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# Spherical maps

by

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**Abstract.** In this work we discuss the class of multi-valued upper semi-continuous maps $\varphi\colon M \to R^n$ of topological space $M$. Their values $\varphi(x)$ are non-empty continua of such nature that if $B\varphi(x)$ stands for the sum of bounded components $R^n \setminus \varphi(x)$ the graph of the map $B\varphi$ is open in $M \times R^n$ and $(\tilde{\varphi}x) := \varphi(x) \cup B\varphi(x)$ is acyclic for each $x \in M$. For such — so called spherical maps the following theorems are proven: (1) the Brouwer fixed point theorem, (2) the Poincaré type coincidence theorem, (3) the Birkhoff–Kellogg theorem, (4) the theorem on antipodes, (5) the theorem on invariance of domain.

**1.** Although a great number of papers have been published on the fixed point theory of various classes of multi-valued mappings, but some strong conditions about images of points by a multi-valued maps are always assumed. In the articles [4], [5], [9], [10] it is assumed that considered multi-valued map has acyclic images or, more generally, it is admissible multi-valued map (i.e. composition of acyclic maps). In the articles [8], [11] multi-valued maps with images of points having homology of the unit sphere $S^{n-1}$ in the Euclidean space $R^n$ are considered.

In the present paper we consider a class of multi-valued maps into Euclidean space $R^n$, called spherical maps. In this case homological assumptions about images of points are quite weak, although some additional non homological conditions are needed. As a special case, our class contains acyclic maps of $n$-spherical type in the sense of [8].

Next, we generalize from the case of admissible maps or $n$-spherical maps on the case of spherical maps the following results: (1) the Brouwer fixed point theorem, (2) the Poincaré type coincidence theorem, (3) the Birkhoff–Kellogg theorem, (4) the theorem on antipodes, (5) the theorem on invariance of domain. Note that in the case of $n$-spherical maps results (2), (3), (4), (5) have been unknown.

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**2. Spherical maps.** We will consider subsets of the Euclidean space $R^n$. We assume that $n \geqslant 2$. For any set $X \subset R^n$, the unbounded pseudo-component $D(X)$ of the set $R^n \setminus X$ is defined as follows: $x \in D(X)$ iff for every $r > 0$ there exists a continuous function $h\colon I \to R^n \setminus X$ such that $h(0) = x$ and $\|h(1)\| > r$, where

$I = [0, 1]$. If $X$ is a closed subset of $R^n$ then $D(X)$ is open. If $X$ is compact then $D(X)$ is the unique unbounded component of $R^n \setminus X$. We will use the notion of pseudo-component $D(X)$ only for compact $X$. Let us further denote $\tilde{X} := R^n \setminus D(X)$, $B(X) := \tilde{X} \setminus X$. The following proposition is evident

(2.1) PROPOSITION. *Let $X, Y$ be two compact subsets of $R^n$. Then we have*

(2.1.1) $D(X) = D(\tilde{X})$;

(2.1.2) *if $X \subset Y$ then $D(Y) \subset D(X)$;*

(2.1.3) *$\tilde{X}$ is a compact set;*

(2.1.4) *if $X \subset B(Y)$ and $Y$ is connected then $Y \subset D(X)$;*

(2.1.5) *if $X$ is a subset of an open or a closed ball $K$ in $R^n$ then $\tilde{X} \subset K$;*

(2.1.6) *$D(X)$ is arcwise connected;*

(2.1.7) *if $Z$ is a subset of $X$ open in $R^n$ then $\widetilde{X \setminus Z} = \tilde{X}$;*

(2.1.8) *if $X$ is connected then $\tilde{X}$ is connected.*

Let $\varphi: M \to R^n$ be a compact-valued map; we associate with $\varphi$ the multi-valued maps $B\varphi$, $D\varphi$, $\tilde{\varphi}: M \to R^n$ defined by the formulae $\tilde{\varphi}(x) := \widetilde{\varphi(x)}$, $B\varphi(x) := B(\varphi(x))$, $D\varphi(x) := D(\varphi(x))$. By $H_* = \{H_i\}_{i \in Z}(H^* = \{H^i\}_{i \in Z})$ we denote the Čech homology (cohomology) functor with coefficients in the field of rational numbers. Let $\tilde{H}_*$ be the reduced functor of $H_*$. We will need the following well known facts:

(2.2) ([9] I.1.1) On the category of compact subsets of $R^n$, the functors $H_*$ and $\mathrm{Hom}_Q \circ H_*$ are naturally isomorphic, here $\mathrm{Hom}_Q$ is the functor which assigns to a graded vector space $E$ the conjugate graded space $\mathrm{Hom}_Q(E) = \mathrm{Hom}(E, Q)$.

(2.3) The ALEXANDER DUALITY THEOREM [12]. *For compact $X \subset R^n$ and $q \in Z$ we have $\tilde{H}^s_{n-q-1}(R^n \setminus X) \approx H^q(X)$, where $\tilde{H}^s_*$ stands for the reduced functor of singular homology.*

Recall that a set $X$ is acyclic iff $\tilde{H}_*(X) = 0$. The compact-valued map $\varphi: M \to N$ is acyclic iff it is usc and $\varphi(x)$ is acyclic for every $x \in M$.

(2.4) DEFINITION. Let $M$ be an arbitrary topological space. A multi-valued map $\varphi: M \to R^n$ is called *spherical* if the following conditions are satisfied:

(2.4.1) for every $x \in M$ the set $\varphi(x)$ is a nonempty continuum and $\varphi$ is usc,

(2.4.2) the graph of the map $B\varphi$ is an open subset of $M \times R^n$,

(2.4.3) $\widetilde{\varphi(x)}$ is acyclic for each $x \in M$.

(2.5) Remark. A map $\varphi: M \to R^n$ with the images of points having homology of an $(n-1)$-dimensional sphere has been called an $n$-spherical map (comp. [8]). Observe that an acyclic map $\varphi: M \to R^n$ is not $n$-spherical in the sense of [8]. In (2.4) we have defined a class of maps, which in particular contains acyclic maps and maps of $n$-spherical type (see Theorem (2.6)), but for simplicity we use the name "spherical map" in place of "generalized spherical map".

(2.6) THEOREM. *Let $X$ be a compact subset of $R^n$.*

(2.6.1) *If $X$ is acyclic then $X = \tilde{X}$.*



(2.6.2) *If $X$ has homology of an $(n-1)$-dimensional sphere then $\tilde{X}$ is acyclic.*

Proof. (2.6.1). Clearly, $R^n \setminus X \supset D(X)$. In view of (2.3), $\tilde{H}^s_0(R^n \setminus X) = 0$ and the set $R^n \setminus X$ is arcwise connected. Since $D(X)$ is a component, $D(X) \supset R^n \setminus X$. We have $B(X) = \emptyset$ and $X = \tilde{X}$.

(2.6.2) Since $n-1 > 0$, we get $\tilde{H}_{n-1}(\tilde{X}) = H_{n-1}(\tilde{X})$. Using (2.1.6) and (2.3), we obtain $0 = \tilde{H}^s_0(D(X)) = H^{n-1}(\tilde{X})$. Hence, by (2.2), $H_{n-1}(\tilde{X}) = 0$, and so our hypothesis holds for $q = n-1$. Assume that $q \neq n-1$, $q \neq 0$:

$$0 = H^q(X) \approx \tilde{H}^s_{n-q-1}(B(X) \cup D(X)) \approx H^s_{n-q-1}(B(X) \cup D(X)) \approx H^s_{n-q-1}(B(X)) \oplus H^s_{n-q-1}(D(X)),$$

and hence $0 \approx \tilde{H}_{n-q-1}(D(X)) \approx H^q(\tilde{X}) \approx H_q(\tilde{X})$. For $q = 0$ it follows from (2.1.8) that $H_0(X) = Q$ and the proof is completed.

(2.7) COROLLARY. *Every acyclic map is spherical.*

(2.8) THEOREM. *Let $M$ be a topological space and let $\varphi: M \to R^n$ satisfy* (2.4.1). *Then $\tilde{\varphi}$ is usc.*

Proof. Take $\varepsilon > 0$ and $x \in M$. We can cover the set $R^n \setminus O_\varepsilon(\tilde{\varphi}(x))$ by finite by means balls $K(a_i, \varepsilon/2)$ and the set $R^n \setminus \overline{K(0, r)}$, with real $r > 0$, so that $O_\varepsilon(\tilde{\varphi}(x)) \subset K(0, r)$ $(O_\varepsilon(\tilde{\varphi}(x)) := \{a \in R^n;$ there is $b \in \tilde{\varphi}(x)$ such that $\|b - a\| < \varepsilon\})$. We join the points $a_i$ with an arbitrarily fixed point $a \in R^n \setminus \overline{K(0, r)}$ with arcs lying in $R^n \setminus \tilde{\varphi}(x)$, and we obtain a continuum $C$ such that $C \cup (R^n \setminus O_\varepsilon(\tilde{\varphi}(x))) \subset R^n \setminus \tilde{\varphi}(x)$. Let $\delta := \min\{\varepsilon/2, \mathrm{dist}(C, \tilde{\varphi}(x))\}$. We choose $U \ni x$ such that $\varphi(y) \subset O_\delta(\varphi(x))$ for $y \in U$. Moreover, the set $R^n \setminus O_\varepsilon(\tilde{\varphi}(x))$ is unbounded and contained in the unique component of $R^n \setminus O_\delta(\tilde{\varphi}(x))$. We have $R^n \setminus O_\varepsilon(\tilde{\varphi}(x)) \subset D\varphi(y)$ and the proof is completed.

(2.9) LEMMA. *Let $\varphi, T: M \to R^n$ be two usc nonempty-continuum-valued maps. If the graph of $B\varphi$ is open then*

(2.9.1) *$M$ is the disjoint union of the sets*

$$M_D := \{x;\ T(x) \subset D\varphi(x)\}, \quad M_K := \{x;\ T(x) \cap \varphi(x) \neq \emptyset\},$$

$$M_B := \{x;\ T(x) \subset B\varphi(x)\}$$

(2.9.2) *$M_D, M_B$ are open subsets of $M$,*

(2.9.3) *if $M$ is connected and $M_B, M_D$ are nonempty then $M_K$ is nonempty.*

In the sequel we denote by $S^{n-1}$ the unit sphere in $R^n$ and by $K^n$ the unit closed ball in $R^n$. From Lemma (2.9) we get

(2.10) COROLLARY (comp. (3.1) in [8]). *Let $\varphi: K^n \to R^n$ be a spherical map and $\varphi(S^{n-1}) \subset K^n$. Then $\varphi$ has a fixed point.*

Proof. Take $T = \mathrm{id}$ in Lemma (2.9) and assume that $M_K = 0$. Of course, $S^{n-1} \subset M_D$. In view of (2.7) and [9] we infer that the map $\tilde{\varphi}$ has a fixed point $z \in K^n$. Since $z \notin \varphi(z)$, we have $z \in B\varphi(z)$, but this contradicts (2.9.3).

**3. Examples.** In this section we explain the sense of the condition (2.4.2) in the definition of spherical maps. We already know that for acyclic maps condition (2.4.2) is satisfied automatically. First we show that if a map is continuous with respect to the Borsuk continuity metric or with respect to the Borsuk homotopy metric then (2.4.2) holds. Then we show that Hausdorff continuity is not sufficient for (2.4.2) to hold.

We remind the notion of the Borsuk continuity metric (cf. [2]). Let $(M, \nu)$ be a metric space. By $\mathscr{K}_0(M)$ we denote the class of all compact nonempty sets in $M$. We define the *Borsuk continuity metric* on $\mathscr{K}_0(M)$ by putting

$$\varrho_c(X, Y) := \max\{\inf\{|f|;\ f \in C(X, Y)\},\ \inf\{|g|;\ g \in C(Y, X)\}\},$$

where $C(X, Y)$ is the set of all continuous functions from $X$ to $Y$ and $|f|$ is defined by $|f| := \sup\{\nu(x, f(x)),\ x \in X\}$.

A (single-valued) function $f\colon M \to S^n$ is called (topologically) *essential* if it is not homotopic to a constant function from $M$ to $S^n$.

(3.1) Lemma. *Let $X$, $Y$ be two compact subsets of $R^n$. If $a \in D(Y) \cap B(X)$ then*

$$\inf\{|f|, f \in C(X, Y)\} \geqslant \operatorname{dist}(a, X) + \operatorname{dist}(a, Y).$$

Proof. Let $r_X = \operatorname{dist}(a, X)$, $r_Y = \operatorname{dist}(a, Y)$. We may assume without loss of generality that $a = 0$. Define

$$p_X\colon X \to S^{n-1}, \quad p_X(b) = \frac{b}{\|b\|},$$

$$p_Y\colon Y \to S^{n-1}, \quad p_Y(c) = \frac{c}{\|c\|}.$$

It is well known (comp. 11.1.10 in [7]) that $p_X$ is an essential function and $p_Y$ is not. Let $f\colon X \to Y$ be a continuous function. Then the composition $p_Y \circ f$ of $f$ and $p_Y$ is not essential; hence there exists $x \in X$ such that $p_X(x) = -p_Y(f(x))$. This implies $|f| \geqslant \|x - f(x)\| = \|x\| + \|f(x)\| \geqslant r_X + r_Y$ and the proof is completed.

A nonempty-compact-valued map $\varphi\colon (M, \nu) \to (N, \eta)$ is called *continuous with respect to the metric of continuity $\varrho_c$ ($C$-continuous)* if $\varphi\colon (M, \nu) \to (\mathscr{K}_0(N), \varrho_c^\eta)$ is a (single-valued) continuous function.

(3.2) Remark. We now give an example showing that the composition of two $C$-continuous maps need not be $C$-continuous. Let $\varphi\colon I \to S^1$ and $\psi\colon S^1 \to R^2$ be two maps defined as follows,

$$\varphi(t) := \{z \in S^1;\ 0 \leqslant \arg(z) \leqslant t\},$$

$$\psi(z) := \{v \in S^1;\ \arg(z) + 1/2 \leqslant \arg(v) \leqslant \arg(z) + 2\pi\}.$$

The maps $\varphi$ and $\psi$ are obviously $C$-continuous, but their composition

$$\psi \circ \varphi\colon I \to R^2, \psi \circ \varphi(t) = \{v \in S^1;\ \tfrac{1}{2} \leqslant \arg(v) \leqslant t + 2\pi\}$$

is not $C$-continuous at the point $t = 1/2$ (the proof follows from (3.1)).

(3.3) Theorem. *If $\varphi\colon (M, \nu) \to R^n$ is a $C$-continuous map then* (2.4.2) *holds.*



Proof. Let $(x, a) \in B\varphi$, i.e. $a \in B\varphi(x)$. We take two real numbers $\alpha, \beta > 0$ such that $\alpha + \beta = \operatorname{dist}(a, \varphi(x))$. Evidently,

(3.3.1) $K(a, \alpha) \cap O_\beta(\varphi(x)) = \emptyset$.

By the $C$-continuity of $\varphi$ there is an open nbd $V$ of $x$ in $M$ such that $\varrho_C(\varphi(x), \varphi(y)) < \beta$ for every $y \in V$. Therefore

(3.3.2) $\varphi(y) \subset O_\beta(\varphi(x))$ for each $y \in V$.

We will prove that $V \times K(a, \alpha)$ is an open nbd of $(x, a)$ in the graph $\Gamma_{B\varphi}$ of $B\varphi$. Assume on the contrary that there exist $z \in V$ and $b \in K(a, \alpha)$ such that $b \notin B\varphi(z)$. From (3.3.1) and (3.3.2) we infer that $b \notin \varphi(z)$, and hence $b \in D\varphi(z)$. Obviously, $b \in B\varphi(x)$. It follows (cf. 3.1)) that $\varrho_c(\varphi(x), \varphi(y)) \geqslant \inf\{|f|;\ f \in C(\varphi(x), \varphi(y))\} \geqslant \beta$, a contradiction.

(3.4) Corollary. *Every $C$-continuous nonempty-continuum-valued map $\varphi\colon (M, \nu) \to R^n$ satisfying condition* (2.4.3) *is a spherical map.*

Proof. In view of (3.3) it remains to show that (2.4.1) is satisfied. We will show that $\varphi$ is continuous. It is not difficult to see that $\varrho_S(X, Y) \leqslant \varrho_c(X, Y)$ for every $X, Y \in \mathscr{K}_0(M)$, where $\varrho_S(X, Y)$ denotes the classical Hausdorff distance between two compact sets.

(3.5) O' Neill's Example [11]. Let $\varphi\colon K^2 \to K^2$ be the map defined as follows,

$$\varphi(x) = \{y \in K^2;\ \|y - x\| = \xi(x)\} \cup \{y \in S^1;\ \|y - x\| \geqslant \xi(x)\},$$

where

$$\xi(x) = 1 - \|x\| + \|x\|^2.$$

It is evident that $\varphi$ is continuous with respect to $\varrho_S$. Moreover, the graph $\Gamma_{B\varphi}$ of $B\varphi$ is not an open subset of $K^2 \times K^2$. Finally, observe that $\varphi$ has no fixed point. In the preceding section we have pointed out that the Brouwer fixed point theorem (2.10) is true for spherical maps.

**4. A Poincaré type coincidence theorem.** Let $X$ be a subset of $R^n$. The set $\operatorname{Sw}(X) := \bigcup_{\lambda > 1} \lambda \cdot X$ is called the *shadow* of $X$.

(4.1) Proposition. *Let $X$ be a compact subset of $R^n$. Then*

(4.1.1) *for every connected unbounded set $M \subset R^n$ we have $X \cap M = \emptyset$ iff $\tilde{X} \cap M = \emptyset$,*

(4.1.2) *if $X$ is connected then $\operatorname{Sw}(X)$ is connected,*

(4.1.3) *if $0 \in D(X)$ then $B(X) \subset \operatorname{Sw}(X) = \operatorname{Sw}(\tilde{X})$,*

(4.1.4) *for every compact $Y \subset R^n$, $\operatorname{Sw}(X) \cap Y = \emptyset$ implies $X \subset Y \cup D(Y)$.*

For any spherical map $\varphi\colon K^n \to R^n$ such that $0 \notin \varphi(S^{n-1})$ we define the topological degree $\operatorname{Deg}(\varphi, 0)$ of $\varphi$ with respect to 0 by putting

$$\operatorname{Deg}(\varphi, 0) = \begin{cases} \{0\} & \text{if there exists } x \in S^{n-1} \text{ such that } 0 \in B\varphi(x), \\ \operatorname{Deg}(\tilde{\varphi}, 0) & \text{if } 0 \in D\varphi(x) \text{ for every } x \in S^{n-1} \end{cases}$$

where $\mathrm{Deg}(\tilde{\varphi}, 0)$ denotes the topological degree for acyclic maps (see, for example, [5]). The topological degree just defined has the same properties as the Brouwer degree.

Let $\mathscr{A}(K^n, R^n)$ be the class of all multi-valued maps from $K^n$ to $R^n$ which are either admissible or spherical. The notion of an admissible multi-valued map has been first considered in [9]. Also the notion of topological degree for admissible maps was first defined in [9].

(4.2) THEOREM. *Let $\varphi, T \in \mathscr{A}(K^n, R^n)$ be two multi-valued maps. Assume that the following conditions are satisfied*:

(4.2.1) $0 \notin \varphi(S^{n-1})$,

(4.2.2) $0 \notin \mathrm{Deg}(\varphi, 0)$,

(4.2.3) $\mathrm{Sw}(\varphi(x)) \cap T(x) = \emptyset$ *for every* $x \in S^{n-1}$.

*Then there exists $y \in K^n$ such that $\varphi(y) \cap T(y) \neq \emptyset$.*

Proof. (4.2.4) In the case where $\varphi$ and $T$ are admissible maps, Theorem (4.2) was proved in [4].

(4.2.5) Thus, let us assume that $\varphi$ is admissible and $T$ is spherical. Assume, contrary to the claim, that $\varphi(y) \cap T(y) = \emptyset$ for every $y \in K^n$. Let $\psi$ be an $s$-admissible selector of $\varphi$ (i.e. a selector which is, in particular, continuum-valued and usc — comp. [9]). By (2.8) $\tilde{T}$ is admissible; thus, as we have noticed in (4.2.4), there exists $z \in K^n$ such that $\psi(z) \cap \tilde{T}(z) \neq \emptyset$. Consequently $\psi(z) \subset BT(z)$ and the set $M_B = \{z;\ \psi(z) \subset BT(z)\}$ is nonempty. By using (4.2.3) and (4.1) it is not difficult to see that $\psi(x) \subset DT(x)$ for each $x \in S^{n-1}$. This implies that the set $M_D = \{x;\ \psi(x) \subset DT(x)\}$ is nonempty and hence $M_K = \{x;\ \psi(x) \cap T(x) \neq \emptyset\} \neq \emptyset$ (see (2.9.3)), a contradiction.

(4.2.6) In the case where $\varphi$ is spherical and $T \in \mathscr{A}(K^n, R^n)$ we define $T': K^n \to R^n$ to be $T$ if $T$ is spherical and to be certain $s$-admissible selector of $T$ if $T$ is admissible. In view of (4.2.4) and (4.2.5) there exists $z \in K^n$ such that $\tilde{\varphi}(z) \cap T'(z) \neq \emptyset$. Now assume that $\varphi(y) \cap T(y) = \emptyset$ for every $y \in K^n$. Consequently $T'(z) \subset B\varphi(z)$ and the set $M_B$ in Lemma (2.9) is nonempty. By (4.2.3) and (4.1) it is not difficult to see that $T'(x) \subset D\varphi(x)$ for each $x \in S^{n-1}$ and hence the set $M_D$ in Lemma (2.9) is nonempty, a contradiction with (2.9.3).

**5. Birkhoff-Kellogg theorem.** We will say that an admissible map $\varphi: M \to S^n$ is *algebraically essential* iff there exists a selected pair $(p, q) \subset \varphi$ such that $H_n(q)H_n(p)^{-1} \neq 0$ (for details, see [9]). In the case of single-valued maps this notion was introduced by Borsuk in [3].

For $X \subset R^n$, by $\mathrm{LC}(X)$ we will denote the *linear cone* of $X$ in $R^n$, which is defined as $\mathrm{LC}(X) = \bigcup_{\lambda \in R} \lambda \cdot X$. First, we will generalize Theorem (3.1) of [10].

(5.1) THEOREM. *Let $e: M \to S^{2(n-1)}$ be an algebraically essential map and let $\varphi: M \to P^{2n-1} := R^{2n-1} \setminus \{0\}$ be an admissible map. Then there exists a point $x \in M$ such that $\mathrm{LC}(e(x)) \cap \varphi(x) \neq \emptyset$.*



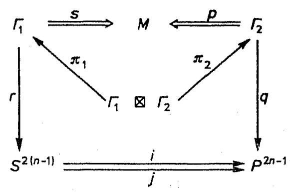
Proof. Let $(s, r)$ be a selected pair of $e$ such that

(5.1.1) $H_{2(n-1)}(r)H_{2(n-1)}(s)^{-1} \neq 0$,

and let $(p, q)$ be a selected pair of $\varphi$. Assume, on the contrary, that

(5.1.2) $\mathrm{LC}(e(x)) \cap \varphi(x) = \emptyset$ for every $x \in M$.

Consider the diagram




in which $\Gamma_1 \boxtimes \Gamma_2 := \{(m_1, m_2) \in \Gamma_1 \times \Gamma_2,\ s(m_1) = p(m_2)\}$, $\pi_1(m_1, m_2) := m_1$, $\pi_2(m_1, m_2) := m_2$, $i(x) := x$, $j(x) := -x$, where the double arrows stand for Vietoris maps (for definition see [8], [9], [5]). It is easy to see that $\pi_1$ and $\pi_2$ are Vietoris maps. Evidently, $(s\pi_1, r\pi_1)$ is a selected pair of $e$. Define two homotopies $F, G$: $(\Gamma_1 \boxtimes \Gamma_2) \times I \to p^{2n-1}$ by putting $F(m, t) := t \cdot ir\pi_1(m) + (1-t) \cdot q\pi_2(m)$, $G(m, t) := t \cdot jr\pi_1(m) + (1-t) \cdot q\pi_2(m)$. The correctness of the above definitions follows from (5.1.2). Therefore $ir\pi_1$ is homotopic to $jr\pi_1$ and, since $H_{2(n-1)}(i) = -H_{2(n-1)}(j)$, we have $H_{2(n-1)}(r) = 0$; but this contradicts (5.1.1).

(5.2) Remark. Theorem (5.1) remains true when $\varphi$ is a spherical map. Indeed, if $0 \in B\varphi(x)$ for some $x \in M$ then our hypothesis is evident. Assume that $0 \notin B\varphi(x)$ and $\mathrm{LC}(e(x)) \cap \varphi(x) = \emptyset$ for each $x \in M$. Then $\mathrm{LC}(e(x)) \subset D\varphi(x)$, and hence $\mathrm{LC}(e(x)) \cap \tilde{\varphi}(x) = \emptyset$, which contradicts (5.1).

**6. Borsuk-Ulam theorem.** Consider two topological $T_3$-spaces $M$ and $N$.

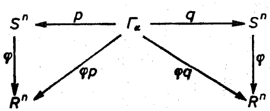
(6.1) PROPOSITION. *Let $\varphi: M \to N$ be an usc nonempty-continuum-valued map. If $M$ is connected then the graph $\Gamma_\varphi$ of $\varphi$ is connected.*

The proof of (6.1) is elementary.

An acyclic map $\alpha: S^n \to S^n$ with symmetric graph $\Gamma_\alpha$ is called *involution*.

(6.2) THEOREM. *Let $\alpha: S^n \to S^n$ be an involution and let $\varphi: S^n \to R^n$ be a spherical map. Then there exists a point $(x, y) \in \Gamma_\alpha$ such that $\varphi(x) \cap \varphi(y) \neq \emptyset$.*

Proof. Consider the natural projections $p, q: \Gamma_\alpha \to S^n$ given by the formulae $p(x, y) = x$, $q(x, y) = y$. Then $\alpha = qp^{-1}$. Assume that $\varphi(x) \cap \varphi(y) = \emptyset$ for every $(x, y) \in \Gamma_\alpha$. Consider the diagram



In virtue of (6.5) of [5], there exists $m_0 \in \Gamma_\alpha$ such that $\tilde{\varphi}p(m_0) \cap \tilde{\varphi}q(m_0) \neq \emptyset$. On the other hand, we know that $\varphi p(m) \cap \varphi q(m) = \emptyset$ for every $m \in \Gamma_\alpha$. We have the following two possibilities:

(6.2.1) $\varphi p(m_0) \subset B\varphi q(m_0)$,

(6.2.2) $\varphi q(m_0) \subset B\varphi p(m_0)$.

Consider the case (6.2.1). From (2.9) it follows that the set $\Gamma_\alpha$ is the disjoint union of the sets

$$\Gamma_B := \{m \in \Gamma_\alpha,\ \varphi p(m) \subset B\varphi q(m)\},\quad \Gamma_D := \{m \in \Gamma_\alpha,\ \varphi p(m) \subset D\varphi q(m)\}$$

and

$$\Gamma_K := \{m \in \Gamma_\alpha,\ \varphi p(m) \cap \varphi q(m) \neq \emptyset\}.$$

From our assumptions we infer that $\Gamma_K = \emptyset$ and $m_0 \in \Gamma_B$. Since $\alpha$ is an involution, there exists $n_0 \in \Gamma_\alpha$ such that $(p(m_0), q(m_0)) = (q(n_0), p(n_0))$; hence $n_0 \in \Gamma_D$ and, in view of (2.9) and (6.1), we get a contradiction. Observe that in the case (6.2.2) the proof is analogous.

**7. Invariance of domain.** Let $(M, \nu)$ be a metric space. A multi-valued map $\varphi\colon M \to Y$ is called an *$\varepsilon$-map* if, for every $x, y \in M$, $\varphi(x) \cap \varphi(y) \neq \emptyset$ implies $\nu(x, y) < \varepsilon$. A map $\varphi\colon M \to Y$ is called *strongly injective* if it is an $\varepsilon$-map for every $\varepsilon > 0$ (cf. [5]).

(7.1) LEMMA. *Let $\varphi\colon R^n \to R^n$ be a spherical map and let $\varepsilon$ be a positive number. Then $\varphi$ is $\varepsilon$-map iff $\tilde{\varphi}$ is $\varepsilon$-map.*

Proof. Assume that $\varphi$ is an $\varepsilon$-map and let $x_1, x_2 \in R^n$ be two points such that $\nu(x_1, x_2) \geq \varepsilon$. Assume further that $\tilde{\varphi}(x_1) \cap \tilde{\varphi}(x_2) \neq \emptyset$. Since $\varphi(x_1) \cap \varphi(x_2) = \emptyset$, we have two possibilities:

(7.1.1) $\varphi(x_1) \subset B\varphi(x_2)$,

(7.1.2) $\varphi(x_2) \subset B\varphi(x_1)$.

Assume that (7.1.1) holds. Applying Lemma (2.9) to the constant map $\varphi(x_1)$ and $\varphi$ on $M := R^n \setminus K(x_1, \varepsilon)$, we obtain $\varphi(x) \subset D\varphi(x_1)$ for every $x \in R^n \setminus K(x_1, \varepsilon)$. Applying Lemma (2.9) to $\varphi$ and the constant map $\varphi(x_2)$ on $M := R^n \setminus K(x_2, \varepsilon)$, we obtain $\varphi(x) \subset B\varphi(x_2)$ for every $x \in R^n \setminus K(x_2, \varepsilon)$. Let $x_3 \in R^n \setminus (K(x_1, \varepsilon) \cup K(x_2, \varepsilon))$. Take $T := \varphi(x_3)$ to be the constant map on $M := R^n \setminus K(x_3, \varepsilon)$. Since $\varphi(x_3) \subset D\varphi(x_1)$ and $\varphi(x_3) \subset B\varphi(x_2)$, we get that $R^n \setminus K(x_3, \varepsilon)$ is not connected, a contradiction.

The proof for the case (7.1.2) is analogous.

(7.2) THEOREM. *Assume that $\varphi\colon R^n \to R^n$ is a spherical $\varepsilon$-map for some $\varepsilon > 0$. Then $\varphi(R^n)$ is an open subset of $R^n$.*

Proof. It follows from (7.1) and (7.5) in [5] that $\tilde{\varphi}(R^n)$ is an open subset of $R^n$. For the proof it is sufficient to show that $\varphi(R^n) = \tilde{\varphi}(R^n)$. Let $a \in \tilde{\varphi}(x)$ for some $x \in R^n$ and assume that $y \notin K(x, \varepsilon)$. Since $\tilde{\varphi}$ is $\varepsilon$-map, we have $a \in D\varphi(y)$. Using (2.9), we obtain a point $z \in R^n$ such that $a \in \varphi(z)$. Since $\varphi(R^n) \subset \tilde{\varphi}(R^n)$, the proof is completed.



Usually the theorem on invariance of domain is formulated for strongly injective multi-valued maps from an open subset $U$ of $R^n$ into $R^n$ (comp. [5]). We will show that if $\varphi\colon U \to R^n$ is a spherical strongly injective map then $\varphi$ must be an acyclic map. Thus the theorem on invariance of domain for strongly injective spherical maps is exactly the same as for strongly injective acyclic maps.

(7.3) THEOREM. *Let $X$ be a subset of $R^n$ and let $\varphi\colon X \to R^n$ be an usc nonempty-continuum-valued strongly injective map such that $B\varphi$ has an open graph. Then $B\varphi(x)$ is empty for every $x \in \mathrm{Int}(X)$.*

Proof. Let $x \in \mathrm{Int}(X)$ and $a \in B\varphi(x)$. From our assumptions it follows that the set $\{y \in X;\ a \in B\varphi(y)\}$ is an open subset of $X$. However, there exists an open ball $K \subset X$ with the center at $x$ such that $a \in B\varphi(y)$ for every $y \in K$. Decompose the ball $K$ into three disjoint subsets,

$$K_B = \{y \in K;\ \varphi(y) \subset B\varphi(x)\},\quad K_D = \{y \in K;\ \varphi(y) \subset D\varphi(x)\}$$

and

$$K_K = \{y \in K;\ \varphi(y) \cap \varphi(x) \neq \emptyset\}.$$

Of course the set $K_K = \{x\}$ is a singleton and it does not separate the ball $K$. There are two possibilities:

(7.3.1) $\varphi(y) \subset B\varphi(x)$ for each $y \in K \setminus \{x\}$,

(7.3.2) $\varphi(y) \subset D\varphi(x)$ for each $y \in K \setminus \{x\}$.

Assume that (7.3.2) holds (the proof in the case (7.3.1) is similar). Since $a \in B\varphi(y) \cap B\varphi(x)$ for $y \in K \setminus \{x\}$, we have $\varphi(x) \subset B\varphi(y)$ for each $y \in K \setminus \{x\}$. Let $y_1, y_2 \in K$ be two points such that $y_1 \neq y_2 \neq x$, $y_1 \neq x$. Consider two different decompositions of $K$; $K_B^1 := \{y;\ \varphi(y) \subset B\varphi(y_1)\}$, $K_D^1 := \{y;\ \varphi(y) \subset D\varphi(y_1)\}$, $K_K^1 := \{y; \varphi(y) \cap \varphi(y_1) \neq \emptyset\}$ and $K_B^2 := \{y;\ \varphi(y) \subset B\varphi(y_2)\}$, $K_D^2 := \{y;\ \varphi(y) \subset D\varphi(y_2)\}$, $K_K^2 := \{y;\ \varphi(y) \cap \varphi(y_2) \neq \emptyset\}$.

Of course, we have $K_K^1 = \{y_1\}$, $K_K^2 = \{y_2\}$. Since $x \in K_B^1$, we have $K_D^1 = \emptyset$, and hence $y_2 \in K_B^1$ and $\varphi(y_2) \subset B\varphi(y_1)$. On the other hand, $x \in K_B^2$ and so $K_D^2 = \emptyset$ and $y_1 \in K_B^2$. Therefore we obtain $\varphi(y_1) \subset B\varphi(y_2)$ and consequently $\varphi(y_1) \cap \varphi(y_2) \neq \emptyset$; but this is impossible.

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# The Hurewicz and Whitehead theorems with compact carriers

by

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**Abstract.** We prove analogues of the classical Hurewicz and Whitehead theorems for Borsuk's weak shape theory or, more generally, for the category generated by the homotopy category of pointed polyhedra. We also give a certain geometrical application of the modified Hurewicz theorem.

**Introduction.** The dual notion to that of a pro-category will be called an *in-category*. By induction, we say that a category pro-$\mathscr{C}'$ or in-$\mathscr{C}'$ is *k-generated* by a category $\mathscr{C}$ whenever provided $\mathscr{C}'$ is $(k-1)$-generated by $\mathscr{C}$ (we assume that $\mathscr{C}$ is 0-generated by $\mathscr{C}$).

The classical Hurewicz and Whitehead theorems have their analogues in shape theory and pro-homotopy theory (for example see [M–S]). We will prove analogues of these theorems in a more general case, for any category generated by the homotopy category of pointed polyhedra $\mathrm{HPol}_*$. As a consequence, we obtain modified Hurewicz and Whitehead theorems for Borsuk's weak shape theory (i.e. shape theory with compact carriers) and for compactly generated shape theory. Under more restrictive assumptions a Whitehead type theorem for compactly generated shape theory has been proved previously by T. J. Sanders [Sa2].

The inspiration to prove a Hurewicz type theorem in Borsuk's weak shape theory was the following question of H. Toruńczyk [T].

Question 1. *Let $A$ be a subset of $R^n$ such that every map $I^2 \to R^n$ is approximable by mappings with images missing $A$. Let $f\colon \partial I^s \to R^n$ be a map which satisfies* $\mathrm{im}(f) \cap A = \emptyset$, *where $s+\dim A < n$. Is there a compact set $C \subset R^n \setminus A$ such that $f$ is null homotopic in every neighborhood $U$ of $C$ in $R^n$?*

If $A$ is $\sigma$-compact then $\{g \in C[I^s, R^n] |\ \mathrm{im}(g) \cap A = \emptyset\}$ is dense in $C[I^s, R^n]$ (see [Š]). This needs be shown for compacta only and follows by induction on $s$ using Alexander duality and Hurewicz theorem. H. Toruńczyk asked if one can prove a Hurewicz type theorem in Borsuk's weak shape theory. He suggested