THEOREM. *Let $F\colon (X,*) \to (Y,*)$ be a shape $n$-equivalence between connected locally compact metric spaces. If $\mathrm{sd}_w X \leqslant n-1$ and $\mathrm{sd}_w Y \leqslant n$, then $F$ is an isomorphism in weak shape theory.*

In a similar way we can obtain counterparts to Theorem 2, 3 and 4. We can also state similar theorems in CG-shape theory (without assumption of local compactness).

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# Equivariant shape

by

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**Abstract.** The paper introduces an equivariant shape category $\mathrm{Sh}^G$. Its objects are $G$-spaces, i.e., topological spaces endowed with an action of a given compact group $G$. The category $\mathrm{Sh}^G$ is constructed using the method of resolutions.

**1. Introduction.** The aim of this paper is to define a shape category for $G$-spaces, i.e., topological spaces endowed with an action of a given compact group $G$. In our development we follow the method of resolutions, introduced in the case of ordinary shape by S. Mardešić [14], [15] (also see [16]).

More precisely, in § 4 we define the notion of a $G$-resolution of a $G$-space and we show that every $G$-space admits a $G$-ANR-resolution, i.e., a $G$-resolution consisting of $G$-ANR's (Theorem 1).

In § 5 we prove that every $G$-ANR-resolution induces in the $G$-homotopy category $[\mathrm{Top}^G]$ a $G$-ANR-expansion in the sense of [16, I, § 2.1]. This means that the full subcategory $[\mathrm{ANR}^G]$ of $[\mathrm{Top}^G]$, which consists of spaces having the $G$-homotopy types of $G$-ANR's, is dense in $[\mathrm{Top}^G]$ [16, I, § 2.2].

In [16, I, § 2] a general procedure is described, which associates a shape category $\mathrm{Sh}_{\mathscr{T}\mathscr{P}}$ with every pair consisting of a category $\mathscr{T}$ and of a dense subcategory $\mathscr{P}$. The equivariant shape category $\mathrm{Sh}^G$ is the shape category associated in this way with the pair $\mathscr{T} = [\mathrm{Top}^G]$, $\mathscr{P} = [\mathrm{ANR}^G]$.

Note that $\mathrm{Sh}^G$ coincides with the ordinary shape category Sh if $G = \{e\}$ is the trivial group.

In the realization of the outlined program (just as in the case of ordinary shape) the crucial tool is a $G$-embedding and $G$-extension theorem (Proposition 1). It asserts that every metric $G$-space $X$ equivariantly embeds as a closed subset in a normed linear $G$-space $L$, which is a $G$-absolute extensor. This fact is the result of the work of several authors (see § 3 and for a detailed proof see [6]). Other results on $G$-ANR's needed in this paper were obtained by considering equivariant versions of appropriate proofs of analogous results in the ordinary case. In several instances the proofs given in [16] were appropriate. However, in some cases we had to change the argument to avoid using polyhedra, because a suitable theory of $G$-polyhedra does not seem to have been developed as yet. For $G = \{e\}$ our approach shows how one can define the ordinary shape category Sh using only ANR's.

In the special case of metrizable $G$-spaces $X$ an equivariant shape category has been previously announced by Yu. M. Smirnov [20], [21], [22], who used an equivariant version of the R. H. Fox approach to shape [12]. This amounts to considering the special $G$-ANR resolution of $X \subseteq L$, which consists of all open invariant neighborhoods of $X$ in $L$. Since shape does not depend on the choice of resolutions, Smirnov's category is the restriction of our category $\mathrm{Sh}^G$ to metrizable $G$-spaces.

Recently, I. Pop [18] has defined a shape category $\mathrm{Sh}^G$ for arbitrary topological spaces. However, he assumes that $G$ is a finite group.

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**2. Basic notions and conventions concerning $G$-spaces.** Throughout this paper $G$ denotes a compact (Hausdorff) group, which we keep fixed. An action of $G$ on a topological space $X$ is a (continuous) map $(g, x) \mapsto gx$ of the direct product $G \times X$ into $X$ such that $(g_1 g_2)x = g_1(g_2 x)$ and $ex = x$, where $x \in X$, $g_1, g_2 \in G$ and $e$ is the unity of $G$. A very special example is the trivial action of $G$ on $X$, where $gx = x$ for all $g \in G$, $x \in X$. Another example is the action of the group $G$ on itself defined by $(g, x) \mapsto xg^{-1}$, $g \in G$, $x \in G$ (Alternatively, one can put $(g, x) \mapsto gx$).

By a *$G$-space* we mean a topological space $X$ together with an action of $G$ on $X$. If $X$ and $Y$ are $G$-spaces then so is $X \times Y$, where $g(x, y) = (gx, gy)$, $g \in G$, $(x, y) \in X \times Y$.

By a *normed linear $G$-space* we mean a real normed vector space $X$ endowed with an action of $G$, which is linear, i.e.,

$$g(\lambda x + \mu y) = \lambda gx + \mu gy, \tag{1}$$

where $g \in G$, $x, y \in X$ and $\lambda$, $\mu$ are real numbers.

A subset $A$ of a $G$-space $X$ is called *invariant* provided $g \in G$, $a \in A$ implies $ga \in A$. Clearly, an invariant subset of a $G$-space is itself a $G$-space. If $X$ is a $G$-space and $A \subseteq X$ is an invariant subset, then every neighborhood of $A$ contains an open and invariant neighborhood of $A$ (see [17], Proposition 1.1.14).

A map $f\colon X \to Y$ between $G$-spaces is called a *$G$-map*, or an *equivariant map*, provided $f(gx) = g(f(x))$ for every $g \in G$, $x \in X$. Note that the identity map is equivariant and the composition of equivariant maps is equivariant. Therefore, $G$-spaces and equivariant maps form a category, which we denote by $\mathrm{Top}^G$.

Let $X$ and $Y$ be $G$-spaces and let $f_0, f_1\colon X \to Y$ be $G$-maps. A *$G$-homotopy* or *equivariant homotopy*, from $f_0$ to $f_1$ is a homotopy $F\colon X \times I \to Y$ from $f_0$ to $f_1$, which is a $G$-map. Hereby we assume that $G$ acts trivially on $I$ so that $g(x, t) = (gx, t)$, $(x, t) \in X \times I$. If for $G$-maps $f_0, f_1$ there is a $G$-homotopy from $f_0$ to $f_1$, we say that $f_0$ and $f_1$ are *$G$-homotopic* and we write $f_0 \simeq_G f_1$.



The relation $\simeq_G$ is an equivalence relation and we denote the class containing a $G$-map $f$ by $[f]$. The relation $\simeq_G$ is compatible with the composition, i.e., $f_0 \simeq_G f_1\colon X \to X'$ and $f_0' \simeq_G f_1'\colon X' \to X''$ implies $f_0' f_0 \simeq_G f_1' f_1$. Therefore, one can define *composition* of classes of $G$-homotopic $G$-maps $[f]\colon X \to X'$, $[f']\colon X' \to X''$ by composing representatives, i.e., $[f'][f] = [f'f]$. In this way one obtains a category $[\mathrm{Top}^G]$, whose objects are $G$-spaces and whose morphisms are classes of $G$-homotopic $G$-maps. There is a homotopy functor: $\mathrm{Top}^G \to [\mathrm{Top}^G]$, which keeps the objects fixed and takes $G$-maps $f$ into their $G$-homotopy classes $[f]$.

For further information concerning $G$-spaces see [17], [9] and [10].

**3. Basic facts concerning $G$-ANR's.** Let $Z$ be a $G$-space and let $Y \subseteq Z$ be an invariant subset. A *$G$-retraction* of $Z$ to $Y$ is a $G$-map $r\colon Z \to Y$ such that $r|Y = 1_Y$.

A $G$-space $Y$ is called a *$G$-absolute neighborhood retract* or a $G$-ANR (*$G$-absolute retract* or a $G$-AR), provided $Y$ is metrizable and whenever $Y$ is a closed invariant subset of a metrizable $G$-space $Z$, then there exist an invariant neighborhood $U$ of $Y$ and a $G$-retraction $r\colon U \to Y$ (there exists a $G$-retraction $r\colon Z \to Y$). For $G = \{e\}$ this definition yields the usual notion of an ANR (AR) for metric spaces.

A $G$-space $Y$ is called a *$G$-absolute neighborhood extensor* or a $G$-ANE (*$G$-absolute extensor* or a $G$-AE), provided for any metrizable $G$-space $X$ and any closed invariant subset $A \subseteq X$, every equivariant map $f\colon A \to Y$ admits an equivariant extension $\tilde{f}\colon U \to Y$, where $U$ is an invariant neighborhood $U$ of $A$ in $X$ ($\tilde{f}\colon X \to Y$).

It is easy to see that an open invariant subset of a $G$-ANE is itself a $G$-ANE. Moreover, the product $Y_1 \times Y_2$ of two $G$-ANE's is a $G$-ANE.

Crucial for our development is the following $G$-embedding and $G$-extension theorem.

PROPOSITION 1. *For every metrizable $G$-space $X$ there exists a normed linear $G$-space $L$ such that the weight*

$$w(L) \leqslant \max\{w(G), w(X), \aleph_0\}, \tag{1}$$

*$L$ is a $G$-AE and there exists an equivariant embedding $i\colon X \to L$, whose image $i(X)$ is closed in $L$.*

The proposition is proved in [6] and it easily follows from the following three lemmas.

LEMMA 1. *Let $T$ be a compact $G$-space and let $Y$ be a metric (normed vector) space. Let $C(T, Y)$ be the space of all continuous maps $\varphi\colon T \to Y$ endowed with the metric (norm)*

(2) $\varrho(\varphi, \varphi') = \sup\{d(\varphi(t), \varphi'(t)) \colon t \in T\}$
((2)′ $\|\varphi\| = \sup\{|\varphi(t)| \colon t \in T\}$).

*Then $C(T, Y)$ is a (linear) $G$-space with the action of $G$ defined by*

(3) $$(g\varphi)(t) = \varphi(g^{-1}t), \quad g \in G,\ \varphi \in C(T, Y),\ t \in T,$$

*and the weight $w(C(T, Y))$ satisfies*

(4) $$w(C(T, Y)) \leqslant \max\{w(T), w(Y), \aleph_0\}.$$

*Moreover, if $Y$ is an* ANR (AR), *then $C(T, Y)$ is a $G$-*ANE *($G$-*AE*) for $p$-paracompact $G$-spaces.*

(4) is proved in [11] (Ch. XII, Theorem 5.2). The last assertion follows from Theorem 1 of [3], using Yu. T. Lisica's theorem ([13], Theorem 1) that ANR's (AR's) are ANE's (AE's) for $p$-paracompact spaces.

LEMMA 2. *Let $X$ be a metrizable $G$-space and let $C(G, X)$ be the $G$-space of continuous maps endowed with the action* (3). *Let $i \colon X \to C(G, X)$ be the map defined by*

(5) $$(i(x))(t) = tx, \quad t \in G,\ x \in X.$$

*Then $i$ is an equivariant embedding and $i(X)$ is closed in $C(G, X)$.*

This was proved by Yu. M. Smirnov ([19], Theorems 4 and 5). Note that the action of $G$ on itself maps $(g, t)$ to $tg^{-1}$, so that (3) becomes $(g\varphi)(t) = \varphi(tg)$, $g, t \in G$, $\varphi \in C(G, X)$.

LEMMA 3. *Every metrizable space $Y$ can be embedded as a closed subset in a normed vector space $L$.*

This is a result due to R. F. Arens and J. Eells, Jr [7] (see also [8], Ch. II, Corollary 1.1).

In order to prove Proposition 1, one first embeds $X$ equivariantly as a closed subset of $C(G, X)$ (Lemma 2). By Lemma 3, one can assume that $X$ is a closed subset of a normed vector space $M$. One can also assume that $X$ spans $M$ and therefore $w(M) \leqslant \max(w(X), \aleph_0)$. Then $C(G, X)$ is a closed and equivariant subset of the normed linear $G$-space $L = C(G, M)$. By the Dugundji extension theorem (see [11], IX, Theorem 6.1. or [16], I, § 3.1, Theorem 3), $M$ is an AR. Therefore, by Lemma 1, $L$ is a $G$-AE satisfying (1).

Remark 1. If instead of Lemma 3 one uses the well-known Kuratowski–Wojdysławski embedding theorem (see [16], I, § 3.1, Theorem 2), one obtains a weaker version of Proposition 1, which however, suffices for all our further arguments.

Remark 2. If $G$ is a compact Lie group, then every normed linear $G$-space is a $G$-AE [5]. This fact and the equivariant version of Lemma 3 (established in [6]) yield for compact Lie groups $G$ a shorter proof of Proposition 1. However, Antonian has recently shown [6] that for any compact group $G$, which is not a Lie group, there exists a normed linear $G$-space which is not even a $G$-ANE.

PROPOSITION 2. *A metrizable $G$-space $Y$ is a $G$-*ANR *($G$-*AR*) if and only if it is a $G$-*ANE *($G$-*AE*).*

This is proved by Antonian in [6] (for complete metrizable spaces see [1], [2]) and it is an immediate consequence of Proposition 1.

Let: $\mathscr{V}$ be a covering of a space $Y$. We say that maps $f, f' \colon X \to Y$ are $\mathscr{V}$-near provided every $x \in X$ admits a $V \in \mathscr{V}$ such that $f(x), f'(x) \in V$. For a homotopy $F \colon X \times I \to Y$ we say that it is a $\mathscr{V}$-homotopy provided every $x \in X$ admits a $V \in \mathscr{V}$ such that $F(x \times I) \subseteq V$.

PROPOSITION 3. *Let $Y$ be a $G$-*ANR. *Then every open covering $\mathscr{V}$ of $Y$ admits an open covering $\mathscr{V}'$ of $Y$ such that whenever $f_0, f_1 \colon X \to Y$ are $\mathscr{V}'$-near $G$-maps from an arbitrary $G$-space $X$ to $Y$, then there exists an equivariant $\mathscr{V}$-homotopy $F$ from $f_0$ to $f_1$. Moreover, if for a given $x \in X$, $f_0(x) = f_1(x)$, then $F|x \times I$ is constant.*

Proof. By Proposition 1, we can assume that $Y$ is an invariant closed subset of a normed linear $G$-space $L$. Since $Y$ is a $G$-ANR, there exists an open invariant neighborhood $U$ of $Y$ in $L$ and an equivariant retraction $r \colon U \to Y$. Let $\mathscr{W}$ be an open covering of $U$, which refines $r^{-1}(\mathscr{V})$ and consists of balls from $L$. Put $\mathscr{V}' = \{W \cap Y \colon W \in \mathscr{W}\}$. We claim that $\mathscr{V}'$ has the desired properties.

Let $f_0, f_1 \colon X \to Y \subseteq L$ be $\mathscr{V}'$-near $G$-maps. We define a homotopy $\Phi \colon X \times I \to L$ from $f_0$ to $f_1$ by putting

(6) $$\Phi(x, t) = (1-t)f_0(x) + tf_1(x), \quad (x, t) \in X \times I.$$

For every $x \in X$ there is a $W \in \mathscr{W}$ which contains $f_0(x)$ and $f_1(x)$. Since $W$ is convex, we conclude that $\Phi(x \times I) \subseteq W \subseteq U$. However, $W$ is contained in a set $r^{-1}(V)$, where $V \in \mathscr{V}$. Therefore $F = r\Phi \colon X \times I \to Y$ is a $\mathscr{V}$-homotopy from $f_0$ to $f_1$. Moreover, if $f_0(x) = f_1(x)$, then $F|x \times I$ is constant. $F$ is equivariant because $f_0, f_1$ and $r$ are equivariant and $G$ acts linearly on $L$, which makes also $\Phi$ equivariant.

PROPOSITION 4. *Let $X$ be a metrizable $G$-space, let $A \subseteq X$ be an invariant closed subset of $X$ and let $Y$ be a $G$-*ANR. *Moreover, let $f_0, f_1 \colon X \to Y$ be equivariant maps and let $F \colon A \times I \to Y$ be an equivariant homotopy from $f_0|A$ to $f_1|A$. Then there exists an invariant neighborhood $V$ of $A$ in $X$ and there exists an equivariant homotopy $\tilde{F} \colon V \times I \to Y$ from $f_0|V$ to $f_1|V$, which extends $F$.*

Proof. The set $T = (A \times I) \cup (X \times 0) \cup (X \times 1)$ is clearly a closed invariant subset of $X \times I$. Consider the equivariant map $f \colon T \to Y$ defined by

(7) $$f(a, t) = F(a, t), \quad (a, t) \in A \times I,$$

(8) $$f(x, 0) = f_0(x), \quad f(x, 1) = f_1(x), \quad x \in X.$$

Since $Y$ is a $G$-ANE, $f$ extends to an equivariant map $\tilde{f} \colon U \to Y$, where $U$ is an invariant neighborhood of $T$ in $X \times I$. Using compactness of $I$ one can find a neighborhood $V$ of $A$ in $X$ such that $V \times I \subseteq U$. One can also achieve that $V$ be an open invariant neighborhood of $A$ in $X$. Then $\tilde{F} = \tilde{f}|V \times I$ has all the desired properties.

For a survey of results on $G$-ANR's see [4].

**4. Equivariant resolutions.** In this section we consider inverse systems $\underline{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$ in the category $\mathrm{Top}^G$. This means that every $X_\lambda$, $\lambda \in \Lambda$, is a $G$-space and every $p_{\lambda\lambda'}\colon X_{\lambda'} \to X_\lambda$, $\lambda \leqslant \lambda'$, is a $G$-map. If every $X_\lambda$, $\lambda \in \Lambda$, is a $G$-ANR, we speak of a $G$-ANR-system. In particular, a single $G$-space $X$ can be viewed as an inverse system in $\mathrm{Top}^G$. A morphism of pro-$\mathrm{Top}^G$ $\underline{p}\colon X \to \underline{X}$ consists of $G$-maps $p_\lambda\colon X \to X_\lambda$, $\lambda \in \Lambda$, such that $p_\lambda = p_{\lambda\lambda'} p_{\lambda'}$, $\lambda \leqslant \lambda'$ (see [16], I. § 1.1).

DEFINITION 1. A morphism $\underline{p}\colon X \to \underline{X}$ of pro-$\mathrm{Top}^G$ is called a *$G$-resolution* of the $G$-space $X$ or an *equivariant resolution*, provided for every $G$-ANR $P$ and every open covering $\mathscr{V}$ of $P$ the following two conditions are satisfied:

(GR1) If $f\colon X \to P$ is a $G$-map, then there is a $\lambda \in \Lambda$ and a $G$-map $h\colon X_\lambda \to P$ such that $hp_\lambda$ and $f$ are $\mathscr{V}$-near.

(GR2) There exists an open covering $\mathscr{V}'$ of $P$ with the following property. Whenever $\lambda \in \Lambda$ and $h_0, h_1\colon X_\lambda \to P$ are equivariant maps such that $h_0 p_\lambda$, $h_1 p_\lambda$ are $\mathscr{V}'$-near maps, then there exists a $\lambda' \geqslant \lambda$ such that $h_0 p_{\lambda\lambda'}$, $h_1 p_{\lambda\lambda'}$ are $\mathscr{V}$-near maps.

If in a $G$-resolution every $X_\lambda$ is a $G$-ANR, then we speak of a *$G$-ANR-resolution*. Generalizing ([14], Theorem 12), we have the following result.

THEOREM 1. *Every $G$-space $X$ admits a $G$-ANR-resolution $\underline{p}\colon X \to \underline{X}$.*

In the proof we need the following lemma (generalizing [14], Lemma 1).

LEMMA 4. *Let $X$ be a $G$-space, $Y$ a $G$-ANR and $f\colon X \to Y$ an equivariant map. Then there exists a $G$-ANR $Z$ of weight*

$$w(Z) \leqslant \max\{w(G), w(X), \aleph_0\} \tag{1}$$

*and there exist equivariant maps $h\colon X \to Z$, $k\colon Z \to Y$ such that $f = kh$.*

Proof. Using Proposition 1, we can assume that $f(X)$ is an invariant closed subset of a normed linear $G$-space $L$, which is a $G$-ANR and satisfies

$$w(L) \leqslant \{\max w(G), w(f(X)), \aleph_0\}. \tag{2}$$

Note that for metric spaces $M$ weight $w(M)$ coincides with the degree of separability $s(M)$ (which is the least cardinal of a dense subset). Therefore,

$$w(f(X)) = s(f(X)) \leqslant s(X) \leqslant w(X). \tag{3}$$

Since $Y$ is a $G$-ANE, the inclusion $f(X) \to Y$ extends to an equivariant map $h\colon Z \to Y$, where $Z$ is an open invariant neighborhood of $f(X)$ in $L$. Consequently, $Z$ is a G-ANR. Let $k\colon X \to Z$ be the composition of $f\colon X \to f(X)$ with the inclusion map $f(X) \to Z$. Clearly, $k$ is also an equivariant map and $hk = f$. Moreover, $w(Z) \leqslant w(L)$ so that (2) and (3) imply (1).

Proof of Theorem 1. Let $P, P'$ be $G$-spaces and let $p\colon X \to P$, $p'\colon X \to P'$ be equivariant maps. We say that $p$ and $p'$ are *equivalent* provided there is an equivariant homeomorphism $h\colon P \to P'$ such that $hp = p'$. Let $\Gamma$ consist of all equivalence classes of $G$-maps $p\colon X \to P$, where $P$ is a $G$-ANR of weight

$$w(P) \leqslant \max\{w(G), w(X), \aleph_0\} = \tau. \tag{4}$$



$\Gamma$ is a set because every metric space of weight $\leqslant \tau$ embeds in the cube $I^\tau$. For every $\gamma \in \Gamma$ we choose a $G$-map $p_\gamma\colon X \to P_\gamma$, where $P_\gamma$ is a $G$-ANR of weight $w(P_\gamma) \leqslant \tau$.

Let $\Delta$ be the set of all finite subsets $\delta = \{\gamma_1, \ldots, \gamma_n\}$ of $\Gamma$. We order $\Delta$ by inclusion and thus obtain a directed set. For $\delta = \{\gamma_1, \ldots, \gamma_n\} \in \Delta$ we put $P_\delta = P_{\gamma_1} \times \ldots \times P_{\gamma_n}$. Letting $G$ act on $P_\delta$ by $g(x_1, \ldots, x_n) = (gx_1, \ldots, gx_n)$, $g \in G$, $P_\delta$ becomes a $G$-ANR of weight $w(P_\delta) \leqslant \tau$. If $\delta = \{\gamma_1, \ldots, \gamma_n\} \leqslant \{\gamma_1, \ldots, \gamma_n, \ldots, \gamma_m\} = \delta'$, we define $p_{\delta\delta'}\colon P_{\delta'} \to P_\delta$ as the natural projection. We also define $p_\delta\colon X \to P_\delta$ as the map $p_\delta = p_{\gamma_1} \times \ldots \times p_{\gamma_n}$. Clearly, $p_{\delta\delta'}$ and $p_\delta$ are $G$-maps, $p_{\delta\delta'} p_{\delta'\delta''} = p_{\delta\delta''}$, $\delta \leqslant \delta' \leqslant \delta''$ and $p_\delta = p_{\delta\delta'} p_{\delta'}$, $\delta \leqslant \delta'$. Therefore, $\underline{P} = (P_\delta, p_{\delta\delta'}, \Delta)$ is a $G$-ANR-system and $\underline{p} = (p_\delta, \Delta)\colon X \to \underline{P}$ is a morphism of pro-$\mathrm{Top}^G$. Using Lemma 4, one readily sees that $\underline{p}$ satisfies condition (GR1).

In order to obtain property (GR2) we must modify $\underline{p}\colon X \to \underline{P}$ as follows. Let $\Lambda$ be the set of all pairs $\lambda = (\delta, U)$, where $\delta \in \Delta$ and $U$ is an invariant open neighborhood of $p_\delta(X)$ in $P_\delta$. We put $X_\lambda = U$ and observe that $X_\lambda$ is a $G$-ANR. We order $\Lambda$ by putting $\lambda = (\delta, U) \leqslant (\delta', U') = \lambda'$ whenever $\delta \leqslant \delta'$ and $p_{\delta\delta'}(U') \subseteq U$. We then define $p_{\lambda\lambda'}\colon X_{\lambda'} \to X_\lambda$ to be the map $p_{\delta\delta'}|U'\colon U' \to U$. Clearly, $p_{\lambda\lambda'}$ is a $G$-map, $\underline{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$ is a $G$-ANR-system and $\underline{p} = (p_\lambda, \Lambda)\colon X \to \underline{X}$ is a morphism of pro-$\mathrm{Top}^G$.

The morphism $\underline{p}\colon X \to \underline{X}$ still has property (GR1), because we have only extended $\underline{P}$ to $\underline{X}$. It has also property (GR2). Indeed, if $P$ is a $G$-ANR and $\mathscr{V}$ is an open covering of $P$, then $\mathscr{V}' = \mathscr{V}$ has the desired property (as seen by an argument similar to the one used in the proof of Theorem 13 of [14]).

**5. Equivariant expansions.** We define *equivariant expansions* of $G$-spaces by specializing the general notion of expansion with respect to a category $\mathscr{T}$ and its full subcategory $\mathscr{P}$ (see [16], I, § 2). In our case this is the category $[\mathrm{Top}^G]$ (see § 2) and its full subcategory $[\mathrm{ANR}^G]$, which consists of $G$-spaces having the $G$-homotopy types of $G$-ANR's.

DEFINITION 2. A *$G$-expansion*, or *equivariant expansion*, of a $G$-space $X$ consists of an inverse system $[\underline{X}] = (X_\lambda, [p_{\lambda\lambda'}], \Lambda)$ in $[\mathrm{Top}^G]$ and of a morphism $[\underline{p}]\colon X \to [\underline{X}]$ in pro-$[\mathrm{Top}^G]$, i.e., a collection of $G$-homotopy classes $[p_\lambda]$ of $G$-maps $p_\lambda\colon X \to X_\lambda$, $\lambda \in \Lambda$ such that $p_{\lambda\lambda'} p_{\lambda'} \simeq_G p_\lambda$, $\lambda \leqslant \lambda'$. Moreover, the following two conditions must be satisfied:

(GE1) If $P$ is a $G$-ANR and $f\colon X \to P$ is a $G$-map, then there is a $\lambda \in \Lambda$ and there is a $G$-map $h\colon X_\lambda \to P$ such that $hp_\lambda \simeq_G f$.

(GE2) If $P$ is a $G$-ANR, $\lambda \in \Lambda$, and $h_0, h_1\colon X_\lambda \to P$ are $G$-maps satisfying $h_0 p_\lambda \simeq_G h_1 p_\lambda$, then there is a $\lambda' \geqslant \lambda$ such that

$$h_0 p_{\lambda\lambda'} \simeq_G h_1 p_{\lambda\lambda'}.$$

A $G$-ANR-expansion $[\underline{p}]$ is a $G$-expansion such that all $X_\lambda$ are $G$-ANR's.

Every inverse system $\underline{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$ in the category $\mathrm{Top}^G$ induces an inverse system $[\underline{X}] = (X_\lambda, [p_{\lambda\lambda'}], \Lambda)$ in the category $[\mathrm{Top}^G]$. Moreover, every morphism $\underline{p} = (p_\lambda, \Lambda)$: $X \to \underline{X}$ in pro-$\mathrm{Top}^G$ induces a morphism $[\underline{p}] = ([p_\lambda], \Lambda)$: $X \to [\underline{X}]$ in pro-$[\mathrm{Top}^G]$. In our development of equivariant shape the next result is fundamental.

THEOREM 2. *Let $X$ be a $G$-space. If $\underline{p}$: $X \to \underline{X}$ is a $G$-resolution of $X$, then the induced morphism $[\underline{p}]$: $X \to [\underline{X}]$ is a $G$-expansion of $X$.*

In order to prove Theorem 2 we need the following lemma which generalizes [16], I, § 4.1, Lemma 1.

LEMMA 5. *Let $X$ be a $G$-space, let $P, P'$ be $G$-ANR's and let $f$: $X \to P'$, $h_0, h_1$: $P' \to P$ be $G$-maps such that*

$$(1) \qquad h_0 f \simeq_G h_1 f.$$

*Then there exist a $G$-ANR $P''$ and $G$-maps $f'$: $X \to P''$, $h$: $P'' \to P'$ such that*

$$(2) \qquad hf' = f,$$

$$(3) \qquad h_0 h \simeq_G h_1 h.$$

Proof. By (1), there exists an equivariant homotopy $Q$: $X \times I \to P$ from $h_0 f$ to $h_1 f$. Consider the space $C(I, P)$ of all continuous maps $\varphi$: $I \to P$ endowed with the metric 3 (2). The action of $G$ on $P$ induces an action of $G$ on $C(I, P)$ given by

$$(4) \qquad (g\varphi)(t) = g(\varphi(t)), \quad g \in G,\ \varphi \in C(I, P),\ t \in I$$

(see [1], Lemma 1). Let $q$: $X \to C(I, P)$ be the map defined by

$$(5) \qquad q(x)(t) = Q(x, t), \quad x \in X,\ t \in I.$$

The continuity of $q$ follows from ([11], XII, Theorem 3.1.1). By (4),

$$(6) \qquad q(gx)(t) = Q(gx, t) = gQ(x, t) = g(q(x)(t)) = (gq(x))(t), \qquad x \in X,\ t \in I,$$

which means that $q$ is an equivariant map. Also notice that

$$(7) \qquad q(x)(0) = h_0 f(x), \quad q(x)(1) = h_1 f(x).$$

We now define $f'$: $X \to P' \times C(I, P)$ by

$$(8) \qquad f'(x) = (f(x), q(x)), \quad x \in X.$$

Clearly, $f'$ is an equivariant map. If we denote by $h$: $P' \times C(I, P) \to P'$ the first projection, then $h$ is also equivariant and (2) holds.

We now define $P'' \subseteq P' \times C(I, P)$ by

$$(9) \qquad P'' = \{(y, \varphi) \in P' \times C(I, P)\colon \varphi(0) = h_0(y),\ \varphi(1) = h_1(y)\}.$$



(7) and (8) show that $f'$ is actually a map of $X$ into $P''$. $P''$ is an invariant subset of $P' \times C(I, P)$, because $(y, \varphi) \in P''$ implies

$$(10) \qquad (g\varphi)(0) = g(\varphi(0)) = g(h_0(y)) = h_0(gy), \quad (g\varphi)(1) = h_1(gy),$$

so that $g(y, \varphi) = (gy, g\varphi) \in P''$, $g \in G$.

We claim that the restrictions $h_0 h|P''$ and $h_1 h|P''$ are $G$-homotopic maps. Indeed, let $K$: $(P' \times C(I, P)) \times I \to P$ be the map given by

$$(11) \qquad K((y, \varphi), t) = \varphi(t), \quad y \in P',\ \varphi \in C(I, P),\ t \in I.$$

The continuity of $K$ follows from ([11], XII, Theorem 2.4.2). $K$ is an equivariant homotopy because

$$(12) \qquad K(g(y, \varphi), t) = K((gy, g\varphi), t) = (g\varphi)(t) = g(\varphi(t)) = gK((y, \varphi), t).$$

Restricting $K$ to $P''$ we obtain an equivariant homotopy $P'' \times I \to P$ from $h_0 h$ to $h_1 h$, because $(y, \varphi) \in P''$ implies

$$(13) \qquad K((y, \varphi), 0) = \varphi(0) = h_0(y) = h_0 h(y, \varphi),$$

$$(14) \qquad K((y, \varphi), 1) = h_1 h(y, \varphi).$$

We have thus also verified (3) and the proof of Lemma 4 will be completed if we show that $P''$ is a $G$-ANR or equivalently a $G$-ANE.

Let $Z$ be a metric $G$-space, let $A \subseteq Z$ be a closed invariant subset of $Z$ and let $k$: $A \to P'' \subseteq P' \times C(I, P)$ be an equivariant map. We must find an invariant neighborhood $V$ of $A$ in $Z$ and an equivariant extension $\tilde{k}$: $V \to P''$ of $k$.

Denote by $h'$: $P' \times C(I, P) \to C(I, P)$ the second projection and observe that it is an equivariant map. Therefore, $h'k$: $A \to C(I, P)$ is also an equivariant map and it induces a homotopy $K$: $A \times I \to P$, defined by

$$(15) \qquad K(a, t) = (h'k(a))(t), \quad (a, t) \in A \times I.$$

The continuity of $K$ follows from ([11], XII, Theorem 3.1.2). $K$ is equivariant, because

$$(16) \qquad K(g(a, t)) = K(ga, t) = (h'k(ga))(t) = (g(h'k(a)))(t) = g(h'k(a)(t)) = gK(a, t), \qquad (a, t) \in A \times I,\ g \in G.$$

By (9), we have

$$(17) \qquad (h'k(a))(0) = h_0 hk(a), \quad (h'k(a))(1) = h_1 hk(a), \quad a \in A,$$

because $k(a) \in P''$. This shows that $K$ is a $G$-homotopy from $h_0 hk$ to $h_1 hk$.

Since $P'$ is a $G$-ANE and $hk$: $A \to P'$ is an equivariant map, there exists an invariant neighborhood $U$ of $A$ in $Z$ and there exists an equivariant map $\tilde{k}'$: $U \to P'$, which extends $hk$. One can now apply Proposition 4 to $h_0 \tilde{k}', h_1 \tilde{k}'$ and $K$ and conclude that there exists an equivariant neighborhood $V$ of $A$ in $U$ and a $G$-homotopy $\tilde{K}$: $V\times I\to P$ from $h_0\tilde{k}'|V$ to $h_1\tilde{k}'|V$. This $G$-homotopy induces a map $\tilde{k}''$: $V\to C(I,P)$, given by

$$\tilde{k}''(z)(t)=\tilde{K}(z,t)\,,\quad z\in V,\ t\in I. \tag{18}$$

The continuity of $\tilde{k}''$ follows from ([11], XII, Theorem 3.1.1). $\tilde{k}''$ is equivariant because

$$\tilde{k}''(gz)(t)=\tilde{K}(gz,t)=g\tilde{K}(z,t)=g(\tilde{k}''(z)(t)) = (g\tilde{k}''(z))(t)\,,\quad g\in G,\ z\in V,\ t\in I. \tag{19}$$

$\tilde{k}''$ is an extension of $h'k$ because

$$(\tilde{k}''(a))(t)=\tilde{K}(a,t)=K(a,t)=(h'k(a))(t)\,,\quad a\in A. \tag{20}$$

Consequently, if we define $\tilde{k}$: $V\to P'\times C(I,P)$ by

$$\tilde{k}(z)=(\tilde{k}'(z),\tilde{k}''(z))\,,\quad z\in V, \tag{21}$$

then $\tilde{k}$ is an equivariant map, which extends $k$.

Finally, $\tilde{k}(z)\in P''$ for every $z\in V$ because

$$\tilde{k}''(z)(0)=\tilde{K}(z,0)=h_0\tilde{k}'(z)\,, \tag{22}$$

$$\tilde{k}''(z)(1)=\tilde{K}(z,1)=h_1\tilde{k}'(z)\,. \tag{23}$$

This completes the proof of Lemma 4.

The proof of Theorem 2 now proceeds in the same way as the proof of the analogous result in the case of ordinary shape, i.e., in the case $G=\{e\}$ (see the proof of I, § 6.1, Theorem 2 of [16]).

**6. The equivariant shape category.** We will now define the $G$-shape category or equivariant shape category $\mathrm{Sh}^G$. We apply the construction of the shape category associated with an arbitrary category $\mathcal{T}$ and a full subcategory $\mathcal{P}$ (see [16], I, § 2.3). The only requirement needed for the construction is that $\mathcal{P}$ be dense in $\mathcal{T}$ in the sense that every object of $\mathcal{T}$ admits a $\mathcal{P}$-expansion.

In the case of equivariant shape we take as $\mathcal{T}$ the category $[\mathrm{Top}^G]$ and as $\mathcal{P}$ the category $[\mathrm{ANR}^G]$ (see § 5). Therefore, we only need the following theorem.

THEOREM 3. *Every $G$-space $X$ admits a $G$-ANR-expansion.*

Theorem 3 is an immediate consequence of Theorems 1 and 2.

According to [16], I, § 2.3, the objects of $\mathrm{Sh}^G$ are all $G$-spaces. The morphisms of $\mathrm{Sh}^G$ between $G$-spaces $X$ and $Y$ are given by triples $([\underline{p}],[\underline{q}],[\underline{f}])$, where $[\underline{p}]$: $X\to\underline{X}$, $[\underline{q}]$: $Y\to\underline{Y}$ are $G$-ANR-expansions of $X$ and $Y$ respectively and $[\underline{f}]$: $[\underline{X}]\to[\underline{Y}]$ is a morphism of pro-$[\mathrm{Top}^G]$ (see [16], I, § 1.1). In particular, one can take for $[\underline{p}]$ and $[\underline{q}]$ morphisms induced by $G$-ANR-resolutions $\underline{p}$ and $\underline{q}$.



One also has a $G$-shape functor $[\mathrm{Top}^G]\to\mathrm{Sh}^G$ (see [16], I, § 2.3). Therefore, if $G$-spaces $X$ and $Y$ have the same $G$-homotopy type, they also have the same $G$-shape, $\mathrm{sh}^G(X)=\mathrm{sh}^G(Y)$, i.e., they are isomorphic objects of $\mathrm{Sh}^G$. Already for $G=\{e\}$ the converse does not hold. However, for $G$-ANR's $X$ and $Y$ $G$-shape morphisms $X\to Y$ coincide with $G$-homotopy classes of $G$-maps and, therefore, for $G$-ANR's classification up to $G$-shape coincides with the classification up to $G$-homotopy type.

Remark 3. There is an alternative proof of Theorem 3 which does not use $G$-resolutions. One uses instead the necessary and sufficient conditions for a subcategory $\mathcal{P}$ to be dense in a category $\mathcal{T}$ (see [16], I, § 2.2, Theorem 2).

Remark 4. We have defined the $G$-shape category $\mathrm{Sh}^G$ for any compact group $G$. However, there are reasons to believe that equivariant shape theory as defined in this paper will prove useful primarily in the case when $G$ is a compact Lie group.

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# Combinatorial aspects of measure and category

by

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**Abstract.** In this paper we study set-theoretical properties of the ideal of meager sets. We prove that the real line is not the union of less than $2^\omega$ meager sets iff for every family of reals of cardinality less than $2^\omega$ there exists an "infinitely equal" real. We also find a characterization of uniformity of the ideal of meager sets.

**0. Preface.** The purpose of this paper is to give combinatorial description of some elementary properties of the ideal of meager sets and the ideal of null sets. In fact, we deal only with the ideal of meager sets. We find a characterization of basic set-theoretical properties of this ideal. For a more complete picture we also formulate, in the same language, the already known characterization of the analoguous properties of the ideal of null sets.

Let us start with the following definition.

DEFINITION. For any ideal $I \subseteq P(R)$ let $c(I)$ denote the smallest $2^\omega$-complete ideal containing $I$.

We define the following sentences.

$$A(I) \equiv c(I) \subseteq I,$$

$$B(I) \equiv R \notin c(I),$$

$$U(I) \equiv \forall_{|X|<2^\omega} X \subseteq R \;\; X \in I,$$

$$C(I) \equiv \forall_{|\mathscr{F}|<2^\omega} \mathscr{F} \subseteq I \; \exists H \in I \; \forall F \in \mathscr{F} \;\; H - F \neq \emptyset .$$

Let $I_c$ and $I_m$ denote the ideal of meager subsets of $R$ and the ideal of Lebesgue measure zero sets, respectively. Let $I_k$ denote the $\sigma$-ideal generated by compact subsets of $\omega^\omega$. We are interested in properties $A$, $B$, $U$ and $C$ for those ideals. For simplicity let $A(c)$ abbreviate $A(I_c)$, $B(k)$ stand for $B(I_k)$ and so on. It is well known that the properties $A$, $B$, $U$ and $C$ are equivalent when stated for the real line $R$, the Baire space $\omega^\omega$ or the Cantor set $2^\omega$.

Throughout the paper we use the standard terminology. For any set $X$ we