}_h^1 \equiv \forall F \subseteq_{|F| < 2^\omega} \omega^{=\omega} \ \exists \varphi \in \mathrm{ST}_h \ \forall f \in F \ \exists^\infty n \ f(n) \in \varphi(n)$$

and

$$\widetilde{\mathrm{In}}^1 \equiv \exists h \in \omega^\omega \ \widetilde{\mathrm{In}}_h^1.$$

The only difference between the sentences $\widetilde{\mathrm{In}}^1$ and In^1 lies in the fact that the space ω^ω is replaced by $\omega^{=\omega}$. It is not very hard to see that the right hand side of Theorem 1.2 is in our terminology equivalent to $\widetilde{\mathrm{In}}_1^1$. So we have

THEOREM 1.3 (Miller). $B(c) \equiv \widetilde{\mathrm{In}}_1^1$. ■

Now we state a combinatorial lemma which will be used later.

LEMMA 1.4. *For any natural numbers* $p, m, k < \omega$ *there exists a number* $b(p, m, k) \in \omega$ *such that for any* $m \times n$*-matrix* $\{a_{i,j}\}_{i<m, j<n}$ *with the properties*

(1) $n \geqslant b(p, m, k)$,
(2) $a_{i,j} \in \omega$ for $i < m, j < n$,
(3) *for every* $i < m$ $\quad a_{i,j_1} \neq a_{i,j_2}$ if $j_1 \neq j_2$; $j_1, j_2 < n$

and for every set $A \subseteq \omega$ *of power less than* p *there exist distinct numbers* $j_1, \ldots, j_k < n$ *such the sets* $B_1 = \{a_{i,j_1}\colon i < m\} \ldots B_k = \{a_{i,j_k}\colon i < m\}$ *and* A *are pairwise disjoint.*

Proof. The lemma is obvious. ■

DEFINITION. A matrix $\{a_{i,j}\}_{i<m, j<n}$ is (p, m, k)*-long* if the conditions (1)–(3) are satisfied.

LEMMA 1.4 says that if $A \subseteq \omega$ has power less than p then every (p, m, k)-long matrix has k columns pairwise disjoint and disjoint with A.

Now we prove the main theorem of this section.

THEOREM 1.5. *The following conditions are equivalent*:

(1) In_1^1,

(2) $\widetilde{\mathrm{In}}^1$,

(3) $\widetilde{\mathrm{In}}_1^1$,

(4) $\forall h \in \omega^\omega\ \widetilde{\mathrm{In}}_h^1$,

(5) $\forall h \in \omega^\omega\ \mathrm{In}_h^1$,

(6) In^1.

Proof. (1) → (2). Let $F \subseteq \omega^{=\omega}$ be any family of power $< 2^\omega$. For any function $f \in \omega^{=\omega}$ take an increasing enumeration $\{x_n^f\colon n < \omega\}$ of the domain of f and define a function $f' \in \omega^\omega$ by

$$f'(n) = f(x_n^f) \quad \text{for } n < \omega.$$

Let $F' = \{f'\colon f \in F\}$.

By our assumption there exists a function $g \in \omega^\omega$ such that

$$\forall f \in F\ \exists^\infty n\ f'(n) = g(n).$$

We define a slalom $\varphi \in \mathrm{ST}_h$, where $h(n) = n+1$ for $n < \omega$, by

$$\varphi(n) = \{g(i)\colon i \leqslant n\} \quad \text{for } n < \omega.$$

Take any function $f \in F$ and a positive integer $n < \omega$ such that $f'(n) = g(n)$. By our definition

$$f(x_n^f) = f'(n) = g(n) \in \{g(i)\colon i \leqslant x_n^f\} = \varphi(x_n^f);$$

thus

$$\forall f \in F\ \exists^\infty n\ f(n) \in \varphi(n).$$

(2) → (3). By our assumption $\widetilde{\mathrm{In}}_h^1$ holds for some function $h \in \omega^\omega$. Let $F \subseteq \omega^{=\omega}$ be any family of power $< 2^\omega$. We will show that there exists a function $g \in \omega^\omega$ such that

$$\forall f \in F\ \exists^\infty n\quad f(n) = g(n).$$

In order to construct such a function we will use the following

CLAIM 1.6. *There exists a family $\{J_{n,k}\colon n < \omega, k \leqslant h(n)\}$ of finite, pairwise disjoint subsets of ω such that*

$$\forall f \in F\ \exists^\infty n\ \forall k \leqslant h(n)\ J_{n,k} \cap \mathrm{dom}(f) \neq \varnothing.$$

Before proving this claim we will use it to get the desired function g. Let

$$\{J_{n,k}\colon n < \omega,\ k \leqslant h(n)\}$$

be a family of sets from Claim 1.6.

We put

$$J_n = \bigcup_{k \leqslant h(n)} J_{n,k} \quad \text{for } n < \omega.$$

For any function $f \in F$ let

$$f'(n) = \begin{cases} f \restriction J_n & \text{if } \forall k \leqslant h(n)\ J_{n,k} \cap \mathrm{dom}(f) \neq \varnothing, \\ \text{undefined otherwise.} \end{cases}$$

It is easy to see that for any function $f \in F$ we have $f' \in H^{=\omega}$ where H is the set of partial mappings from J_n's into ω.

Let $F' = \{f'\colon f \in F\}$. By the assumption $\widetilde{\mathrm{In}}_h^1$ we have a slalom $\psi \in \mathrm{ST}_h$ such that

$$\forall f \in F\ \exists^\infty n\ f'(n) \in \psi(n).$$

Without loss of generality we can assume that for every $n < \omega$

$$\psi(n) = \{w_1^n, \ldots, w_{h(n)}^n\}$$

where w_i^n is a partial function from J_n into ω for $i \leqslant h(n)$.

Define

$$g_n = \bigcup_{k \leqslant h(n)} w_k^n \restriction J_{n,k} \quad \text{for } n < \omega$$

and

$$g = \bigcup_{n < \omega} g_n.$$

Take any function $f \in F$. For any $n < \omega$ such that $f'(n) \in \psi(n)$, we have

$$f \restriction J_n = w_{k_0}^n \quad \text{for some } k_0 \leqslant h(n).$$

Thus by the definition of f'

$$\forall k \leqslant h(n)\ J_{n,k} \cap \mathrm{dom}(f) \neq \varnothing.$$

Let $x \in J_{n,k_0} \cap \mathrm{dom}(f)$. In this case

$$f(x) = w_{k_0}^n(x) = g(x).$$

Therefore

$$\forall f \in F\ \exists^\infty n\ f(n) = g(n).$$

Thus in order to finish the proof we have to prove Claim 1.6.

Proof of the claim. For every $f \in F$ we choose an enumeration $r_f \in \omega^\omega$ of the domain of f:

$$\mathrm{dom}(f) = \{r_f(n)\colon n < \omega\}.$$

Let $\{m_n\colon n < \omega\}$, $\{k_n\colon n < \omega\}$ and $\{p_n\colon n < \omega\}$ be sequences defined by

$$\left.\begin{aligned} m_n &= k_n = h(n) \\ p_n &= \sum_{i<n} h(i)^2 \end{aligned}\right\} \quad \text{for } n < \omega.$$

Let $\{a_n: n<\omega\}$ be a sequence such that

$$a_{n+1}-a_n \geqslant b(p_n, m_n, k_n) \quad \text{for } n<\omega.$$

For every function $f \in F$ define

$$\hat{r}_f(n) = r_f \upharpoonright [a_n, a_{n+1}) \quad \text{for } n<\omega.$$

Using the assumption $\overline{\mathrm{In}_h^1}$ we will find a slalom $\varphi \in \mathrm{ST}_h$ such that

$$\forall f \in F\ \exists^\infty n\ r_f(n) \in \varphi(n).$$

As before, we can assume without loss of generality that for $n<\omega$

$$\varphi(n) = \{u_1^n, \ldots, u_{h(n)}^n\}$$

where u_i^n is a 1—1 mapping from $[a_n, a_{n+1})$ into ω for $i \leqslant h(n)$. We will construct a family $\{J_{n,k}: n<\omega, k \leqslant h(n)\}$ by induction. Assume that the sets

$$\{J_{i,j}: i<n, j \leqslant h(i)\}$$

are already defined. We define $\{J_{n,i}: i \leqslant h(n)\}$. Assume also that $|J_{i,j}| \leqslant h(i)$ for $i<n, j<h(i)$. So

$$|\bigcup_{i<n} \bigcup_{j<h(i)} J_{i,j}| \leqslant \sum_{i<n} h(i)^2 = p_n.$$

Notice that $a_{n+1}-a_n \geqslant b(p_n, m_n, k_n)$; hence the matrix

$$U_n = \{u_j^n(a_n+i)\}_{j \leqslant h(n), i<a_{n+1}-a_n}$$

is (p_n, m_n, k_n)-long.

By Lemma 1.4 there exist k_n distinct columns of U_n pairwise disjoint and disjoint with

$$\bigcup_{i<n} \bigcup_{j \leqslant h(i)} J_{i,j}.$$

We define the set of elements of the j-th column to be $J_{n,j}$, for $j \leqslant k_n = h(n)$. Notice that $|J_{n,j}| \leqslant h(n)$ for $j \leqslant h(n)$. We now show that for any function $f \in F$

$$\exists^\infty n\ \forall k \leqslant h(n)\ J_{n,k} \cap \mathrm{dom}(f) \neq \emptyset.$$

Take any function $f \in F$. Let $n<\omega$ be such that $r_f(n) \in \varphi(n)$, which means that

$$r_f \upharpoonright [a_n, a_{n+1}) \in \varphi(n).$$

In this case there exists $j \leqslant h(n)$ such that

$$r_f \upharpoonright [a_n, a_{n+1}) = u_j^n.$$

Take any $k \leqslant h(n)$ and consider the set $J_{n,k}$. By the definitions, $J_{n,k}$ is one of the columns of the matrix U_n, say the i-th column. Consider the element $x = u_j^n(a_n+i)$. Then $x = r_f(a_n+i) \in \mathrm{dom}(f)$ and x belongs to the i-th column: therefore $x \in J_{n,k}$. Thus $x \in J_{n,k} \cap \mathrm{dom}(f)$.

Implications (3) → (4) → (5) → (6) are obvious.

(6) → (1) Assume that In_h^1 holds for some function $h \in \omega^\omega$. Let $F \subseteq \omega^\omega$ be any family of power $< 2^\omega$. We have to find a function $g \in \omega^\omega$ such that

$$\forall f \in F\ \exists^\infty n\ f(n) = g(n).$$

Just as in the proof of the implication (2) → (3), we are going to find a family $\{J_{n,k}: n<\omega, k \leqslant h(n)\}$ of finite pairwise disjoint subsets of ω such that

$$\forall f \in F\ \exists^\infty n\ \forall k \leqslant h(n)\ J_{n,k} \cap \mathrm{dom}(f) \neq \emptyset.$$

But each function from the family F has a domain equal to ω. Hence it is enough to put

$$J_{n,k} = \{k + \sum_{i<n} h(i)\} \quad \text{for } n<\omega,\ k \leqslant h(n).$$

Now, as before, for $f \in F$ and $n<\omega$ define

$$J_n = \bigcup_{k \leqslant h(n)} J_{n,k} \quad \text{and} \quad f'(n) = f \upharpoonright J_n.$$

Functions f' are also complete; so, applying assumption In_h^1 to the family $F' = \{f': f \in F\}$ and arguing as in the proof that (2) → (3), we get the required function $g \in \omega^\omega$. ■ ■

From Theorem 1.3 and 1.5 we immediately get the following

THEOREM 1.7. *The following conditions are equivalent*:

(1) $B(c)$,

(2) $\forall F \subseteq \omega^\omega_{|F|<2^\omega}\ \exists g \in \omega^\omega\ \forall f \in F\ \exists^\infty n\ f(n) = g(n)$,

(3) In^1. ■

In fact we have proved

COROLLARY 1.8. *R is not the union of κ meager sets iff*

$$\forall F \subseteq_{|F|<\kappa} \omega^\omega\ \exists g \in \omega^\omega\ \forall f \in F\ \exists^\infty n\ f(n) = g(n). \ ■$$

Define

κ_B = the least cardinal κ such that R can be covered by κ many meager sets.

From Corollary 1.8 immediately follows

THEOREM 1.9 (Miller). $\mathrm{cf}(\kappa_B) > \omega$.

Proof. Assume that $\mathrm{cf}(\kappa_B) = \omega$.

By Corollary 1.8, in order to get a contradiction, it is sufficient to show that for every family $F \subseteq \omega^\omega$ of size κ_B there exists a function $g \in \omega^\omega$ such that

$$\forall f \in F\ \exists^\infty n\ f(n) = g(n).$$

Let $F \subseteq \omega^\omega$ be any family of power κ_B. Using the fact that $\mathrm{cf}(\kappa_B) = \omega$ we can find a sequence $\{F_n: n<\omega\}$ such that

$$F = \bigcup_n F_n \quad \text{and} \quad |F_n| < \kappa_B \quad \text{for } n<\omega.$$

Now fix a sequence $\{A_n: n<\omega\}$ of pairwise disjoint, infinite subsets of ω. Define for $n<\omega$:

$$F_n' = \{f \upharpoonright A_n: f\in F_n\}.$$

By Corollary 1.9, for every $n<\omega$ we have a function $g_n \in \omega^{A_n}$ such that

$$\forall f\in F_n\ \exists^\infty m\in A_n\quad f'(m) = g_n(m).$$

Let

$$g = \bigcup_n g_n.$$

It is easy to see that

$$\forall f\in F\ \exists^\infty n\ f(n) = g(n). \ \blacksquare$$

2. Uniformity of the ideal of meager sets. In this section we show that $U(c) \equiv \mathrm{Out}_2$. This will allow us to give a positive answer to the question posed by A. Miller in [Mi1] and [Mi2], whether

$$U(c) \equiv \forall F\subseteq_{|F|<2^\omega} \omega^\omega\ \exists g\in\omega^\omega\ \forall f\in F\ \forall^\infty n\ f(n)\neq g(n).$$

Consider the following combinatorial principles.

DEFINITION. For any function $h\in\omega^\omega$

$$\mathrm{Out}^h_{1\frac{1}{2}} \equiv \forall \Phi\subseteq_{|\Phi|<2^\omega} \mathrm{ST}_h\ \exists g\in\omega^\omega\ \exists X\in[\omega]^\omega\ \forall\varphi\in\Phi\ \forall^\times n\in X\ g(n)\notin\varphi(n).$$

and

$$\mathrm{Out}_{1\frac{1}{2}} \equiv \forall\, h\in\omega^\omega\ \mathrm{Out}^h_{1\frac{1}{2}}.$$

This terminology is motivated by the following obvious implications:

$$\mathrm{Out}_2\to\mathrm{Out}_{1\frac{1}{2}}\to\mathrm{Out}_1.$$

In the paper [Mi1] A. Miller proved

THEOREM 2.1 (Miller). $U(c)\equiv\mathrm{Out}^1_{1\frac{1}{2}}$. $\blacksquare$

THEOREM 2.2. *The following conditions are equivalent*:

(1) Out_2,

(2) Out_2^1,

(3) $\mathrm{Out}^1_{1\frac{1}{2}}$,

(4) $\mathrm{Out}_{1\frac{1}{2}}$.

Proof. Implications (1) → (2) → (3) are obvious.

(3) → (4). Assume that condition (4) does not hold. This means that for some function $h\in\omega^\omega$ there exists a family of slaloms $\Phi\subseteq\mathrm{ST}_h$, $|\Phi|<2^\omega$ such that

$$\forall g\in\omega^\omega\ \forall X\in[\omega]^\omega\ \exists\varphi\in\Phi\ \exists^\infty n\in X\ g(n)\in\varphi(n).$$

We will show that under this assumption there exists a family $F\subseteq\omega^\omega$ of power $<2^\omega$ such that

$$\forall g\in\omega^\omega\ \forall X\in[\omega]^\omega\ \exists f\in F\ \exists^\infty n\in X\ g(n)=f(n).$$

Assume that $|\Phi| = \lambda < 2^\omega$. We will need the following lemma.

CLAIM 2.3. *For every $\xi<\lambda$ there exists a family $\{J^\xi_{n,k}: n<\omega, k\leqslant h(n)\}$ of finite, pairwise disjoint subsets of ω such that*

$$\forall X\in[\omega]^\omega\ \exists\xi<\lambda\ \exists^\infty n\ \forall k\leqslant h(n)\ J^\xi_{n,k}\cap X\neq\emptyset.$$

Proof. As in the proof of Claim 1.6, we define sequences $\{m_n: n<\omega\}$, $\{k_n: n<\omega\}$ and $\{p_n: n<\omega\}$ by

$$\left.\begin{aligned} m_n &= k_n = h(n)\\ p_n &= \sum_{i<n} h(i)^2\end{aligned}\right\}\quad \text{for } n<\omega.$$

Let $\{a_n: n<\omega\}$ be a sequence of positive integers such that

$$a_{n+1}-a_n = b(p_n, m_n, k_n)\quad \text{for } 1\leqslant n<\omega\quad (a_0=0).$$

Let

$$H_n = \{s\in\omega^{[a_n, a_{n+1})}: s \text{ is } 1\text{—}1\}\quad \text{for } n<\omega.$$

There exists a family of slaloms $\Phi' = \{\varphi'_\xi: \xi<\lambda\}\subseteq\mathrm{ST}_h$, such that

$$\forall f\in\prod_n H_n\ \forall X\in[\omega]^\omega\ \exists\xi<\lambda\ \exists^\infty n\ \big((f(n)\in\varphi'_\xi(n)\ \&\ n\in X\big).$$

Without loss of generality we can assume that

$$|\varphi'_\xi(n)| = h(n)\quad\text{and}\quad \varphi'_\xi(n)\subseteq H_n\quad\text{for } \xi<\lambda, n<\omega.$$

By the above remarks, for $\xi<\lambda$ and $n<\omega$ the set $\varphi'_\xi(n)$ can be interpreted as a matrix having $h(n)$ rows and $a_{n+1}-a_n$ columns.

Moreover, by the definition of the sequence $\{a_n: n<\omega\}$ this matrix is (p_n, m_n, k_n)-long.

Now, using Lemma 1.4 and arguing as in the proof of Claim 1.6, we can define families $\{J^\xi_{n,k}: n<\omega, k\leqslant h(n)\}$ of finite, pairwise disjoint subsets of ω for $\xi<\lambda$. Let $X\in[\omega]^\omega$ be any infinite subset of ω and $r_X\in\omega^\omega$ its increasing enumeration. Put

$$\hat r_X(n) = r_X\upharpoonright[a_n, a_{n+1})\quad\text{for } n<\omega.$$

Notice that $\hat r_X(n)\in H_n$ for $n<\omega$. Therefore we have

$$\exists^\infty_n\ \hat r_X(n)\in\varphi'_\xi(n)\quad\text{for some } \xi<\lambda.$$

Take the family $\{J^\xi_{n,k}: n<\omega, k\leqslant h(n)\}$ defined from φ'_ξ. Repeating the argument from Claim 1.6 we immediately get

$$\exists^\infty n\ \forall k\leqslant h(n)\ J^\xi_{n,k}\cap X\neq\emptyset.$$

Since X was arbitrary, this proves the claim. $\blacksquare$

Let $\{\{J^\xi_{n,k}: n<\omega, k\leqslant h(n)\}: \xi<\lambda\}$ be the family of partitions from the claim. For $\xi<\lambda$ define

$$\left.\begin{aligned} J^\xi_n &= \bigcup_{k\leqslant h(n)} J^\xi_{n,k}\\ H^\xi_n &= \omega^{J^\xi_n}\end{aligned}\right\}\quad\text{for } n<\omega.$$

By our assumption, to every $\xi<\lambda$ we can find a family $\Psi_\xi \subseteq \mathrm{ST}_h$ of size λ such that

$$\forall f \in \prod_n H_n^\xi \ \forall X \in [\omega]^\omega \ \exists \psi \in \Psi_\xi \ \exists^\infty n \in X \ f(n) \in \psi(n).$$

For $\xi<\lambda$ and $\psi \in \Psi_\xi$ define a function $f_\psi^\xi \in \omega^\omega$ in the following way:

As before, assume that

$$\psi(n) = \{w_1^n, \ldots, w_{h(n)}^n\} \quad \text{where } w_j^n\colon J_n^\xi \to \omega \text{ for } j \leqslant h(n).$$

Put

$$f_{\psi,n}^\xi = \bigcup_{k\leqslant h(n)} w_k^n \restriction J_{n,k}^\xi \quad \text{for } n<\omega \quad \text{and } \bar{f}_\psi^\xi = \bigcup_n f_{\psi,n}^\xi.$$

Finally,

$$f_\psi^\xi(n) = \begin{cases} \bar{f}_\psi^\xi(n) & \text{if } n \in \mathrm{dom}(\bar{f}_\psi^\xi),\\ 0 & \text{if } n \notin \mathrm{dom}(\bar{f}_\psi^\xi).\end{cases}$$

Let $F = \{f_\psi^\xi\colon \xi<\lambda,\ \psi \in \Psi_\xi\}$. We will show that F is a family we are looking for. Take any function $g \in \omega^\omega$ and a subset $X \in [\omega]^\omega$. By Claim 2.3 there exists $\xi<\lambda$ such that

$$\exists^\infty n \forall k \leqslant h(n) \ J_{n,k}^\xi \cap X \neq \emptyset.$$

Let $\bar{g}(n) = g \restriction J_n^\xi$ for $n<\omega$ and

$$Y = \{n\colon \forall k \leqslant h(n)\ J_{n,k}^\xi \cap X \neq \emptyset\}.$$

We can find a slalom $\psi \in \Psi_\xi$ such that

$$\exists^\infty n \in Y \ \bar{g}(n) \in \psi(n).$$

Arguing as in the proof of Theorem 1.5 we get

$$\exists^\infty n \in X \ f_\psi^\xi(n) = g(n)$$

and this finishes the proof.

(4) → (1). Take any function $h \in \omega^\omega$ and a family $\Phi \subseteq \mathrm{ST}_h$ of size $<2^\omega$. For every slalom $\varphi \in \mathrm{ST}_h$ define a slalom φ' by

$$\varphi'(n) = \bigcup_{k\leqslant n} \varphi(k).$$

Let $h'(n) = \sum_{i\leqslant n} h(i)$ for $n<\omega$. Obviously $\varphi' \in \mathrm{ST}_{h'}$. Let $\Phi' = \{\varphi'\colon \varphi \in \Phi\}$. By $\mathrm{Out}_{1\frac{1}{2}}$ there exist a set $X \in [\omega]^\omega$ and a function $\bar{g} \in \omega^\omega$ such that

$$\forall \varphi \in \Phi \ \forall^\infty n \in X \ \bar{g}(n) \notin \varphi'(n).$$

Let $\{x_n\colon n<\omega\}$ be an increasing enumeration of X. Put

$$g(n) = \bar{g}(x_n) \quad \text{for } n<\omega.$$

For every slalom $\varphi \in \Phi$ and for almost every $n<\omega$ we have

$$g(n) = \bar{g}(x_n) \notin \varphi'(x_n) = \bigcup_{k\leqslant x_n} \varphi(k) \supseteq \varphi(n).$$

Therefore

$$\forall \varphi \in \Phi \ \forall^\infty n \ g(n) \notin \varphi(n). \ \blacksquare$$

From Theorems 2.1 and 2.2 immediately follows

THEOREM 2.4. *The following conditions are equivalent*:

(1) $U(c)$,

(2) $\forall F \subseteq \omega^\omega, |F|<2^\omega \ \exists g \in \omega^\omega \ \forall f \in F \ \forall^\infty n \ f(n) \neq g(n)$,

(3) Out_2. ■

For $f \in \omega^\omega$ let I_f be the σ-ideal generated by the sets of branches of trees on ω whose n-th level is bounded by $f(n)$. It is easy to see that I_1 is the ideal of countable sets.

In the paper [Ba1] the author asked whether

$$c(I_f) \subseteq I_c \equiv U(c).$$

By an easy generalization of the proof of Theorem 2.1 we get that $\forall f \in \omega^\omega$ $c(I_f) \subseteq I_c \equiv \mathrm{Out}_{1\frac{1}{2}}$. Thus Theorem 2.2 implies

THEOREM 2.5. $\forall f \in \omega^\omega \ c(I_f) \subseteq I_c \equiv U(c)$. ■

Notice that $I_k = \bigcup_{f\in\omega^\omega} I_f$ and for this ideal we have

THEOREM 2.6. $c(I_k) \subseteq I_c \equiv c(I_k) \subseteq I_k$ ($\not\equiv U(c)$). ■

(For the proof see [Ba1])

3. Bases of the ideal of meager sets. In this section we show that $C(c) \equiv \mathrm{Out}^1 \vee \mathrm{Out}_2$. Let us start with

THEOREM 3.1. $C(c) \equiv U(c) \vee \mathrm{In}_1$.

Proof. ← This implication was proved by A. Miller. (see [Mi3]).

→ Assume $\neg U(c)$ & $\neg \mathit{In}_1$.

Let $F \subseteq \omega^\omega$ be a nonmeager sets of power less than 2^ω.

CLAIM 3.2. $\forall g \in \omega^\omega \ \exists f \in F \ \exists^\infty n \ (g(n) = f(n) \ \& \ \forall i<n \ f(i)<n)$.

Proof. Let $Z^g = \{y \in \omega^\omega\colon \forall^\infty n \ (g(n) \neq y(n) \text{ or } \exists i<n \ y(i) \geqslant n)\}$. It is easy to see that Z^g is meager. Any element of $F - Z^g$ has the required properties. ■

It is also not hard to see that the assumption $\neg \mathrm{In}_1$ is equivalent to the existence of a family $G \subseteq \omega^\omega$ of power less than 2^ω such that

$$\forall g \in \omega^\omega \ \exists f \in G \ \forall^\infty n \ g(n) < f(n).$$

Let G be any such family. We can assume that G consists of increasing functions.

We will show that the existence of two such families of functions allows us to construct a base of the ideal of meager sets whose size is less than 2^ω.

Let $\{s_n\colon n<\omega\}$ be an enumeration of $\omega^{<\omega}$. For functions $f \in F$ and $g \in G$ define

$$U_{g,m}^f = \bigcup_{n\geqslant m} [s_n{}^\frown s_{f(n)}{}^\frown s_{f(n+1)}{}^\frown \ldots {}^\frown s_{f(g(n)+m)}] \quad \text{for } m<\omega$$

and

$$U_g^f = \bigcap_m U_{g,m}^f .$$

For $m<\omega$ the sets $U_{g,m}^f$ are open and dense in ω^ω. Let

$$H_g^f = \omega^\omega - U_g^f \quad \text{and} \quad \mathscr{C} = \{H_g^f\colon f\in F,\ g\in G\}.$$

We will show that $\mathscr{C}$ is a base of the ideal I_c. Let $C\subseteq\omega^\omega$ be any meager set and let a sequence $\{C_n\colon n<\omega\}$ be a covering of C by closed and nowhere dense sets. Define a function $f_C\in\omega^\omega$ as follows:

$$f_C(n) = \min\{m\colon \forall i,k<n\ \forall i_1\ldots i_k<n\ [s_{i_1}{}^\frown s_{i_2}\ldots{}^\frown s_{i_k}{}^\frown s_m]\cap C_i = \emptyset\} \quad \text{for } n<\omega .$$

We can find a function $f\in F$ such that

$$\exists^\infty n\big(f_C(n) = f(n)\ \&\ \forall i<n\ f(i)<n\big).$$

Let

$$X = \{n\colon f_C(n) = f(n)\ \&\ \forall i<n\ f(i)<n\}.$$

Let $\hat{X}\in\omega^\omega$ be an increasing enumeration of X. By the properties of the family G we can find a function $g\in G$ such that

$$\forall^\infty n\ \hat{X}(n)<g(n).$$

We will show that

$$C\subseteq \bigcup_n C_n \subseteq H_g^f = \omega^\omega - U_g^f .$$

It is enough to show that for every $n<\omega$ there exists $m<\omega$ such that

$$C_n\cap U_{g,m}^f = \emptyset .$$

Fix $n<\omega$. Let $m>n$ be a positive integer such that

$$\forall n\ \hat{X}(n)<g(n)+m .$$

The set $U_{g,m}^f$ is the union of basic intervals of the form

$$[s_k{}^\frown s_{f(k)}{}^\frown s_{f(k+1)}{}^\frown\ldots{}^\frown s_{f(g(k)+m)}] \quad \text{for } k\geqslant m>n .$$

By the choice of $m<\omega$, for every $k<\omega$ there exists $i\in[k, g(k)+m)$ such that

$$f_C(i) = f(i)\ \&\ \forall j<i\ f(j)<i .$$

Hence, by the definition of f_C,

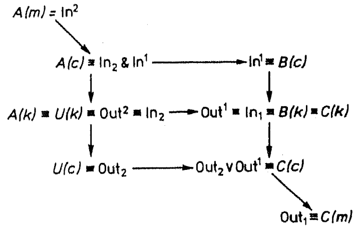
$$[s_k{}^\frown s_{f(k)}{}^\frown\ldots{}^\frown s_{f(g(k)+m)}]\cap C_n = \emptyset \quad \text{for } k>n .$$

This means that $C_n \cap U_{g,m}^f = \emptyset$. Since $n<\omega$ was arbitrary, this finishes the proof. ■

Theorems 3.1 and 2.4 immediately imply

THEOREM 3.2. $C(c)\equiv \mathrm{Out}_2\vee\mathrm{Out}^1$. ■

The following diagram summarizes the contents of this paper.



Remarks. (1) A more general version of this diagram (without its combinatoria part) is called Cichon's diagram (see [Fr]).

(2) Problem (D. Fremlin): Suppose M is a model of ZFC. Assume that there exists a function $g\in\omega^\omega$ such that

$$\forall f\in M\cap\omega^\omega\exists^\infty n\ f(n) = g(n).$$

Does this mean that there exists Cohen real over M?

(3) This paper is a part of my Ph. D. thesis. I would like to express my deepest gratitude to my advisor Wojciech Guzicki.

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