THEOREM 4. *If* $2^{\varkappa} > 2^{\omega}$ *then* $\neg Q_1((2^{\varkappa})^+)$.

Proof. Let $\lambda = (2^{\varkappa})^+$. For each $\alpha < \lambda$ choose a one-to-one function $h_\alpha\colon \alpha \to \mathscr{P}(\varkappa)$ and define

$$S_\alpha = \{\langle \xi, \eta\rangle\colon \eta < \alpha \ \&\ \xi \in h_\alpha(\eta)\} \subset \varkappa \times \lambda .$$

Now suppose that there is a countable family $\mathscr{D} = \langle D_n\colon n < \omega\rangle$ of subsets of $\kappa \times \lambda$ s.t. $\{S_\alpha\colon \alpha < \lambda\} \subset [C_1^2(\lambda) \cup \mathscr{D}]$. Choose $X \subset \lambda$ s.t. $|X| = (2^{\omega})^+$ and for each $n < \omega$ $D_n \cap (\varkappa \times X) = D_n' \times X$ for some $D_n' \subset \varkappa$ (X is a subset of a counter-image of the point $g\colon \lambda \to {}^{\varkappa \times \omega}2$ defined by $g(\eta)(\xi, n) = 0$ iff $\langle \xi, \eta\rangle \in D_n$). Thus

$$\{S_\alpha \cap (\varkappa \times X)\colon \alpha < \lambda\} \in [C_1^2(\lambda)] .$$

Fix $\alpha < \lambda$ s.t. $X \subset \alpha$. Now, if $\eta_1 \eta_2 \in X$ and $\eta_1 \neq \eta_2$ then

$$h_\alpha(\eta_1) \neq h_\alpha(\eta_2), \quad \text{i.e.} \quad \{\xi\colon \langle \xi, \eta_1\rangle \in S_\alpha\} \neq \{\xi\colon \langle \xi, \eta_2\rangle \in S_\alpha\}.$$

Since $|X| > 2^{\omega}$, it is easy to see that $S_\alpha \cap (\kappa \times X) \notin [C_1^2(\lambda)]$, because otherwise each set $\{\xi\colon \langle \xi, \eta\rangle \in S_\alpha\}$ would be determined by some real number. This gives contradiction.

An easy corollary to this theorem is that $2^{\omega} = \omega_2$, $2^{\omega_1} = \omega_3$ and $2^{\omega_2} \geqslant \omega_4$ imply $\neg Q_1((2^{\mathfrak{c}})^+)$. Another consequence is that $\neg Q_1((2^{\mathfrak{c}})^+)$. Let us also notice that an easy modification of the proof of Theorem 4 (using the fact that $\neg P_n(\beth_n^+)$) gives also that, for any $n < \omega$, $Q_n(\kappa)$ implies $\kappa \leqslant 2^{\beth_n^+}$. So the following problem might be mentioned in this context:

"does $Q_n(\varkappa)$ imply $\varkappa \leqslant \beth_{n+1}$ for $2 \leqslant n < \omega$?"

Let us finally note that in a model of ZFC obtained by adding at least ω_ω Cohen reals, for every $n < \omega$ we have $Q_n(\kappa)$ iff $\kappa \leqslant \omega_{n+1}$.

The proof is similar to that of Theorem 3.

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Terminal continua and the homogeneity *

by

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Abstract. In the paper we prove the following statements: (1) every hereditarily indecomposable and continuously homogeneous continuum is one-dimensional; (2) every proper terminal subcontinuum of a homogeneous curve is tree-like; (3) every homogeneous hereditary θ-continuum is atriodic.

1. Terminal continua. Definitions which are not recalled here can be found in [13]. All spaces in this paper are metric.

A compact space X has *Kelley's property at* $x \in X$ if for every continuum $Y \subset X$ containing x and for every sequence x_n of points of X converging to x, there exists a sequence of continua $Y_n \subset X$ converging to Y such that $x_n \in Y_n$. A space X has *Kelley's property* if it has Kelley's property at each point (see [21]).

A space is said to be *homogeneous with respect to the class M of mappings* if for every two points p and q of X, there exists a continuous surjection f from X onto itself such that $f \in M$ and $f(p) = q$. A continuum homogeneous with respect to homeomorphisms (continuous maps) will be simply called *homogeneous* (*continuously homogeneous*).

Charatonik has observed in [2] that

(1.1) *Continua which are homogeneous with respect to open mappings have Kelley's property.*

A subcontinuum Q of X is called *terminal* if $K \in C(X)$ and $K \cap Q \neq \emptyset$ imply either $K \subset Q$ or $Q \subset K$, where $C(X)$ denotes as usually the space of all subcontinua of X with the Hausdorff distance. We will denote the collection of all terminal subcontinua of X by $T(X)$ and the collection of all indecomposable subcontinua of X by $IN(X)$. The following proposition is an immediate consequence of above definitions.

(1.2) *If a continuum X has Kelley's property, then $T(X)$ is closed in $C(X)$.*

We have (see [10])

(1.3) *If f is a continuous mapping from a continuum X onto Y, $K \in T(Y)$ and C is a component of $f^{-1}(K)$, then $f(C) = K$.*

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In fact, let C be a component of $f^{-1}(K)$ and $b \in K \setminus f(C)$. Consider an open neighbourhood U of C such that $\overline{U} \cap f^{-1}(b) = \emptyset$ and take a component M of $\overline{U}$ containing C. Then $f(M) \setminus K \neq \emptyset$ and $b \notin f(M)$ which contradict the terminalit of K in X. Hence $f(C) = K$.

We have (see [13], (6.13), p. 18)

(1.4) *If X is a homogeneous continuum, then* $T(X) \setminus \{X\} \subset IN(X)$.

Recall that a mapping $f\colon X \to Y$ from a space X onto Y is *completely regular* if for given $\varepsilon > 0$ and $y \in Y$ there exists an open set V containing y such that if $y' \in V$, then there is a homeomorphism h from $f^{-1}(y)$ to $f^{-1}(y')$ such that $d(x, h(x)) < \varepsilon$ for $x \in f^{-1}(y)$. It is known (see [15], Corollary 3.3)

(1.5) *If f is a completely regular monotone mapping of a metric curve onto a nondegenerate continuum Y, then* $\dim Y = 1$ *and for each* $y \in Y$, $f^{-1}(y)$ *is tree-like.*

Let $H(X)$ denote the group of all homeomorphisms of a space X onto itself. Every metric σ on X such that if $\sigma(x, y) < \varepsilon$ then $\sigma(h, e) < \varepsilon$ for some $h \in H(X)$ with $h(x) = y$ where e denotes the identity on X, will be called an *Effros metric* on X. If ϱ is an arbitrary bounded metric on X, then the formula

$$\sigma(x, y) = \inf\{\varrho(h, e)\colon h \in H(X) \text{ and } h(x) = y\}$$

gives an Effros metric. It easily follows from Effros' Theorem ([7])

(1.6) (ε-push property). *If X is a homogeneous compact space and $\varepsilon > 0$, then there is $\delta > 0$ such that if $\varrho(x, y) < \delta$, then there is $h \in H(X)$ with $h(x) = y$ and $\varrho(h, e) < \varepsilon$.*

In particular, (1.6) implies (compare [3])

(1.6′) *A continuum X with a metric ϱ is homogeneous if and only if X possesses an Effros metric equivalent to ϱ.*

We will always assume that a homogeneous continuum is equipped with an Effros metric. We claim that

(1.7) *If a continuum X is homogeneous and $Y \in C(X) \setminus T(X)$, then maximal terminal continua contained in Y form a completely regular monotone decomposition of Y.*

In fact, let $\varepsilon = \frac{1}{2}\inf\{\operatorname{dist}(K, Y)\colon K \in T(X) \text{ and } K \subset Y\}$. Since $T(X) = \overline{T(X)}$ ((1.1) and (1.2)) and $Y \notin T(X)$, we have $\varepsilon > 0$. Let K be a maximal terminal subcontinuum of X contained in Y. If $y \in Y$ and $\sigma(y, K) < \delta < \varepsilon$, then there is $h \in H(X)$ such that $y \in h(K)$ and $\sigma(h, e) < \delta$. Therefore $h(K) \in T(X)$ and $h(K) \subset Y$. Suppose $L \in T(X)$ and $h(K) \subset L \subset Y$. Then $\operatorname{dist}(L, Y) > \varepsilon$. We have $K \subset h^{-1}(L) \subset Y$ and $h^{-1}(L) \in T(X)$; thus $K = h^{-1}(L)$, i.e., $L = h(K)$. This means that (1.7) holds.

Since every subcontinuum of a tree-like continuum is a tree like continuum, from (1.5) and (1.7) we obtain

(1.8) *If a curve X is homogeneous, $A \in T(X)$, $A \subset Y \in C(X) \setminus T(X)$, then A is tree-like.*

Recall that a connected space X is *discoherent* if and only if the complement of each closed connected subset C of X is connected (see [11], p. 163). The collection of all discoherent subcontinua of X we will denote by $D(X)$.

Recall that a mapping $f\colon X \to Y$ is *essential* provided that f is not homotopic to a constant map. It is well-known that

(1.9) *Let $f\colon X \to K$ be an essential map from a compact space X into $K \in$* ANR. *Then there is a subcontinuum C of X such that $f|C$ is irreducibly essential.*

Moreover

(1.10) *Let $f\colon X \to K$ be an irreducibly essential map from a continuum X into a connected graph K. Then X is discoherent.*

In fact, suppose $X = A \cup B$ where A and B are proper subcontinua of X and $A \cap B$ is connected. Let $p\colon \tilde{K} \to K$ be the universal covering projection (see [20], p. 80). Since p has the homotopy lifting property and the maps $f|A\colon A \to K$ and $f|B\colon B \to K$ are inessential, there are maps $g_A\colon A \to \tilde{K}$ and $g_B\colon B \to \tilde{K}$ such that $f|A = pg_A$, $f|B = pg_B$ and $g_A(a) = g_B(a)$ for some $a \in A \cap B$. Then $g_A|A \cap B = g_B|A \cap B$, because $A \cap B$ is connected. Hence we have a map $g\colon X \to \tilde{K}$ such that $f = pg$. Since $\tilde{K}$ is a connected, simply connected, one dimensional polyhedron, it is contractible in itself, therefore g is homotopic to a constant; thus f is homotopic to a constant, a contradiction.

We have

(1.11) *If X is a homogeneous curve with $D(X) \subset T(X)$, then every proper subcontinuum of X is a tree-like continuum.*

Proof. Suppose Y is a proper subcontinuum of X which is not tree-like. Then there is a mapping f from Y into a connected graph K which is essential (see [1]). According to (1.9) we can assume that f is irreducibly essential. Hence Y is discoherent by (1.10). Thereby Y is terminal. Corollary 3.6 from [8] (compare [18]) implies that f can be extended to a mapping f^* defined on the whole space X. Since $K \in$ ANR and K is compact, there is a positive number ε such that if $g, h\colon X \to K$ and $\varrho(g, h) < \varepsilon$, then g and h are homotopic (see [11], p. 379).

Let $\mathscr{F}$ denote the collection of all maps from X into K which are homotopic to f^*. Now, if $g, h \in \mathscr{F}$, $g|Y_g$ is irreducibly essential and $h|Y_h$ is irreducibly essential, then either $Y_g \cap Y_h = \emptyset$ or $Y_g = Y_h$, because continua Y_g and Y_h are terminal.

Since f is uniformly continuous, there is $\delta > 0$ such that if $\sigma(x, x') < 2\delta$, then $\varrho(f^*(x), f^*(x')) < \varepsilon$. Let $w \in H(X)$ and $\sigma(w, e) < \delta$, then $f^*w, f^*w^{-1} \in \mathscr{F}$. Since $f^*|Y$ is irreducibly essential, we obtain that $f^*w|Y$ is irreducibly essential; thus $f^*|w(Y)$ is essential. Now, if $S \subset w(Y)$, $g \in \mathscr{F}$ and $g|S$ is irreducibly essential, then $g|w^{-1}(S)$ is essential; thus $w^{-1}(S) = Y$, i.e. $S = w(Y)$.

Above considerations imply that $\{Z\in C(X)$: there is $g\in\mathscr{F}$ such that $g|Z$ is irreducibly essential$\}$ gives a completely regular monotone decomposition of a δ-neighbourhood of Y with Y as an element of this decomposition. It follows from (1.5) that Y is tree-like, a contradiction.

If X is an atriodic continuum, then $D(X)\subset T(X)$ (see [13], (13.2)). Therefore, from (1.11) we obtain (compare [9])

(1.12) COROLLARY. *If X is a homogeneous atriodic curve, then every proper subcontinuum of X is a tree-like continuum.*

We have

(1.13) COROLLARY. *If X is a homogeneous curve, then every proper terminal subcontinuum of X is a tree-like continuum.*

In fact, let $A\in T(X)\setminus\{X\}$. If there is $Y\in C(X)\setminus T(X)$ such that $A\subset Y$, then A is tree-like by (1.8). Suppose that if $A\subset Y\in C(X)$, then $Y\in T(X)$. Let B be a minimal continuum in A with respect to the property that if $B\subset Y\in C(X)$, then $Y\in T(X)$. Then homeomorphic copies of B form a completely regular monotone decomposition of X. Denote the quotient map by φ. The quotient space $\varphi(X)$ is hereditarily indecomposable; thus every its proper subcontinuum is tree-like by (1.11). Since every point inverse of φ is tree-like, we conclude, by (6.14) in [1], p. 18, that every proper subcontinuum of X is tree-like. In particular, A is a tree-like continuum.

If $Q\subset L\subset K\neq Q$, $Q,L,K\in T(X)$ imply $Q=L$ or $L=K$, then we say that Q, K form a *jump* and Q is called the *beginning* of a jump and K is called a *sequel* of Q and will be denoted by $S(Q)$.

(1.14) *If $Q\in T(X)$ is a beginning of a jump in a homogeneous continuum X, then Q is homogeneous.*

In fact, let $\varepsilon=\frac{1}{2}\operatorname{dist}(Q,S(Q))$. If $h\in H(X)$, $\sigma(h,e)<\varepsilon$ and $Q\cap h(Q)\neq\emptyset$, then either $Q\subset h(Q)$ or $h(Q)\subset Q$. Since $h(Q)\subset S(Q)$, we conclude $h(Q)\subset Q$. But then $h^{-1}(Q)\subset S(Q)$, $Q\subset h^{-1}(Q)$, which imply $h(Q)=Q$. Now, (1.14) follows easily (see [13], (6.9), p. 17).

2. Hereditary θ-continua. Recall that a continuum X is a *θ-continuum* if the complement of every subcontinuum of X has a finite number of components.

We have (compare [4])

(2.1) *If X is a hereditary θ-continuum, then the intersection of every two subcontinua of X has a finite number of components.*

In fact, suppose that A and B are subcontinua of X such that the set $A\cap B$ has an infinite number of components. Then we can find infinitely many pairwise disjoint nonempty closed and open sets C_i in $A\cap B$. Let U_i be open sets in $A\cup B$ such that $C_i\subset U_i$, $\overline{U}_i\cap\overline{U}_j=\emptyset$, $U_i\subset B\left(C_i,\frac{1}{i}\right)$ for $i\neq j$ and $i,j=1,2,\ldots$ where $B(C,\varepsilon)$ denotes a ball around C with the radius ε. Consider the component K of $A\cup\bigcup_{i=1}^{\infty}U_i$ containing A. From Janiszewski's theorem we obtain that $K\setminus A$ has infinitely many components, a contradiction.

We claim that

(2.2) *Let X be a hereditary θ-continuum. If X is homogeneous, $A\cap B\neq\emptyset$ and A and B are proper subcontinua of $A\cup B$, then both A and B are locally connected at every component of $A\cap B$.*

According to (2.1) the set $A\cap B$ has a finite number of components. Let C be a component of $A\cap B$. We can find an open set U in $A\cup B$ such that $C\subset U\subset\overline{U}\subset((A\cup B)\setminus(A\cap B))\cup C$. Then $(\overline{U}\setminus U)\cap A\cap B=\emptyset$. Let $0<\varepsilon<\frac{1}{2}\inf\{\sigma(x,y)\colon x\in A,\ y\in(\overline{U}\setminus U)\cap B\}$. Put

$$K=\operatorname{cl}\left(\bigcup\{h(A)\colon h\in H(X),\ \sigma(h,e)<\varepsilon,\ h(A)\cap U\neq\emptyset\}\right).$$

The set $K\cup B$ is a continuum and $B(C,\varepsilon)\cap B\subset K$. The set K has a finite number of components (otherwise $K\setminus B$ has infinitely many components). Therefore, by (2.11), the set $B(C,\varepsilon)\cap B$ is contained in a finite union of continua which is contained in $B(C,2\varepsilon)$. Thereby B is locally connected at C.

Immediately from (2.2) we obtain

(2.3) *If X is a homogeneous hereditary θ-continuum, then*

(i) *$IN(X)\subset T(X)$,*

(ii) *every irreducible and decomposable subcontinuum of X is of type λ,*

(iii) *every layer of an irreducible and decomposable subcontinuum of X is a terminal continuum in X.*

One can obtain also the following generalization of (13.3) from [13], p. 31.

(2.4) *If X is a hereditary θ-continuum with Kelley's property, then $IN(X)\subset T(X)$.*

Now, we will prove

(2.5) *If a homogeneous curve X is a hereditary θ-continuum, and every proper subcontinuum of X is decomposable, then X is a solenoid.*

In fact, since $T(X)\setminus\{X\}\subset IN(X)$ and every layer of an irreducible decomposable subcontinuum of X is a terminal subcontinuum of X, we obtain that every proper subcontinuum of X is an arc provided that it is an irreducible continuum by (2.3) (ii). If X is not atriodic, then X contains a simple triod (a union of three arcs pairwise disjoint except of one end-point). Then every point of X is a vertex of a simple triod. Fix an arc pq. We construct a sequence of pairwise disjoint arcs p_nb_n such that $\lim\operatorname{diam}p_nb_n=0$, $\lim p_n=p$ and $p_nb_n\cap pq=\{p_n\}$. This is impossible, because X is a hereditary θ-continuum. Therefore X is an atriodic continuum containing an arc. Hence X is a solenoid by (14.4) in [13].

(2.6) *If a homogeneous tree-like continuum X is a hereditary θ-continuum, then it is atriodic.*

Suppose that X contains a triod. Then applying (2.3) we easily find decomposable irreducible continua pv, qv and rv such that each two of them intersect on a common end-layer V containing v. Let $\varepsilon_1 < \varepsilon_0 = \min\{\sigma(p, V), \sigma(q, V), \sigma(r, V), \sigma(p, q), \sigma(q, r), \sigma(r, p)\}$ and take $v_1 \in rv \setminus V$ with $\sigma(v_1, v) < \varepsilon_1$. Let $h_1 \in H(X)$ be such that $\sigma(h_1, e) < \varepsilon_1$ and $h_1(v) = v_1$. Then $h_1(V) \cap V = \emptyset$; thus either $h_1(pv) \cap \cap (pv \cup vq) = \emptyset$ or $h_1(qv) \cap (pv \cup vq) = \emptyset$ because h_1 is a homeomorphism and X is hereditarily unicoherent. Say $h_1(pv) \cap (pv \cup vq) = \emptyset$. Let

$$\varepsilon_2 < \min\{\varepsilon_1, \inf\{\sigma(x, y)\colon x \in pv \cup vq \text{ and } y \in h_1(pv)\}\}.$$

Take $v_2 \in rv \setminus V$ with $\sigma(v_2, v) < \varepsilon_2$. Let $h_2 \in H(X)$ be such that $\sigma(h_2, e) < \varepsilon_2$ and $h_2(v) = v_2$. Then $h_2(V) \cap V = \emptyset$; thus either $h_2(pv) \cap (pv \cup vq) = \emptyset$ or $h_2(qv) \cap \cap (pv \cup vq) = \emptyset$ because h_2 is a homeomorphism. Say $h_2(pv) \cap (pv \cup vq) = \emptyset$. Moreover $h_2(pv) \cap h_1(pv) = \emptyset$ or $h_2(rv) \cap h_1(pv) = \emptyset$ and so on. In this way we can construct a continuum K containing vr such that $K \setminus vr$ has infinitely many components, a contradiction, because K is a θ-continuum.

(2.7) *If a homogeneous curve X is a hereditary θ-continuum, then X is atriodic.*

Proof. Let A be a proper decomposable subcontinuum of X. It suffices to show that A is not a triod. Observe firstly that A is not terminal. Consider three cases.

(a) There is a beginning of a jump B containing A. Then B is a homogeneous tree-like continuum by (1.13) and (1.14). According to (2.6) the continuum B is atriodic; thus A is not a triod.

(b) There is a proper terminal continuum containing A and no terminal subcontinuum of X containing A is a beginning of a jump. Consider a minimal terminal continuum B containing A. Then B is a sequel of some jump and every subcontinuum of X containing B is terminal. Therefore B is a maximal proper terminal subcontinuum of X which is a sequel of some jump. These properties of B are preserved by homeomorphic surjections. If K and L are copies of B under homeomorphic surjections of X and $K \cap L \neq \emptyset$, then either $K \subset L$ or $L \subset K$, because K and L are terminal. By the maximality of K and L we infer that $K = L$. Thereby the copies of B under homeomorphic surjections of X form a decomposition of X. It is easy to verify that it is a completely regular monotone decomposition of X with homogeneous layers. Therefore, by (1.5), the layers of this decomposition are tree-like continua. In particular, B is a homogeneous tree-like continuum. Hence A is not a triod by (2.6).

(c) There is no terminal proper subcontinuum of X containing A. Then the maximal proper terminal subcontinua can be determined and they form a completely regular monotone decomposition of X onto a homogeneous curve Y which is a hereditary θ-continuum containing no nondegenerate proper terminal subcontinuum. In particular, every proper subcontinuum of Y is decomposable (compare (2.3) (i)); thus Y is a solenoid by (2.5). The continuum A is mapped by a quotient map φ onto a nondegenerate proper subcontinuum of a solenoid, i.e. $\varphi(A)$ is an arc. Denote the end-points of $\varphi(A)$ by a_1 and a_2.

Let $b_i \in \varphi^{-1}(a_i) \cap A$ for $i = 1, 2$ and let b_1b_2 be a subcontinuum of A irreducible between b_1 and b_2. Since the point inverses of φ are terminal subcontinua of X and $\varphi(b_1b_2)$ is nondegenerate we infer that $\varphi^{-1}\varphi(b_1b_2) = b_1b_2$. Therefore $A = b_1b_2$, because $A \subset \varphi^{-1}\varphi(b_1b_2) = \varphi^{-1}\varphi(A)$. Hence A is an irreducible continuum; thus it is not a triod.

3. Corollaries. Now we will apply the construction and denotations from [13], pp. 49–50 to obtain a little bit more general result than (20.3) in [13], p. 50; namely, we have

(3.1) THEOREM. *Suppose $K = \langle k_i \rangle_{i=0}^{\infty}$, $L = \langle l_i \rangle_{i=0}^{\infty} \in \mathscr{S}$ and $K \neq L$. Let $h_K\colon X \to W_K$ and $h_L\colon X \to W_L$ be continuous mappings from a Hausdorff continuum X into the spirals W_K and W_L, respectively. If either h_K or h_L is onto, then $d(h_K, h_L) \geq 1$.*

Proof. Suppose that $d(h_K, h_L) < 1$. Then $\pi\hat{F}_K(2) = \pi\hat{F}_L(2) \in h_K(X) \cap h_L(X)$. Firstly observe that the proof of (20.3) in [13] give us that

(3.1.1) $h_K(X) \cap h_L(X) \cap (\{0\} \times T) = \emptyset$.

Suppose now that h_K is onto. From (3.1.1) we obtain that $h_L(X) \subset \pi\hat{F}_L[t_0, 3]$ for some $t_0 > 0$. Take a positive integer m such that $a_{m-1} < t_0$. Choose $p \in X$ such that $h_K(p) = \pi(3, 0)$ and let J denote the component of $h_K^{-1}([a_{m+1}, 3] \times T)$ containing p. Then $h_K(J) = \pi\hat{F}_K[a_{m+1}, 3]$ which is an arc. Moreover $h_L(J)$ is also an arc. Therefore $|\gamma_K^J(x) - \gamma_L^J(x)| < 3$ for $x \in J$ by (20.3.1) in [13], p. 50. Now, by Janiszewski's boundary-bumping theorem there is $q \in J$ such that $h_K(q) = \pi\hat{F}_K(a_{m+1})$ and hence $H_K^J h_K(q) = (a_{m+1}, k_{m+1})$, hence $\gamma_K^J(q) = k_{m+1}$. But $\gamma_L^J(q) < l_{m-1} \leq 6^m$ and $6^{m+1} < k_{m+1}$; thus $|\gamma_K^J(q) - \gamma_L^J(q)| > 6 > 3$, a contradiction.

We will now prove the following

(3.2) THEOREM. *Let X be a continuum such that for some $x \in X$ and for each $A \in \overline{IN(X)}$ there are $B \in T(X)$ and a continuous map $f\colon B \to A$ such that $x \in B$ and $f(B) = A$. Then there is no weakly confluent map from X onto the unit square I^2. In particular,* $\dim X \leq 1$.

Proof. Suppose firstly that g is a weakly confluent map from X onto I^2. Then

(3.2.1) *for each $Q \in C(I^2)$ there is $A \in \overline{IN(X)}$ such that $g(A) = Q$.*

In fact, it suffices to prove that if $Q \in IN(I^2)$ then there is $A \in IN(X)$ with $g(A) = Q$. Let $Q \in IN(I^2)$. Since g is weakly confluent, there is $A \in C(X)$ with $g(A) = Q$. We can assume that A is minimal with respect to this property. Then A is indecomposable, because any decomposition of A into two proper subcontinua gives a decomposition of Q, which is impossible.

Now, we can assume that every spiral W_L constructed in [13], p. 49 is embedded into I^2. According to (3.2.1) we find a continuum $V_L \in \overline{IN(X)}$ such that $g(V_L) = W_L$.

Fix $x \in X$. The assumptions imply the existence of $U_L \in T(X)$ and of a continuous map $f_L\colon U_L \to V_L$ such that $x \in U_L$ and $f_L(U_L) = V_L$. By the Tietze extension theorem there is a map h_L from X into I^2 such that $h_L|U_L = gf_L$. Since $L \in \mathscr{S}$ and

$\mathscr{S}$ is uncountable, we will obtain a contradiction provided we will show that $d(h_L, h_K) > 1$ for $L \neq K$ and $K, L \in \mathscr{S}$.

Fix $K, L \in \mathscr{S}$ with $K \neq L$. Since continua U_K and U_L are terminal and $x \in U_K \cap \cap U_L$, we obtain either $U_K \subset U_L$ or $U_L \subset U_K$. Assume $U_K \subset U_L$. The mappings gf_K and $gf_L|U_K$ map the continuum U_K into W_K and W_L, respectively, and gf_K is onto. Therefore $d(gf_K, gf_L|U_K) \geqslant 1$ by Theorem (3.1). Hence $d(h_L, h_K) \geqslant 1$.

To complete the proof observe that if $\dim X \geqslant 2$, then there is a weakly confluent map from X onto I^2 (see [14]; compare [16]).

(3.3) Corollary. *If a hereditarily indecomposable continuum X is continuously homogeneous, then* $\dim X \leqslant 1$.

Since X is hereditarily indecomposable, we have the equalities $IN(X) = T(X) = C(X)$. Therefore, if $A \in \overline{IN(X)}$, then $A \in T(X)$ and for an arbitrary point $x \in X$ we have a continuous map $f\colon X \to X$ such that $f(x) \in A$. According to (1.3), the component B of $f^{-1}(A)$ containing x has the property $f(B) = A$. But $B \in T(X)$; thus X has all required properties from Theorem (3.2).

(3.4) Corollary. *If every subcontinuum of X is a continuous image of X, then* $\dim X \leqslant 1$.

(3.5) Corollary. *If X is a homogeneous hereditary θ-continuum, then* $\dim X \leqslant 1$.

According to (1.1), (1.2) and (2.3) (i) we have $IN(X) \subset T(X) = \overline{T(X)}$. These relations imply that assumptions of Theorem (3.2) are satisfied, because homeomorphisms preserve terminal continua.

(3.6) Corollary. *If every indecomposable subcontinuum of a homogeneous continuum X is terminal, then* $\dim X \leqslant 1$.

From (2.7) and (3.5) we obtain

(3.7) Corollary. *If X is a homogeneous hereditary θ-continuum, then X is an atriodic curve.*

From (3.7) and Hagopian's result (see [9]) we obtain

(3.8) Corollary. *If a homogeneous continuum X is a hereditary θ-continuum, then either X is a hereditarily indeceomposable tree-like continuum, or X has a decomposition onto a solenoid with the layers of the decomposition being homeomorphic hereditarily indecomposable tree-like continua which are terminal in X.*

4. Remarks. Corollaries (3.3), (3.5) and (3.6) generalize Theorem 11 from [17], (13.5) from [13] and Corollary 1 from [9]. The nonexistence of hereditarily indecomposable homogeneous continuum of the infinite dimension was proved by J. T. Rogers, Jr. in [17]. Recently he has found another simple proof in [19] which is based, similarly to the proof of Corollary (3.3) here, on the Waraszkiewicz spirals and Mazurkiewicz theorem.

In particular, from Corollary (3.4) we obtain the Cook's result (see [6]) that if every two nondegenerate subcontinua of a continuum X are homeomorphic then X is a curve. On the other hand, it is worth to note that if no two disjoint nondegenerate subcontinua of a continuum X are comparable by continuous maps, then X is again of low dimension. An example of a continuum with such property was found by H. Cook in [5] (for a plane example see [12]) and continua of this type are called Cook continua (see [12]). We have

(4.1) *If C is a subcontinuum of a compactum X and* $\dim(X \setminus C) \geqslant 3$, *then C is a continuous image of some subcontinuum of $X \setminus C$.*

In fact, let Y be a compactum contained in $X \setminus C$ with $\dim(X \setminus C) \geqslant 3$. Then there is a weakly confluent mapping f from X onto the cube I^3 (see [16]). Let M be a copy of the Menger's universal curve contained in I^3. According to Theorem 1 in [22] there is an open monotone mapping g from M onto the Hilbert cube Q. Let h be an embedding of C into Q. Since the mapping gf is weakly confluent, there is a continuum D contained in Y such that $gf(D) = h(C)$, i.e. the mapping $h^{-1}gf$ maps D onto C.

The following question remains open.

(4.2) *Is it true that Cook continua are curves?*

It follows from Theorem 8 in [19] that some continuous terminal decompositions are impossible (compare Corollary 10 in [19]). Remark that this theorem can be formulated more generally (the assumption of the continuity of the decomposition is inessential); namely

(4.3) *Let $\mathscr{D}$ be a terminal decomposition of a continuum X into nondegenerate continua. If* $\dim X \geqslant n$, *then the dimension of some element of $\mathscr{D}$ is $\geqslant n$.*

In fact, if $\dim X \geqslant n$, then X contains a decreasing sequence K_n of continua with the dimension $\geqslant n$ and with the degenerate intersection. Since an element of the decomposition $\mathscr{D}$ containing $\bigcap K_n$ is nondegenerate and terminal, it contains sufficiently small K_n. Therefore the dimension of this element of $\mathscr{D}$ is $\geqslant n$.

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Spherical maps

by

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Abstract. In this work we discuss the class of multi-valued upper semi-continuous maps $\varphi\colon M \to R^n$ of topological space M. Their values $\varphi(x)$ are non-empty continua of such nature that if $B\varphi(x)$ stands for the sum of bounded components $R^n \setminus \varphi(x)$ the graph of the map $B\varphi$ is open in $M \times R^n$ and $(\tilde{\varphi}x) := \varphi(x) \cup B\varphi(x)$ is acyclic for each $x \in M$. For such — so called spherical maps the following theorems are proven: (1) the Brouwer fixed point theorem, (2) the Poincaré type coincidence theorem, (3) the Birkhoff–Kellogg theorem, (4) the theorem on antipodes, (5) the theorem on invariance of domain.

1. Although a great number of papers have been published on the fixed point theory of various classes of multi-valued mappings, but some strong conditions about images of points by a multi-valued maps are always assumed. In the articles [4], [5], [9], [10] it is assumed that considered multi-valued map has acyclic images or, more generally, it is admissible multi-valued map (i.e. composition of acyclic maps). In the articles [8], [11] multi-valued maps with images of points having homology of the unit sphere S^{n-1} in the Euclidean space R^n are considered.

In the present paper we consider a class of multi-valued maps into Euclidean space R^n, called spherical maps. In this case homological assumptions about images of points are quite weak, although some additional non homological conditions are needed. As a special case, our class contains acyclic maps of n-spherical type in the sense of [8].

Next, we generalize from the case of admissible maps or n-spherical maps on the case of spherical maps the following results: (1) the Brouwer fixed point theorem, (2) the Poincaré type coincidence theorem, (3) the Birkhoff–Kellogg theorem, (4) the theorem on antipodes, (5) the theorem on invariance of domain. Note that in the case of n-spherical maps results (2), (3), (4), (5) have been unknown.

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2. Spherical maps. We will consider subsets of the Euclidean space R^n. We assume that $n \geq 2$. For any set $X \subset R^n$, the unbounded pseudo-component $D(X)$ of the set $R^n \setminus X$ is defined as follows: $x \in D(X)$ iff for every $r > 0$ there exists a continuous function $h\colon I \to R^n \setminus X$ such that $h(0) = x$ and $\|h(1)\| > r$, where