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Received 30 January 1985;
in revised form 18 June 1985

The Hurewicz and Whitehead theorems with compact carriers

by

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Abstract. We prove analogues of the classical Hurewicz and Whitehead theorems for Borsuk's weak shape theory or, more generally, for the category generated by the homotopy category of pointed polyhedra. We also give a certain geometrical application of the modified Hurewicz theorem.

Introduction. The dual notion to that of a pro-category will be called an *in-category*. By induction, we say that a category pro-$\mathscr{C}'$ or in-$\mathscr{C}'$ is *k-generated* by a category $\mathscr{C}$ whenever provided $\mathscr{C}'$ is $(k-1)$-generated by $\mathscr{C}$ (we assume that $\mathscr{C}$ is 0-generated by $\mathscr{C}$).

The classical Hurewicz and Whitehead theorems have their analogues in shape theory and pro-homotopy theory (for example see [M–S]). We will prove analogues of these theorems in a more general case, for any category generated by the homotopy category of pointed polyhedra HPol_*. As a consequence, we obtain modified Hurewicz and Whitehead theorems for Borsuk's weak shape theory (i.e. shape theory with compact carriers) and for compactly generated shape theory. Under more restrictive assumptions a Whitehead type theorem for compactly generated shape theory has been proved previously by T. J. Sanders [Sa2].

The inspiration to prove a Hurewicz type theorem in Borsuk's weak shape theory was the following question of H. Toruńczyk [T].

Question 1. *Let A be a subset of R^n such that every map $I^2 \to R^n$ is approximable by mappings with images missing A. Let $f\colon \partial I^s \to R^n$ be a map which satisfies $\operatorname{im}(f) \cap A = \emptyset$, where $s+\dim A < n$. Is there a compact set $C \subset R^n \setminus A$ such that f is null homotopic in every neighborhood U of C in R^n?*

If A is σ-compact then $\{g \in C[I^s, R^n] \mid \operatorname{im}(g) \cap A = \emptyset\}$ is dense in $C[I^s, R^n]$ (see [Š]). This needs be shown for compacta only and follows by induction on s using Alexander duality and Hurewicz theorem. H. Toruńczyk asked if one can prove a Hurewicz type theorem in Borsuk's weak shape theory. He suggested a positive answer to Question 1 by using such a modified Hurewicz type theorem and Sitnikov duality [M].

Recall that a pointed metric space $(X, x_0) \subset (M, x_0) \in \mathrm{AR}(\mathfrak{M})$ is *approximatively k-connected* [B] if for every compactum B in X, $x_0 \in B$, there is a compactum B' in X such that for every neighborhood U' of B' in M there is a neighborhood U of B in M such that every map of a pointed k-dimensional sphere (S^k, s_0) into (U, x_0) is homotopically trivial in (U', x_0).

Using Theorem 11.19 of [M] and the Hurewicz type theorem for Borsuk's weak shape theory (§ 3, Theorem 1) one can prove the following proposition.

PROPOSITION 1. *Let A be a subset of R^n. If $R^n \setminus A$ is approximatively 0-connected and 1-connected then $R^n \setminus A$ is approximatively s-connected for $s < n - \dim A - 1$.*

If A is a subset of R^n such that every map $I^2 \to R^2$ with $f(\dot{I}^2) \subset R^n \setminus A$ is approximable relative $\dot{I}^2$ by mappings with images missing A, then $R^n \setminus A$ is approximatively 1-connected. Thus the above proposition gives the positive answer to Toruńczyk's question slightly modified (see § 4, Corollary 2).

In our paper we use the notation of [M–S].

1. Category theory. Let $\mathscr{C}$ be a category. The category pro-$\mathscr{C}$ (see [G], [A–S], [M–S]) is the category whose objects are all inverse systems (over all directed sets) and whose morphisms are equivalence classes of morphisms of inverse systems. By in-$\mathscr{C}$ we denote the category defined dually. Objects of in-$\mathscr{C}$ are all direct systems in $\mathscr{C}$ over all directed sets. Observe that $\mathscr{C}$ can be considered as a full subcategory of pro-$\mathscr{C}$ and as a full subcategory of in-$\mathscr{C}$. We say that a category $\mathscr{C}'$ is *k-generated* (or *generated*) by $\mathscr{C}$ if $\mathscr{C}' = \mathscr{C}$ or $\mathscr{C}' = \tau_n\text{-} \ldots \text{-}(\tau_1\text{-}\mathscr{C})$ where $\tau_i = $ pro or $\tau_i = $ in for $i = 1, 2, \ldots, n$, $n \leqslant k$.

Let $\mathscr{C}$ be an arbitrary category. A diagram in the category $\mathscr{C}$ consists of a set $\mathscr{O}$ of objects of $\mathscr{C}$ and a set $\mathscr{M}$ of morphism of $\mathscr{C}$. Moreover, it is required that both $\mathrm{Dom} f$ and $\mathrm{Codom} f$ belong to $\mathscr{O}$ for each f in $\mathscr{C}$. We denote a diagram by $\mathscr{D} = (\mathscr{O}, \mathscr{M})$. If there exists a sequence $f_1, \ldots, f_n$, $n \geqslant 1$, of morphisms in $\mathscr{M}$ such that $\mathrm{Dom}(f_n \ldots f_1) = \mathrm{Codom}(f_n \ldots f_1)$, we say that the diagram $\mathscr{D} = (\mathscr{O}, \mathscr{M})$ contains a loop. If both $\mathscr{O}$ and $\mathscr{M}$ are finite we say that the diagram $\mathscr{D} = (\mathscr{O}, \mathscr{M})$ is finite.

The following theorem is a generalization of Theorem 3, [M–S], p. 12 (see also [A–M]).

THEOREM 1. *Let $\mathscr{D} = (\mathscr{O}, \mathscr{M})$ be a finite diagram in* pro-$\mathscr{C}$ *(or in* in-$\mathscr{C}$*) without loops. Then there exists a diagram $\mathscr{D}' = (\mathscr{O}', \mathscr{M}')$ in* pro-$\mathscr{C}$ *(resp.* in-$\mathscr{C}$*) and there exist one-to-one correspondences $i\colon \mathscr{O} \to \mathscr{O}'$ and $j\colon \mathscr{M} \to \mathscr{M}'$ such that*

(i) $\mathrm{Dom}\, j(f) = i(\mathrm{Dom} f)$ *and* $\mathrm{Codom}\, j(f) = i(\mathrm{Codom} f)$ *for each morphism f in $\mathscr{M}$;*

(ii) *all objects in $\mathscr{O}'$ are indexed over the same cofinite directed ordered set A;*

(iii) *every term and bonding morphism of $i(X)$ is also one in X, for each X in $\mathscr{O}$;*

(iv) *there exists an isomorphism $i_X\colon X \to i(X)$ in* pro-$\mathscr{C}$ *(resp. in* in-$\mathscr{C}$*), where $X \in \mathscr{O}$, such that for every morphism $f\colon X \to Y$ from $\mathscr{M}$ the following diagram commutes*

$$\begin{array}{ccc} X & \xrightarrow{\;i_X\;} & i(X) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle j(f)} \\ Y & \xrightarrow[\;i_Y\;]{} & i(Y) \end{array}$$

Moreover, $j(f)$ admits a representative f' which is a level morphism of systems.

Let $\mathscr{C}^*$ be the category dual to a category $\mathscr{C}$. The canonical contravariant bijective functor from $\mathscr{C}$ to $\mathscr{C}^*$ induces a contravariant bijective functor from pro-$\mathscr{C}$ to in-$\mathscr{C}^*$. This yields

PROPOSITION 1. *Let $\mathscr{P}$ be a property of categories and $\mathscr{P}^*$ be the property dual to $\mathscr{P}$. Then the following two conditions are equivalent for any category $\mathscr{C}$:*

(i) pro-$\mathscr{C}$ *has property $\mathscr{P}$ provided $\mathscr{C}$ has $\mathscr{P}$;*

(ii) in-$\mathscr{C}$ *has property $\mathscr{P}^*$ provided $\mathscr{C}$ has $\mathscr{P}^*$.*

A *zero-object* 0 in a category $\mathscr{C}$ is an object of $\mathscr{C}$ which is initial and terminal, i.e. for every object A of $\mathscr{C}$ there are unique morphisms $0 \to A$ and $A \to 0$. We say that $\mathscr{C}$ is a *category with zero-objects* if there exists at least one zero object in $\mathscr{C}$. (Any two zero-objects are isomorphic and any object isomorphic to a zero-object is itself a zero-object).

A zero-object in a category $\mathscr{C}$ can be considered as a zero-object in the category pro-$\mathscr{C}$ and as a zero-object in the category in-$\mathscr{C}$. Thus we have

PROPOSITION 2. *If $\mathscr{C}$ is a category with zero-objects then both* pro-$\mathscr{C}$ *and* in-$\mathscr{C}$ *are categories with zero-objects.*

In a category $\mathscr{C}$ with zero-objects a morphism $X \to Y$ is a zero-morphism provided it factors through a zero-object 0. For any two objects there is a unique zero-morphism $X \to Y$ which is denoted by $0\colon X \to Y$.

One can easily prove (compare [M–S], Theorem 7, p. 116):

PROPOSITION 3. *Let $\mathscr{C}$ be a category with zero-objects. An object $\underline{X} = (X_\lambda, p_{\lambda\lambda'}, A)$ of* pro-$\mathscr{C}$ *(respectively of* in-$\mathscr{C}$*) is a zero-object of* pro-$\mathscr{C}$ *(respectively of* in-$\mathscr{C}$*) if and only if every $\lambda \in A$ admits $\lambda' \geqslant \lambda$ such that $p_{\lambda\lambda'} = 0$.*

A *kernel* of a morphism $f\colon X \to Y$ in a category $\mathscr{C}$ with zero-objects is defined as a morphism $i\colon N \to X$ with the following properties

(i) $fi = 0$

(ii) whenever $g\colon Z \to X$ is a morphism with $fg = 0$ then there is a unique morphism $h\colon Z \to N$ such that $ih = g$.

Usually a kernel of $f\colon X \to Y$ is denoted by $\mathrm{Ker} f \to X$.

Kernels of f are unique up to natural isomorphisms. A kernel is always a monomorphism and the kernel of a monomorphism $f\colon X \to Y$ is the morphism $0\colon X \to Y$. If, in a category $\mathscr{C}$ with zero-objects, every morphism has a kernel, we say that $\mathscr{C}$ is a *category with zero-objects and kernels.*

Dually, one can define the *cokernel* $Y \to \operatorname{Coker} f$ of a morphism $f\colon X \to Y$ in a category with zero-objects. If, in a category $\mathscr{C}$ with zero-objects, every morphism has a cokernel, we say that $\mathscr{C}$ is a *category with zero-objects and cokernels.*

One can easily prove the following

PROPOSITION 4. *If $\mathscr{C}$ is a category with zero-objects and kernels (respectively cokernels) then* pro-$\mathscr{C}$ *is a category with zero-objects and kernels (respectively cokernels).*

By Propositions 1 and 4 we obtain

PROPOSITION 4*. *If $\mathscr{C}$ is a category with zero-objects and kernels (respectively cokernels) then* in-$\mathscr{C}$ *is a category with zero-objects and kernels (respectively cokernels).*

The group category Grp is a category with zero-objects and kernels and cokernels. Thus, a category generated by Grp is a category with zero-objects and kernels and cokernels.

Let $\mathscr{C}$ be a category with zero-objects and kernels and let $i_k\colon \operatorname{Ker} f_k \to X$ be the kernel of a morphism $f_k\colon X \to Y_k$ for $k = 0, 1$. We say that $\operatorname{Ker} f_1 \subseteq \operatorname{Ker} f_0$ if $f_0 i_1\colon \operatorname{Ker} f_1 \to Y_0$ is the zero morphism or equivalently, if there exists a morphism $h\colon \operatorname{Ker} f_1 \to \operatorname{Ker} f_0$ such that $i_1 = i_0 h$. Let $\mathscr{C}$ be a category with zero-objects and cokernels and let $j_k\colon X \to \operatorname{Coker} f_k$ be the cokernel of a morphism $f_k\colon Y_k \to X$ for $k = 0, 1$. Dually, we say that $\operatorname{Im} f_0 \subseteq \operatorname{Im} f_1$ if $j_1 f_0\colon Y_0 \to \operatorname{Coker} f_1$ is the zero-morphism.

We say that a category $\mathscr{C}$ with zero-objects and kernels has property α if for every two morphisms $f_0\colon X \to Y_0$ and $f_1\colon X \to Y_1$ with $\operatorname{Ker} f_1 \subseteq \operatorname{Ker} f_0$ and every two morphisms k and l from Z to X such that $f_1 k = f_1 l$ we have $f_0 k = f_0 l$. It is easy to see that $\mathscr{C}$ has property α if and only if for every commutative diagram in $\mathscr{C}$

$$\begin{array}{ccccc}
\operatorname{Ker} f_1 & \xrightarrow{i_1} & X_1 & \xrightarrow{f_1} & Y_1 \\
{\scriptstyle h}\downarrow & & \downarrow{\scriptstyle g} & & \\
\operatorname{Ker} f_0 & \xrightarrow[i_0]{} & X_0 & \xrightarrow[f_0]{} & Y_1
\end{array}$$

where i_k is the kernel of f_k for $k = 0, 1$, and every two morphisms k and l from Z to X_1 such that $f_1 k = f_1 l$ we have $f_0 g k = f_0 g l$. The dual property of categories with zero-objects and cokernels will be denoted by α^*.

We say that a category $\mathscr{C}$ with zero-objects, kernels and cokernels has property β if for every commutative diagram D in $\mathscr{C}$

$$D\colon\quad \begin{array}{ccccc}
X_1 & \xleftarrow{p_1} & X_2 & \xleftarrow{p_2} & X_3 \\
{\scriptstyle f_1}\downarrow & & {\scriptstyle f_2}\downarrow & & {\scriptstyle f_3}\downarrow \\
Y_1 & \xleftarrow[q_1]{} & Y_2 & \xleftarrow[q_2]{} & Y_3
\end{array}$$

with $\operatorname{Ker} f_2 \subseteq \operatorname{Ker} p_1$ and $\operatorname{Im} q_2 \subseteq \operatorname{Im} f_2$ there exists a unique morphism $g_D\colon Y_3 \to X_1$ such that $g_D f_3 = p_1 p_2$ and $f_1 g_D = q_2 q_1$. Moreover, for every commutative diagram D' in $\mathscr{C}$ with $\operatorname{Ker} f'_2 \subseteq \operatorname{Ker} p'_1$ and $\operatorname{Im} q'_2 \subseteq \operatorname{Im} f'_2$

$$D'\colon\quad \begin{array}{ccccc}
X'_1 & \xleftarrow{p'_1} & X'_2 & \xleftarrow{p'_2} & X'_3 \\
{\scriptstyle f'_1}\downarrow & & {\scriptstyle f'_2}\downarrow & & {\scriptstyle f'_3}\downarrow \\
Y'_1 & \xleftarrow[q'_1]{} & Y'_2 & \xleftarrow[q'_2]{} & Y'_3
\end{array}$$

and every morphism $D \to D'$ given by morphisms $k_i\colon X_i \to X'_i$ and $l_i\colon Y_i \to Y'_i$ for $i = 1, 2, 3$ we have $k_1 g_D = g_{D'} l_3$.

Let us observe that the group category Grp has properties α, α^* and β.

One can prove

PROPOSITION 5. *Let $\mathscr{C}$ be a category with zero-objects and kernels. If $\mathscr{C}$ has property α then both* pro-$\mathscr{C}$ *and* in-$\mathscr{C}$ *have property α (a category generated by $\mathscr{C}$ has property α).*

By Propositions 1 and 5 we obtain

PROPOSITION 5*. *Let $\mathscr{C}$ be a category with zero-objects and cokernels. If $\mathscr{C}$ has property α^* then both* pro-$\mathscr{C}$ *and* in-$\mathscr{C}$ *have α^* (a category generated by $\mathscr{C}$ has property α^*).*

One can also prove the following propositions

PROPOSITION 6. *Let $\mathscr{C}$ be a category with zero objects, kernels and cokernels. If $\mathscr{C}$ has property β then categories generated by $\mathscr{C}$ have the property β.*

(Since the property β is semidual, it suffices to prove that pro-$\mathscr{C}$ has property β provided $\mathscr{C}$ has property β.)

PROPOSITION 7. *Let a category $\mathscr{C}$ with zero-objects and kernels have property α. Then a morphism $f\colon X \to Y$ in $\mathscr{C}$ is a monomorphism if and only if the morphism $0 \to X$ is the kernel of f.*

PROPOSITION 7*. *Let a category $\mathscr{C}$ with zero-objects and cokernels have property α^*. Then a morphism $f\colon X \to Y$ in $\mathscr{C}$ is an epimorphism if and only if the morphism $Y \to 0$ is the cokernel of f.*

Propositions 7 and 7*, along with Propositions 5 and 5*, yield the following corollary

COROLLARY 1. *Let $\mathscr{C}$ be a category generated by* Grp. *Then a morphism $f\colon X \to Y$ in $\mathscr{C}$ is a monomorphism (respectively an epimorphism) if and only if the morphism $0 \to X$ (respectively $Y \to 0$) is the kernel (respectively cokernel) of f.*

One can easily obtain Theorems 1, 2, 3 and 4 of [M–S], p. 107–110, from the above Corollary 1.

COROLLARY 2. *Let a category $\mathscr{C}$ with zero-objects and kernels have property α and let $\underline{f}\colon \underline{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda) \to \underline{Y} = (Y_\lambda, q_{\lambda\lambda'}, \Lambda)$ be a morphism of* pro-$\mathscr{C}$ *(re-*

spectively of in-$\mathscr{C}$) *given by a level morphism of systems* (f_λ). *Then the morphism* $\underline{f}$ *is a monomorphism if and only if the following condition holds*

(M) *For every* $\lambda \in \Lambda$ *there is a* $\lambda' \geqslant \lambda$ *such that* $\operatorname{Ker} f_{\lambda'} \subseteq \operatorname{Ker} p_{\lambda\lambda'}$ (*respectively* $\operatorname{Ker} f_\lambda \subseteq \operatorname{Ker} p_{\lambda\lambda'}$).

COROLLARY 2*. *Let a category* $\mathscr{C}$ *with zero-objects and cokernels have property* α^* *and let* $\underline{f}\colon \underline{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda) \to \underline{Y} = (Y_\lambda, g_{\lambda\lambda'}, \Lambda)$ *be a morphism in* pro-$\mathscr{C}$ (*respectively in* in-$\mathscr{C}$) *given by a level morphism of systems* $(f_\lambda)\colon \underline{X} \to \underline{Y}$. *Then the morphism* $\underline{f}$ *is an epimorphism in* pro-$\mathscr{C}$ (*respectively in* in-$\mathscr{C}$) *if and only if the following condition holds*

(E) *For every* $\lambda \in \Lambda$ *there is a* $\lambda' \geqslant \lambda$ *such that* $\operatorname{Im} q_{\lambda\lambda'} \subseteq \operatorname{Im} f_\lambda$ (*respectively* $\operatorname{Im} q_{\lambda\lambda'} \subseteq \operatorname{Im} f_{\lambda'}$).

Corollaries 2 and 2* generalize Theorems 1 and 3, [M–S], p. 107 and p. 109. One can also generalize, in a similar way, Theorems 2 and 4, [M–S], p. 108 and p. 112.

One can prove

PROPOSITION 8. *Let* $\mathscr{C}$ *be a category with zero-objects, kernels and cokernels. If* $\mathscr{C}$ *has property* β *then* $\mathscr{C}$ *is a balanced category* (*i.e. a bimorphism must be an isomorphism*).

From Propositions 6 and 8 follows

COROLLARY 3. *Let a category* $\mathscr{C}$ *with zero-objects, kernels and cokernels have property* β. *Then* (*both categories* pro-$\mathscr{C}$ *and* in-$\mathscr{C}$ *being balanced*), *a category generated by* $\mathscr{C}$ *is balanced.*

The above Corollary generalizes Theorem 6, [M–S], p. 114

COROLLARY 4. *A category* $\mathscr{C}$ *generated by* Grp *is balanced.*

2. Hurewicz theorem. For every Abelian group G the kth singular homology group $H_k(-;\, G)$ is a functor from the homotopy category HTop of topological spaces into the category Ab of abelian groups. By induction this functor extends to a functor $H_k(-;\, G)$ from a category n-generated by HTop into a category n-generated by Ab (for all $n \geqslant 0$) as follows (compare [M–S], pp. 120–121). Assume that this is done for $n \leqslant m$. Let $\mathscr{C}''$ be a category $(m+1)$-generated by HTop. Then $\mathscr{C}'' = \text{pro-}\mathscr{C}'$ or $\mathscr{C}'' = \text{in-}\mathscr{C}'$, where $\mathscr{C}'$ is a category m-generated by HTop. We can consider an object of $\mathscr{C}''$ as an inverse or direct system $\underline{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$ in $\mathscr{C}'$. Then, by definition, $H_n(\underline{X};\, G) = (H_n(X_\lambda;\, G),\, H_n(p_{\lambda\lambda'};\, G);\, \Lambda)$. Furthermore, if $\underline{f}\colon \underline{X} \to \underline{Y} = (Y_\mu, q_{\mu\mu'}, \Lambda)$ is a morphism of $\mathscr{C}''$ given by (f_μ, f) (here $f_\mu\colon X_{f(\mu)} \to Y_\mu$ are morphisms in $\mathscr{C}'$) then $H_k(\underline{f};\, G)\colon H_k(\underline{X};\, G) \to H_k(\underline{Y};\, G)$ is given by $(H_k(f_\mu, G), f)$. If G is the group of integers $\mathbf{Z}$, we suppress G in the above notations.

In a similar way we can define a functor $\pi_k(-)$ from a category generated by the homotopy category HTop_* of pointed topological spaces into a category generated by the category Grp of groups.

With every pointed space $(X, *)$ and integer $k \geqslant 1$ there is associated the Hurewicz homeomorphism $\varphi_{k,(X,*)}\colon \pi_k(X, *) \to H_k(X)$. Recall that φ is a natural transformation, i.e. if $f\colon (X, *) \to (Y, *)$ is an H-map, then $\varphi_{k,(Y,*)}\pi_k(f) = H_k(f)\varphi_{k,(X,*)}$. By induction, we define the Hurewicz homomorphism φ_k on a category $\mathscr{C}$ generated by HTop_*. Let $\underline{f}\colon (\underline{X}, *) \to (\underline{Y}, *)$ be a morphism in $\mathscr{C}$. By the naturality of the Hurewicz homomorphism the following diagram commutes,

$$\begin{array}{ccc}
\pi_k(\underline{X}, *) & \xrightarrow{\pi_k(\underline{f})} & \pi_k(\underline{Y}, *) \\
\downarrow{\scriptstyle \varphi_{k,(\underline{X},*)}} & & \downarrow{\scriptstyle \varphi_{k,(\underline{X},*)}} \\
H_k(\underline{X}) & \xrightarrow[H_k(\underline{f})]{} & H_k(\underline{Y})
\end{array}$$

There are unique homomorphisms $K_k(\underline{f})$ and $C_k(\underline{f})$ which make the following diagram commutative,

$$\begin{array}{ccc}
\operatorname{Ker}\varphi_{k,(\underline{X},*)} & \xrightarrow{K_k(\underline{f})} & \operatorname{Ker}\varphi_{k,(\underline{Y},*)} \\
\downarrow{\scriptstyle i_{k,(X,*)}} & & \downarrow{\scriptstyle i_{k,(Y,*)}} \\
\pi_k(\underline{X}, *) & \xrightarrow{\pi_k(\underline{f})} & \pi_k(\underline{Y}, *) \\
\downarrow{\scriptstyle \varphi_{k,(\underline{X},*)}} & & \downarrow{\scriptstyle \varphi_{k,(Y,*)}} \\
H_k(\underline{X}) & \xrightarrow{H_k(\underline{f})} & H_k(\underline{Y}) \\
\downarrow{\scriptstyle j_{k,(X,*)}} & & \downarrow{\scriptstyle j_{k,(Y,*)}} \\
\operatorname{Coker}\varphi_{k,(\underline{X},*)} & \xrightarrow{C_k(\underline{f})} & \operatorname{Coker}\varphi_{k,(\underline{Y},*)}
\end{array}$$

where $i_{k,(\underline{X},*)}$ and $i_{k,(\underline{Y},*)}$ are kernels and $j_{k,(\underline{X},*)}$ and $j_{k,(\underline{Y},*)}$ are cokernels of $\varphi_{k,(\underline{X},*)}$ and $\varphi_{k,(\underline{Y},*)}$, respectively.

We need the following

LEMMA 1 ([M–S], p. 136). *Let* $n \geqslant 1$ *and let* $p_k\colon (X_k, *) \to (X_{k+1}, *)$ *be maps of pointed polyhedra such that the induced homomorphism* $\pi_k(p_k)\colon \pi_k(X_k, *) \to \pi_k(X_{k+1}, *)$ *equals* 0 *for* $k = 0, 1, \ldots, n-1$. *Then the map*

$$p = p_{n-1} \ldots p_1 p_0\colon (X_0, *) \to (X_n, *)$$

factors through an $(n-1)$-*connected polyhedron.*

By the classical Hurewicz isomorphism theorem we obtain

COROLLARY 1. *Under the assumptions of Lemma* 1, *the homomorphisms*

$$K_k(p)\colon \operatorname{Ker}\varphi_{k,(X_0,*)} \to \operatorname{Ker}\varphi_{k,(X_n,*)} \quad \text{for } k \leqslant n,$$

$$C_k(p)\colon \operatorname{Coker}\varphi_{k,(X_0,*)} \to \operatorname{Coker}\varphi_{k,(X_n,*)} \quad \text{for } k \leqslant n+1$$

induced by $p = p_{n-1} \ldots p_1 p_0$ *equal* 0 *provided* $n \geqslant 2$. *If* $n = 1$ *then* $C_1(p) = 0$.

By induction (on l) one can prove the following

LEMMA 2. *Let $\mathscr{C}$ be a category l-generated by* HPol_*, $l \geq 0$. *Let* $\underline{p_k}$: $(\underline{X_k}, *) \to (\underline{X_{k+1}}, *)$ *be a morphism of $\mathscr{C}$ such that the induced morphism* $\pi_k(\underline{p_l})$: $\underline{\pi_k(X_k, *)} \to \pi_k(\underline{X_{k+1}}, *)$ *equals* 0 *for* $k = 0, 1, \ldots, n-1$. *If* $n \geq 2$ *then the morphisms*

$$K_k(\underline{p})\colon \operatorname{Ker}\varphi_{k,\underline{(X_0,*)}} \to \operatorname{Ker}\varphi_{k,\underline{(X_n,*)}} \quad \text{for } k \leq n,$$

$$C_k(\underline{p})\colon \operatorname{Coker}\varphi_{k,\underline{(X_0,*)}} \to \operatorname{Coker}\varphi_{k,\underline{(X_n,*)}} \quad \text{for } k \leq n+1$$

induced by $p = p_{n+1} \ldots p_1 p_0$ *equal* 0. *If* $n = 1$ *then* $C_1(p) = 0$.

Let $\mathscr{C}$ be a category generated by HPol_*. We say that an object $(\underline{X}, *)$ of $\mathscr{C}$ is *n-connected*, $n \geq 0$, if $\pi_k(\underline{X}, *)$ is a zero-object in an appropriate category generated by Grp for $k = 0, 1, \ldots, n$. A direct consequence of Lemma 2 and § 1, Proposition 3 is the following

COROLLARY 2. *Let $\mathscr{C}$ be a category generated by* HPol_*. *Let* $(\underline{X}, *)$ *be an $(n-1)$-connected object of $\mathscr{C}$. If $n \geq 2$ then* $\operatorname{Ker}\varphi_{k,\underline{(X,*)}} = 0$ *for $k \leq n$ and* $\operatorname{Coker}\varphi_{k,\underline{(X,*)}} = 0$ *for* $k \leq n+1$. *If $n = 1$ then* $\operatorname{Coker}\varphi_{1,\underline{(X,*)}} = 0$.

By the above corollary and § 1, Corollaries 1 and 4, we obtain a Hurewicz isomorphism theorem for categories generated by HPol_*

THEOREM 1. *Let $\mathscr{C}$ be a category generated by HPol_* and $\mathscr{C}_{\mathrm{Grp}}$ be an appropriate category generated by* Grp. *Let* $(\underline{X}, *)$ *be an $(n-1)$-connected object of $\mathscr{C}$. If $n \geq 2$, then*

(i) $H_k(\underline{X}) = 0$, $1 \leq k \leq n-1$;

(ii) φ_n: $\pi_n(\underline{X}, *) \to H_n(\underline{X})$ *is an isomorphism of* $\mathscr{C}_{\mathrm{Grp}}$;

(iii) φ_{n+1}: $\pi_{n+1}(\underline{X}, *) \to H_{n+1}(\underline{X})$ *is an epimorphism* $\mathscr{C}_{\mathrm{Grp}}$.

If $n = 1$, then

(iv) φ_1: $\pi_1(\underline{X}, *) \to H_1(\underline{X})$ *is an epimorphism* $\mathscr{C}_{\mathrm{Grp}}$.

The above theorem generalizes the Hurewicz isomorphism theorem for pro-HPol_* (e.g. [M–S], Theorem 2, p. 136).

The Hurewicz isomorphism theorem is also proved for pro-HPol^2_*. (e.g. [M–S], Theorem 7, p. 140). Analogous result can be obtained for a category generated by HPol^2_*.

3. Hurewicz theorem for Borsuk's weak shape theory. To any pointed Hausdorff space $(X, *)$ we assign an object of in-pro-HPol_* in the following way. We consider $(X, *)$ as the limit of the direct system $((X_\lambda, *), j_{\lambda\lambda'}, \Lambda)$, where $(X_\lambda, *)_{\lambda\in\Lambda}$ is the family of all pointed compact subsets of $(X, *)$ and $j_{\lambda\lambda'}$: $(X_\lambda, *) \to (X_{\lambda'}, *)$ is the inclusion whenever $\lambda \leq \lambda'$. Every $(X_\lambda, *)$ is the limit of an inverse system $((X^\alpha_\lambda, *), p^{\alpha\beta}_\lambda, A_\lambda)$. Let $(\underline{X_\lambda}, *) = ((X^\alpha_\lambda, *), [p^{\alpha\beta}_\lambda], A_\lambda) \in$ pro-HPol_*. A map $j_{\lambda\lambda'}$ induces a morphism $\underline{j_{\lambda\lambda'}}$: $(\underline{X_\lambda}, *) \to (\underline{X_{\lambda'}}, *)$ in pro-HPol_*. We assign to $(X, *)$ the system $(\underline{X}, *) = ((\underline{X_\lambda}, *), \underline{j_{\lambda\lambda'}}, \Lambda) \in$ in-pro-HPol_*. One can easily see that any other object assigned to $(X, *)$ in such a way is isomorphic to $(\underline{X}, *)$ in the category in-pro-HPol_*. We denote $\pi_k(\underline{X}, *)$ by in-pro-$\pi_k(X, *)$, $H_k(\underline{X})$ by in-pro-$H_k(X)$ and $\varphi_{k,\underline{X}}$ by $\varphi_{k,X}$.

If $(X, *)$ is a pointed metric space, it can be easily checked that in-pro-$\pi_k(\tilde{X}, *)$ is the trivial element in the category in-pro-Grp if and only if $(\tilde{X}, *)$ is approximatively k-connected in the sense of Borsuk. Thus from § 2, Theorem 1 we obtain the following theorem.

THEOREM 1. *Let* $(X, *)$ *be a pointed metric space which is approximatively k-connected for $k \leq n-1$. If $n \geq 2$, then*

(i) in-pro-$H_k(X) = 0$, $1 \leq k \leq n-1$;

(ii) *the Hurewicz morphism* $\varphi_{n,X}$: in-pro-$\pi_n(X, *) \to$ in-pro-$H_n(X)$ *is an isomorphism*;

(iii) $\varphi_{n+1,X}$ *is an epimorphism.*

If $n = 1$ then

(iv) $\varphi_{1,X}$ *is an epimorphism.*

One can also state a similar theorem in CG-shape theory.

4. Approximative connectivity. In this section we use the notation of [M] for homology groups. We say that a Hausdorff space X is *approximatively homologically k-connected* if in-pro-$H_k(X)$ is trivial. If X is a subspace of $M \in \mathrm{AR}(\mathfrak{M})$ then X is approximatively homologically k-connected in M if for every compactum C in X there is a compactum D in X, $C \subset D$, such that for any neighborhood V of D in M there is a neighborhood U of C in M, $U \subset V$, such that the homomorphism $H_k(U) \to H_k(V)$ induced by the inclusion $U \to V$ is trivial.

LEMMA 1. *Let X be a subspace of R^m and* $\dim X = k < m$. *Then the space* $Y = R^m \setminus X$ *is approximatively homologically l-connected for every $l < m-k-1$.*

Proof. Let A be a compactum in Y. There is a sequence $\{M_i\}$ of m-manifolds in R^m such that $M_{i+1} \subset M_i$, M_1 is a ball and $\bigcap M_i = A$. Let l be an integer less then $m-k-1$.

Since $k < m-l-1$, we have $H^{m-l-1}(X \cap M_i) = H^{m-1}(X \cap M_i) = 0$. Thus by Theorem 11.19 of [M], the homomorphism $h_{i,*}$: $H^C_l(Y \cap M_i) \to H_l(M_i)$ induced by the inclusion is an isomorphism. The group $H^C_l(Y \cap M_i)$ is finitely generated, so there is a compactum B_i in $Y \cap M_i$ such that the homomorphism $H_l(B_i) \to H^C_l(Y \cap M_i)$ induced by the inclusion is an epimorphism. We define

$$C_n = \bigcup_{i \geq n} B_i \cup A.$$

Then C_n is a compactum and the homomorphism $k_{n,*}$: $H_l(C_n) \to H^C_l(Y \cap M_n)$ induced by the inclusion is also an epimorphism.

Let $\alpha^1_n, \alpha^2_n, \ldots, \alpha^{r_n}_n$ be generators of $H^C_l(Y \cap M_n)$ and $\beta^1_n, \beta^2_n, \ldots, \beta^{r_n}_n$ be elements of $H_l(C_n)$ such that $k_{n,*}(\beta^i_n) = \alpha^i_n$ for $i = 1, 2, \ldots, r_n$. Let

$$i_{n,*}(\alpha^k_{n+1}) = \sum_{i=1}^{r_n} a^k_{n+1,i}\alpha^i_n \quad \text{for } k = 1, 2, \ldots, r_{n+1}$$

where $i_{n,*}\colon H_l^C(Y\cap M_{n+1})\to H_l^C(Y\cap M_n)$ is the homomorphism induced by the inclusion. We define

$$\gamma_{n+1}^k = j_{n,*}(\beta_{n+1}^k)-\sum_{i=1}^{r_n} a_{n+1,i}^k\beta_n^i \quad \text{for } k = 1, 2, \ldots, r_{n+1}$$

where $j_{n,*}\colon H_l(C_{n+1})\to H_l(C_n)$ is the homomorphism induced by the inclusion.

The image of γ_{n+1}^k under the homomorphism $k_{n,*}$ is trivial. Thus there is a compactum D_n^k such that $C_n\subset D_n^k\subset Y\cap M_n$ and that the image of γ_{n+1}^k under the homomorphism $H_l(C_n)\to H_l(D_n^k)$ induced by the inclusion is trivial. We define compact sets

$$D_n = \bigcup_{k=1}^{r_{n+1}} D_n^k \quad\text{and}\quad D=\bigcup_{k=1}^{\infty} D_n.$$

Let us observe that the image of β_n^k under the homomorphism $H_l(C_n)\to H_l(D)$ induced by the inclusion is trivial for every n.

We will show that for any neighborhood U of D in R^m there is a neighborhood V of A in R^m such that the homomorphism $H_l(V)\to H_l(U)$ induced by the inclusion is trivial. We may assume that U is a polyhedron.

There is an integer n such that $M_n\subset U$. We take $V=M_n$. We have the following commutative diagram,

$$\begin{array}{ccc} C_n & \longrightarrow & D \\ \downarrow & & \downarrow \\ M_n = V & \longrightarrow & U \end{array}$$

The image of β_n^k (for $k = 1, 2, \ldots, r_n$) under the homomorphism $H_l(C_n)\to H_l(D)$ induced by the inclusion is trivial. Thus the image of β_n^k under the homomorphism $H_l(C_n)\to H_l(U)$ induced by the inclusion is trivial. Hence also image of $\hat{\alpha}_n^k = h_{n,*}(\alpha_n^k)$ under the homomorphism $H_l(M_n)\to H_l(U)$ induced by the inclusion is trivial. Since $\hat{\alpha}_n^1, \hat{\alpha}_n^2, \ldots, \hat{\alpha}_n^{r_n}$ are generators of the group $H_l(M_n)$, the inclusion $M_n\to U$ induces the trivial homomorphism. This finishes the proof of the lemma.

Lemma 1 and § 3, Theorem 1 imply the following corollary.

Corollary 1. *Let X be a subspace of R^m with $\dim X = k < m$. If the space $Y = R^m\setminus X$ is approximatively* 1*-connected then Y is approximatively l-connected for every $l<m-k-1$.*

We will also need the following lemma.

Lemma 2. *Let Y be a subset of R^m such that every map $f\colon I^2\to R^m$ with $f(\dot I^2)\subset Y$ is approximable relative $\dot I^2$ by mappings with images in Y. Then Y is approximatively* 1*-connected.*

Proof. Let us observe that any map $g\colon I\to R^m$ with $g(0)\in Y$ is approximable relatively $\{0\}$ by mappings with images in Y. It follows that any map $g\colon I\to R^m$ with $f(\dot I)\subset Y$ is approximable relatively $\dot I$ by mappings with images in Y.

Let A be any compact subset of Y. There is a sequence $\{M_i\}$ of m-manifolds in R^m such that $M_{i+1}\subset \mathring{M}_i$, M_1 is a ball and $\bigcap M_i = A$. Let $a\in A$. Let $[f_i^1], [f_i^2], \ldots, [f_i^{r_i}]$, where $f_i^k\colon (I,\dot I)\to (M_i, a)$, be generators of the 1-homotopy group $\pi_1(M_i, a)$; if $i=1$ then $r_1 = 1$ and f_1^1 is the constant map. We may assume that $f_i^k(I)\subset Y$ for every i and k. Let $\hat f_{i+1}^k\colon (I,\dot I)\to (M_i, a)$ be a product of paths $f_i^1, f_i^2, \ldots, f_i^{r_i}$ such that the homotopy class $[\hat f_{i+1}^k]\in\pi_1(M_i, a)$ is the image of $[f_{i+1}^k]$ under the homomorphism $\pi_1(M_{i+1}, a)\to\pi_1(M_i, a)$ induced by the inclusion. Observe that the product $f_{i+1}^k * (\hat f_{i+1}^k)^{-1}$ represents the trivial element of $\pi_1(M_i, a)$. Thus there exists a map $g_{i+1}^k\colon (I^2, 0)\to (M_i, a)$ such that $g_{i+1}^k|\dot I^2$ is defined by $f_{i+1}^k * (\hat f_{i+1}^k)$. By our assumption we may assume that $g_{i+1}^k(I^2)\subset Y$. We define

$$B = \bigcup_{i=1}^{\infty}\bigcup_{k=1}^{r_i} g_{i+1}^k(I^2)\cup A.$$

Let V be any neighborhood of B in R^m. There is an integer i such that $M_i\subset V$. It is easy to check that the homomorphism $\pi_1(M_i, a)\to\pi_1(V, a)$ induced by the inclusion is trivial.

By Corollary 1 and Lemma 2 we obtain

Corollary 2. *Let X be a subset of R^m such that every map $f\colon I^2\to R^m$ with $f(\dot I^2)\subset R^m\setminus X$ is approximable relative $\dot I^2$ by mappings with images missing X. If $\dim X = k<m$ then $R^m\setminus X$ is approximatively l-connected for every $l<m-k-1$. In particular, if $g\colon \partial I^s\to R^n$ is a map which satisfies $\operatorname{im}(g)\cap X=\emptyset$, where $s+\dim X<m$, then there is a compact set $X\subset R^m\setminus X$ such that g is null homotopic in every neighborhood of C in R^m.*

The above corollary gives a positive answer to a modified Toruńczyk's question.

Recently D. W. Curtis [C] introduced the definition of a locally approximatively l-connected space. In a similar way one can introduce the notion of a locally approximatively homologically l-connected space. Both these notions can be described in the category pro-in-pro-HPol$_*$. One can prove the following modification of Lemma 1.

Lemma 3. *Let X be a k-dimensional subspace of R^m, $k<m$. Then the space $Y = R^m\setminus X$ is locally approximatively homologically k-connected for $l<m-k-1$.*

Lemma 3 and § 2, Theorem 1 (for the category pro-in-pro-HPol$_*$) implies the following corollary.

Corollary 3. *Let X be a k-dimensional subspace of R^m, $k<m$. If the space $Y = R^m\setminus X$ is locally approximatively* 1*-connected then Y is locally approximatively l-connected for $l<m-k-1$.*

In a similar way one can also modify Lemma 2 and Corollary 2.

5. Whitehead theorem. Let $p_k\colon (X_k, A_k, *)\to (X_{k+1}, A_{k+1}, *)$ be a map of pointed pairs of topological spaces. By $p_k'\colon (X_k, *)\to (X_{k+1}, *)$ and $q_k\colon (A_k, *)\to (A_{k+1}, *)$ we denote the maps induced by p_k. By $i_k\colon (A_k, *)\to (X_k, *)$ and

$j_k\colon (X_k, *) \to (X_k, A_k, *)$ we denote the inclusions. One can find a proof of the following lemma in [D–S], pp. 104–105.

LEMMA 1. *If* $\operatorname{im} \pi_n(p_2') \subset \operatorname{im}\pi_n(i_3)$ *and* $\ker\pi_{n-1}(i_1) \subset \ker\pi_{n-1}(q_1)$ *then the homomorphism* $\pi_n(p_2 p_1)$ *equals* 0, $n \geqslant 1$.

We will need the following two lemmas (see [D–S], p. 104, [M-S], p. 140 and p. 145).

LEMMA 2. *Let* $p_k\colon (X_k, A_k, *) \to (X_{k+1}, A_{k+1}, *)$, $k = 0, 1, \ldots, n$, $n \geqslant 1$ *be maps of pointed polyhedral pairs such that* X_0 *is connected and let* $\pi_k(p_k)\colon \pi_k(X_k, A_k, *) \to \pi_k(X_{n+1}, A_{n+1}, *)$ *equal* 0 *for* $k = 0, 1, \ldots, n$. *Then the map*

$$p_n \ldots p_1 p_0\colon (X_0, A_0, *) \to (X_{n+1}, A_{n+1}, *)$$

factors up to homotopy through the pair $(X_0, A_0 \cup X_0^{(n)}, *)$ *where* $X_0^{(m)}$ *is an m-skeleton of* X_0.

LEMMA 3. *Suppose that the following diagram in* Pol_* *is commutative up to a pointed homotopy,*

$$\begin{array}{ccc} (X,*) & \xleftarrow{\;p\;} & (X',*) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} \\ (Y,*) & \longleftarrow & (Y',*) \end{array}$$

Let $(Z, *)$ *be the reduced mapping cylinder of f and let* $i\colon (X, *) \to (Z, *)$ *and* $j\colon (Y, *) \to (Z, *)$ *be the canonical embeddings. Let* $(Z', *)$, i' *and* j' *be analogously defined by* f'. *Then there exists a* $s\colon (Z', *) \to (Z, *)$ *such that the following diagram commutes in* Pol_*,

$$\begin{array}{ccc} (X,*) & \xleftarrow{\;p\;} & (X',*) \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle i'} \\ (Z,*) & \xleftarrow{\;s\;} & (Z',*) \\ {\scriptstyle j}\downarrow & & \downarrow{\scriptstyle j'} \\ (Y,*) & \xleftarrow{\;q\;} & (Y',*) \end{array}$$

Identifying X with $i(X)$ and X' with $i(X')$ one can view s as a map of pairs $s\colon (Z', X', *) \to (Z, X, *)$.

Using Lemmas 1, 2 and 3 one can prove (compare [D–S], Corollary 8.13, p. 105) the following proposition.

PROPOSITION 1. *Let the following diagram in* Pol_* *be commutative up to a pointed homotopy* $(m \geqslant 1)$

$$\begin{array}{ccccccccc} (X_0,*) & \xrightarrow{p_0} & (X_1,*) & \xrightarrow{p_1} & \ldots & \xrightarrow{p_{2m-1}} & (X_{2m},*) & \xrightarrow{p_{2m}} & (X_{2m+1},*) \\ \downarrow{\scriptstyle f_0} & & \downarrow{\scriptstyle f_1} & & & & \downarrow{\scriptstyle f_{2m}} & & \downarrow{\scriptstyle f_{2m+1}} \\ (Y_0,*) & \xrightarrow{q_0} & (Y_1,*) & \xrightarrow{q_1} & \ldots & \xrightarrow{q_{2m-1}} & (Y_{2m},*) & \xrightarrow{q_{2m}} & (Y_{2m+1},*) \end{array}$$

Assume that $\ker\pi_{n-1}(f_{2n-1}) \subset \ker\pi_{n-1}(p_{2n-1})$ *and* $\operatorname{im}\pi_n(q_{2n}) \subset \operatorname{im}\pi_n(f_{2n+1})$ *for* $n = 1, 2, \ldots, m$ *and that* Y_0 *is connected. Then the map* (p, q) *from* f_0 *to* f_{2n+1}, *where* $p = p_{2n} \ldots p_1 p_0$ *and* $q = q_{2n} \ldots q_1 q_0$, *factors in* HPol_* *through an inclusion* $f\colon (P, *) \to (Q, *)$, *where Q is a connected polyhedron and the n-skeletons of P and of Q are equal (with respect to some triangulations).*

Let $\underline{Y}$ be an object of a category $\mathscr{C}$ generated by the category HPol_*. We say that $\underline{Y}$ is *connected* if $(\underline{Y}, y)$ is 0-connected for some base point $y \in \underline{Y}$. It can be proved that $\underline{Y}$ is connected if and only if $\underline{Y}$ is equivalent to an object $\underline{Y}'$ of $\mathscr{C}$ with every term connected.

We now prove the following proposition.

PROPOSITION 2. *Let the following diagram in* pro-HPol_* *be commutative* $(n \geqslant 1)$

$$\begin{array}{ccccccccc} (\underline{X}^0,*) & \xrightarrow{\underline{p}^0} & (\underline{X}^1,*) & \xrightarrow{\underline{p}^1} & \ldots & \longrightarrow & (\underline{X}^{2m},*) & \xrightarrow{\underline{p}^m} & (\underline{X}^{2m+1},*) \\ \downarrow{\scriptstyle \underline{f}^0} & & \downarrow{\scriptstyle \underline{f}^1} & & & & \downarrow{\scriptstyle \underline{f}^{2m}} & & \downarrow{\scriptstyle \underline{f}^{2m+1}} \\ (\underline{Y}^0,*) & \xrightarrow{\underline{q}^0} & (\underline{Y}^1,*) & \xrightarrow{\underline{q}^1} & \ldots & \longrightarrow & (\underline{Y}^{2m},*) & \xrightarrow{\underline{q}^m} & (\underline{Y}^{2m+1},*) \end{array}$$

where all inverse systems are indexed over the same cofinite directed ordered set Λ *i.e.* $\underline{X}^k = (X_\lambda^k, p_{\lambda\lambda'}^k, \Lambda)$ *and* $\underline{Y}^k = (Y_\lambda^k, q_{\lambda\lambda'}^k, \Lambda)$ *and all morphisms are given by level, morphisms of systems,* $\underline{f}^k$ *by* (f_λ^k), $\underline{p}^k$ *by* (p_λ^k) *and* $\underline{q}^k$ *by* (q_λ^k). *Assume that* $\operatorname{Ker}\pi_{k-1}(\underline{f}^{2k-1}) \subseteq \operatorname{Ker}\pi_{k-1}(\underline{p}^{2k-1})$ *and* $\operatorname{im}\pi_k(\underline{q}^{2k}) \subseteq \operatorname{im}\pi_k(\underline{f}^{2k})$ *for* $k = 1, 2, \ldots, m$ *and that* $\underline{Y}^0$ *is connected. Then for every* $\lambda \in \Lambda$ *there exists* $\lambda' \in \Lambda$ *such that the map* $(p_\lambda, q_\lambda) = (p_\lambda^{2m} \ldots p_\lambda^1 p_\lambda^0 p_{\lambda\lambda'}^0, q_\lambda^{2m} \ldots q_\lambda^1 q_\lambda^0 q_{\lambda\lambda'})$ *from* $f_{\lambda'}^0\colon X_{\lambda'}^0 \to Y_{\lambda'}^0$ *to* $f_\lambda^{2m+1}\colon X_\lambda^{2m+1} \to X_\lambda^{2m+1}$ *factors through an inclusion* $f_\lambda\colon P_\lambda \to Q_\lambda$, *where* Q_λ *is a connected polyhedron and the n-skeletons of* P_λ *and* Q_λ *are equal (with respect to some triangulations).*

Proposition 2 follows from Proposition 1 and the following two lemmas.

LEMMA 4. *Let* $\mathscr{C}$ *be a category with zero-objects and kernels, and let* $\underline{A} = (A_\lambda, p_{\lambda\lambda'}, \Lambda)$, $\underline{A}' = (A_\lambda', p_{\lambda\lambda'}', \Lambda)$ *and* $\underline{B} = (B_\lambda, q_{\lambda\lambda'}, \Lambda)$ *be objects of* pro-$\mathscr{C}$. *Let* $\underline{f}\colon \underline{A} \to \underline{A}'$ *and* $\underline{g}\colon \underline{A} \to \underline{B}$ *be morphisms in* pro-$\mathscr{C}$ *given by level morphisms of systems* (f_λ) *and* (g_λ), *respectively. If* $\operatorname{Ker}\underline{g} \subset \operatorname{Ker}\underline{f}$ *then there is a morphism of systems* $(f_\lambda', f')\colon \underline{A} \to \underline{A}'$ *(equivalent to* (f_λ)*) such that*

(i) $f_\lambda' = f_\lambda p_{\lambda f'(\lambda)}\colon A_{f'(\lambda)} \to A_\lambda'$ *for* $\lambda \in \Lambda$

(ii) $\operatorname{Ker} g_{f'(\lambda)} \subset \operatorname{Ker} f_\lambda'$ *for* $\lambda \in \Lambda$

LEMMA 4*. *Let* $\mathscr{C}$ *be a category with zero-objects and cokernels and let* $\underline{A} = (A_\lambda, p_{\lambda\lambda'}, \Lambda)$, $\underline{B} = (B_\lambda, q_{\lambda\lambda'}, \Lambda)$ *and* $\underline{B}' = (B_\lambda', q_{\lambda\lambda'}', \Lambda)$ *be objects of* pro-$\mathscr{C}$. *Let* $\underline{h}\colon \underline{B}' \to \underline{B}$ *and* $\underline{g}\colon \underline{A} \to \underline{B}$ *be morphisms in* pro-$\mathscr{C}$ *given by level morphisms of systems* (h_λ) *and* (g_λ), *respectively. If* $\operatorname{Im}\underline{h} \subset \operatorname{Im}\underline{g}$ *then there is a morphism of systems* $(h_\lambda', h')\colon \underline{B}' \to \underline{B}$ *(equivalent to* (g_λ)*) such that*

(i) $h_\lambda' = h_\lambda q_{\lambda h'(\lambda)}'\colon B_{h'(\lambda)}' \to B_\lambda$ *for* $\lambda \in \Lambda$

(ii) $\operatorname{im} h_\lambda' \subset \operatorname{im} f_\lambda$ *for* $\lambda \in \Lambda$

One can easily prove the following lemma.

LEMMA 5. *Let the following diagram in* HPol$_*$ *be commutative,*

$$\begin{array}{ccc} (X,*) & \xrightarrow{p} & (X',*) \\ f\downarrow & & \downarrow f' \\ (Y,*) & \xrightarrow[q]{} & (Y',*) \end{array}$$

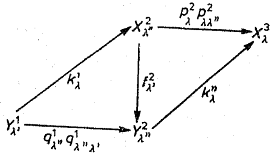
If $\dim X \leqslant n-1$ *and* $\dim Y \leqslant n$ *and the map* (p, q) *from* f *to* f' *factors in* HPol$_*$ *through an inclusion* $i\colon (P,*) \to (Q,*)$ *where the n-skeletons of* P *and* Q *are equal, then there is a map* $h\colon (Y,*) \to (X',*)$ *such that* $p = hf$ *and* $q = f'h$ *(in* HPol$_*$*).*

We will also need the following

LEMMA 6. *Let* $\mathscr{C}$ *be a category and suppose that* $\underline{X}^i = (X^i_\lambda, p^i_{\lambda\lambda'}, \Lambda)$ *and* $\underline{Y}^i = (Y^i_\lambda, q^i_{\lambda\lambda'}, \Lambda)$ *are inverse systems in* $\mathscr{C}$ *for* $i = 1, 2, 3$. *Let the following diagram in* $\mathscr{C}$ *be commutative,*

$$\begin{array}{ccccc} \underline{X}^1 & \xrightarrow{p^1} & \underline{X}^2 & \xrightarrow{p^2} & \underline{X}^3 \\ f^1\downarrow & & f^2\downarrow & & f^3\downarrow \\ \underline{Y}^1 & \xrightarrow[q^1]{} & \underline{Y}^2 & \xrightarrow[q^2]{} & \underline{Y}^3 \end{array}$$

where the morphisms p^i, q^i *and* f^i *are given by the level morphisms* (p^i_λ), (q^i_λ) *and* (f^i_λ), *respectively. Assume that for every* $\lambda \in \Lambda$ *there are* λ' *and* λ'', $\lambda \leqslant \lambda'' \leqslant \lambda'$, *and there exist morphisms* $k'_\lambda\colon Y^1_{\lambda'} \to X^2_{\lambda''}$ *and* $k''_\lambda\colon Y^2_{\lambda''} \to X^3_{\lambda'}$ *such that the following diagram in* $\mathscr{C}$ *is commutative,*



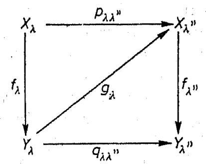
Then there is a morphism $\underline{k}\colon \underline{Y}^1 \to \underline{X}^3$ *in* pro-$\mathscr{C}$ *given by the family* $(p^2_\lambda p^2_{\lambda\lambda''} k'_\lambda)_{\lambda\in\Lambda}$ *of morphisms in* $\mathscr{C}$. *Moreover, if* $p^1_{\lambda''} p^1_{\lambda''\lambda'} = k'_\lambda f^1_{\lambda'}$, *(respectively* $q^2_\lambda q^2_{\lambda\lambda''} = f^3_\lambda k''_\lambda$*) then* $\underline{p}^2\underline{p}^1 = \underline{k}\underline{f}^1$ *(respectively* $\underline{f}^3\underline{k} = \underline{q}^2\underline{q}^1$*).*

We say that a morphism $\underline{f}\colon (\underline{X},*) \to (\underline{Y},*)$ in a category generated by HPol$_*$ is an *n-equivalence* if $\pi_k(\underline{f})$ is an isomorphism for $k = 0, 1, \ldots, n-1$ and $\pi_n(\underline{f})$ is an epimorphism. We say that $\dim \underline{X} \leqslant n$ if and only if the dimension of each term in $\underline{X}$ is $\leqslant n$.

Now we can state the Whitehead theorem in in-pro-HPol$_*$ (compare [M–S], p. 149).

THEOREM 1. *Let* $\underline{f}\colon (\underline{X},*) \to (\underline{Y},*)$ *be a morphism of* in-pro-HPol$_*$ *which is an n-equivalence. If* $\dim \underline{X} \leqslant n-1$, $\dim \underline{Y} \leqslant n$ *and if* $\underline{X}$ *and* $\underline{Y}$ *are connected then* $\underline{f}$ *is an isomorphism of* in-pro-HPol$_*$.

Proof. By § 1 Theorem 1, we may assume that $\underline{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$, $\underline{Y} = (Y_\lambda, q_{\lambda\lambda'}, \Lambda)$ are directed systems in pro-HPol$_*$ with $\Lambda = \Lambda'$ and that $\underline{f}$ is given by a level morphism (f_λ), where $f_\lambda\colon X_\lambda \to Y_\lambda$, $\lambda \in \Lambda$, are morphisms in pro-HPol$_*$. By § 1, Corollaries 1.2 and 1.2*, for every $\lambda_0 \in \Lambda$ we can find a sequence $\lambda_0 \leqslant \lambda_1 \leqslant \ldots \ldots \leqslant \lambda_{2n+1} = \lambda'$ such that $\operatorname{Ker}\pi_{k-1}(f_{2k-1}) \subset \operatorname{Ker}\pi_{k-1}(p_{\lambda_{2k-1}\lambda_{2k}})$ and $\operatorname{im}\pi_k(q_{\lambda_{2k}\lambda_{2k+1}}) \subset \operatorname{im}\pi_k(f_{\lambda_{2k+1}})$ for $k = 1, \ldots, n$. In the same way we assign to λ' a $\lambda'' \geqslant \lambda'$. By Proposition 2, Lemma 5 and Lemma 6 there is a morphism $g_\lambda\colon Y_\lambda \to X_{\lambda''}$ such that the following diagram commutes in pro-HPol$_*$,



By the dual version of Morita's lemma (see [M–S]) it follows that $\underline{f}$ is an isomorphism.

In a similar way one can prove the following theorems (compare [M–S], Theorems 4, 5 and 6, pp. 149–151).

THEOREM 2. *Let* $\underline{f}\colon (\underline{X},*) \to (\underline{Y},*)$ *be an* $(n+1)$*-equivalence in* in-pro-HPol$_*$. *If* $\dim \underline{Y} \leqslant n$ *and* $\underline{Y}$ *is connected, then there is a morphism* $\underline{g}\colon (\underline{Y},*) \to (\underline{X},*)$ *such that* $\underline{f}\underline{g} = 1$

THEOREM 3. *Let* $\underline{f}\colon (\underline{X},*) \to (\underline{Y},*)$ *be an n-equivalence in* in-pro-HPol$_*$. *If* $\dim \underline{Y} \leqslant n$ *and* $\underline{Y}$ *is connected then* $\underline{f}$ *is an epimorphism of* pro-HPol$_*$.

THEOREM 4. *Let* $\underline{f}\colon (\underline{X},*) \to (\underline{Y},*)$ *be a morphism in* in-pro-HPol$_*$, *which induces an isomorphism* $\pi_k(\underline{f})$ *for* $k \leqslant n$. *If* $\dim \underline{X} \leqslant n$, $\dim \underline{Y} \leqslant n$ *and* $\underline{X}$ *and* $\underline{Y}$ *are connected then* $\underline{f}$ *is an isomorphism of* in-pro-HPol$_*$.

Remark. The above theorems can be proved in any category generated by HPol$_*$.

Let $X \subset M \in \mathrm{AR}(\mathfrak{M})$ be a metric space. We say that $\mathrm{sd}_w X \leqslant k$ if and only if for any compact set B there is a compact set B' such that for every neighborhood U' of B' in M there is a homotopy $H\colon B \times [0,1] \to U'$ such that $H_0 = \mathrm{id}_B$ and $\dim H_1(B) \leqslant k$. It can be proved that $\mathrm{sd}_w X \leqslant k$ if and only if the object $\underline{X}$ of in-pro-HPol$_*$ assigned to X is isomorphic to an object $\underline{X}'$ with $\dim \underline{X}' \leqslant k$. A morphism $F\colon (X,*) \to (Y,*)$ in weak shape theory for locally compact metric spaces can be considered as a morphism $\underline{f}\colon (\underline{X},*) \to (\underline{Y},*)$ in-pro-HPol$_*$ where $(\underline{X},*)$ and $(\underline{Y},*)$ are assigned to $(X,*)$ and $(Y,*)$, respectively (see [Sa1]).

We now can state the Whitehead theorem in weak shape theory.

THEOREM. *Let* $F: (X, *) \to (Y, *)$ *be a shape n-equivalence between connected locally compact metric spaces. If* $\mathrm{sd}_w X \leqslant n-1$ *and* $\mathrm{sd}_w Y \leqslant n$, *then F is an isomorphism in weak shape theory.*

In a similar way we can obtain counterparts to Theorem 2, 3 and 4. We can also state similar theorems in CG-shape theory (without assumption of local compactness).

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Received 1 March 1985

Equivariant shape

by

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Abstract. The paper introduces an equivariant shape category Sh^G. Its objects are G-spaces, i.e., topological spaces endowed with an action of a given compact group G. The category Sh^G is constructed using the method of resolutions.

1. Introduction. The aim of this paper is to define a shape category for G-spaces, i.e., topological spaces endowed with an action of a given compact group G. In our development we follow the method of resolutions, introduced in the case of ordinary shape by S. Mardešić [14], [15] (also see [16]).

More precisely, in § 4 we define the notion of a G-resolution of a G-space and we show that every G-space admits a G-ANR-resolution, i.e., a G-resolution consisting of G-ANR's (Theorem 1).

In § 5 we prove that every G-ANR-resolution induces in the G-homotopy category $[\mathrm{Top}^G]$ a G-ANR-expansion in the sense of [16, I, § 2.1]. This means that the full subcategory $[\mathrm{ANR}^G]$ of $[\mathrm{Top}^G]$, which consists of spaces having the G-homotopy types of G-ANR's, is dense in $[\mathrm{Top}^G]$ [16, I, § 2.2].

In [16, I, § 2] a general procedure is described, which associates a shape category $\mathrm{Sh}_{\mathscr{T}\mathscr{P}}$ with every pair consisting of a category $\mathscr{T}$ and of a dense subcategory $\mathscr{P}$. The equivariant shape category Sh^G is the shape category associated in this way with the pair $\mathscr{T} = [\mathrm{Top}^G]$, $\mathscr{P} = [\mathrm{ANR}^G]$.

Note that Sh^G coincides with the ordinary shape category Sh if $G = \{e\}$ is the trivial group.

In the realization of the outlined program (just as in the case of ordinary shape) the crucial tool is a G-embedding and G-extension theorem (Proposition 1). It asserts that every metric G-space X equivariantly embeds as a closed subset in a normed linear G-space L, which is a G-absolute extensor. This fact is the result of the work of several authors (see § 3 and for a detailed proof see [6]). Other results on G-ANR's needed in this paper were obtained by considering equivariant versions of appropriate proofs of analogous results in the ordinary case. In several instances the proofs given in [16] were appropriate. However, in some cases we