Cylinder problem

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Abstract. S. Ulam in the Scottish Book (see [Ma2]) posed the so-called rectangle problem. As a generalization F. Galvin (compare [Ga]) formulated the n-dimensional cylinder problem \( P_n(x) \) where \( x \) is a cardinal, \( n > 1 \). The 2-dimensional case is original Ulam’s problem. In this paper we consider the question for which cardinals \( x \) the problem \( P_n(x) \) has a positive solution.

We use standard set theoretic notation. The reference for forcing is Kunen [Kn2]. For the \( \sigma \)-algebra \( [F] \) generated by a family \( F \) of subsets of a set \( E \) closed under complements, we define the following hierarchy: \( [F]_0 = F \) and for each \( \alpha < \omega_1 \) (\( \alpha \neq 0 \)), \( [F]_\alpha \) is the family of all countable unions (intersections) of sets from \( \bigcup [F]_\beta \) if \( \alpha \) is odd (even). Finally, \( [F] = \bigcup \{ [F]_\alpha \} \).

If \( n \leq m < \omega \) and \( i_0 < i_1 < \ldots < i_{m-1} < m \), let \( C^m_{i_0 \ldots i_{m-1}}(X) \) denote the family of all sets of the form
\[
\{(x_{i_0}, \ldots, x_{i_{m-1}}) \in {}^m X : (x_{i_0}, \ldots, x_{i_{m-1}}) \in S\}
\]
where \( S \subseteq {}^m X \), and let
\[
C^m_{i_0 \ldots i_{m-1}}(X) = \bigcup \{ C^m_{i_0 \ldots i_{m-1}}(X) : i_0 < \ldots < i_{m-1} < m \}.
\]

For \( 1 \leq n \leq m < \omega \), let \( P^m_n(X) \) denote the sentence
\[
\varphi^m_n(X) = [C^m_{i_0 \ldots i_{m-1}}(X)]
\]
and let \( P_n(X) \) stand for \( P^{n+1}_n(X) \). The sentence \( P_n(X) \) is called the \((n+1)\)-dimensional cylinder problem for \( X \).

Let us note the following simple facts:

**Proposition 1.** Let \( X \) be a set, let \( \omega \leq \lambda \leq \kappa \) be cardinals and let \( 1 \leq n < m < \omega \). Then

1. \( P^m_n(X) \) if \( P^m_n(X) \).
2. \( P^m_n(X) \).
3. If \( P_n(x) \) then \( P_n(\lambda) \).
4. \( P^m_n(X) \) if \( P^m_n(X) \).

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(5) If $P_1(x)$ then $P_2(y)$.
(6) If $P_3(x)$ then there exists $z < \omega_1$ s.t.
$$\mathcal{P}(\omega + 1) = \{z \in \mathcal{P}(\omega) : z < \omega_1\}.$$

Proof. (1), (3), and (4) are obvious.

(4) Let $P_4(x)$ and let $A \in \mathcal{P}(\omega^x)$. Then $A \times \{0\} < \omega^x \in \mathcal{P}(\omega)$, so
$$A \in \mathcal{P}(\omega^x)$$
and hence $P_5(x)$.

For the reverse implication it is enough to show that for every $m > n$, $P_4(z)$ implies $P_n(x)$. By $P_n(x)$ we have
$$\mathcal{C}_n(x) = \mathcal{R}(\omega^n) = \mathcal{R}(\omega^n \times \omega^n).$$
Thus, identifying $x$ with $x \times x$ we obtain
$$\mathcal{C}_m(x) = \mathcal{P}(\omega^m) = \mathcal{P}(\omega^m \times \omega^m).$$

However, by $P_6(x)$, $\mathcal{C}_m(x) = \mathcal{C}_n(x)^{\mathcal{C}_n(x)}$. Hence $\mathcal{C}_m(x) = \mathcal{P}(\omega^m \times \omega^m)$, i.e. $P_n(x)$.

(5) follows immediately from (4).

(6) We may assume $x > \omega_1$.

If there is no $z$ with the property asserted, then for each $z < \omega_1$ there exists $A_z \subseteq \mathcal{C}_m(x)$ for $z < \omega_1$. Thus, using the natural bijection between $X$ and $(\omega + 1) \times \omega$ we may assume that for $x < \omega_1$, we have $A_x \in (\omega + 1) \times \omega$. Then $\bigcup_{x < \omega_1} A_x = \mathcal{C}_m(x)$, i.e. $P_5(x)$, which contradicts $P_5(x)$.

From the above proposition it follows, in particular, that in order to check whether $P_4(x)$ holds it is enough to verify a suitable cylinder problem $P_4(x)$ where $x = |X|$.

Now we are going to study the interrelation between the $\omega + 1$- and $(\omega + 2)$-dimensional cylinder problems. This will yield the positive solutions for $P_4(x)$.

**Theorem 1.** Let $1 < \eta < \omega$ and let $\lambda$ be a cardinal s.t. $\lambda \neq \omega_1$. If $P_4(x)$ holds for every $\eta < \lambda$, then $P_4(\lambda)$.

**Proof.** By Proposition 1 (6) and our assumption on the cofinality of $\lambda$ there exists an ordinal $\zeta < \omega_1$ s.t.
$$\mathcal{P}(\omega + 2) = \{z \in \mathcal{P}(\omega) : z < \omega_1\}.$$

Let $T \in \mathcal{P}(\omega + 2)$. We show that $T \in \mathcal{C}_m(x)$. For any $i < \eta + 2$ write
$$F_i = \{x_0, \ldots, x_{i+1} \in \mathcal{C}_m(x) : x_j \in x_0 \text{ for } j < i + 2\}.$$
Then $\mathcal{C}_m(x) = \bigcup_{i < \eta + 2} F_i$. So it is enough to show that, for any $i < \eta + 2$
$$T' = T \cap \mathcal{C}_m(x)$$
By symmetry the proof can be reduced to the case of $i = \eta + 1$. Then
$$T' = T \cap \{x \in \mathcal{C}_m(x) : x_0 \in T\}.$$

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Let us define for each $\zeta < \omega_1$ a set-theoretical operation $\varphi_\zeta$ describing the inductive definition of the hierarchy $[\lambda]; \varphi_\zeta$ is the identity operation; for $0 < \xi < \omega_1$ let $\xi \colon \mathcal{C}_m(x) \rightarrow \mathcal{P}(\omega)$ be a bijection and let
$$\varphi_\zeta(I \colon \omega^2 \rightarrow \omega) = \{\bigcup_{i < \zeta} \varphi_\zeta(I \setminus \omega, \omega^i) : i < \zeta, \omega \in \omega^i\} \text{ if } \zeta \text{ is odd},$$
$$\varphi_\zeta(I \colon \omega^2 \rightarrow \omega) = \{\bigcup_{i < \zeta} \varphi_\zeta(I \setminus \omega, \omega^i) : i < \zeta, \omega \in \omega^i\} \text{ if } \zeta \text{ is even}.$$Then, for every $\zeta < \omega_1$ and every sequence $\langle \mathcal{S}_i : i < \zeta \rangle$ of families of subsets of a set $E$, closed under complements, $\varphi_\zeta$ defines a function from $\prod_{i < \zeta} \mathcal{S}_i$ onto
$$\bigcup_{i < \zeta} \mathcal{S}_i.$$

Let for $\eta < \lambda$
$$T_\eta = T_{\eta + 1} \cap \{x \in \omega^2 \times \omega : x \in \omega^2 \times \omega\} \text{ if } \eta \text{ is odd},$$
$$T_\eta = T_{\eta + 1} \cap \{x \in \omega^2 \times \omega : x \in \omega^2 \times \omega\} \text{ if } \eta \text{ is even}.$$Hence, by $P_4(\eta + 1)$, for each $i < \eta + 1$ there exists a sequence $\langle A_i : \omega \in \omega \rangle$ of elements of $\mathcal{P}(\omega^2 \times \omega)$ s.t.
$$\varphi_\eta(I \colon \omega^2 \rightarrow \omega), \text{ for } i < \eta + 1, \omega \in \omega,$$s.t.

Let
$$A_{i, \omega} = \bigcup_{i < \eta + 1} A_i,$$Hence $A_{i, \omega} \in \mathcal{C}_m(x)$ and
$$T_{\eta + 1} = \bigcup_{i < \eta + 1} A_{i, \omega} \cap \{x \in \omega^2 \times \omega : x \in \omega^2 \times \omega\} \text{ if } \eta \text{ is odd},$$
$$T_{\eta + 1} = \bigcup_{i < \eta + 1} A_{i, \omega} \cap \{x \in \omega^2 \times \omega : x \in \omega^2 \times \omega\} \text{ if } \eta \text{ is even}.$$Cylinder problem

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Let
$$A_{i, \omega} = \bigcup_{i < \eta + 1} A_i,$$Hence $A_{i, \omega} \in \mathcal{C}_m(x)$ and
$$T_{\eta + 1} = \bigcup_{i < \eta + 1} A_{i, \omega} \cap \{x \in \omega^2 \times \omega : x \in \omega^2 \times \omega\} \text{ if } \eta \text{ is odd},$$
$$T_{\eta + 1} = \bigcup_{i < \eta + 1} A_{i, \omega} \cap \{x \in \omega^2 \times \omega : x \in \omega^2 \times \omega\} \text{ if } \eta \text{ is even}.$$
Proof. We use two partition properties, \( z_n^+ \to (n+2)^\omega \) and \( z_n^+ \to (n+2)^\omega_1 \).

The first of them is a weak statement of the Erdős–Rado theorem: \( \langle z_n(x) \rangle^+ \to (\kappa^+)^2 \) for \( 1 < \kappa < \aleph \) (see [Ku3], Theorem 6.4. p. 392). The second one follows from the theorem: \( z_n(x) \to (n+2)^\omega_1 \) for \( 1 < n < \omega \) (see [Ku3], Theorem 6.5. p. 393).

Let us suppose that \( P_\kappa(x) \) holds for \( x = z_n^+ \) and choose a partition \( f : [\kappa]^{n+1} \to \{0, 1\} \) witnessing \( n \mapsto (n+2)^\omega_1 \).

For each \( k < \omega \) let

\[
A_k = \{ \langle x_0, ..., x_n \rangle \in [\kappa]^{n+1} : x_0 < x_1 < ... < x_n \}.
\]

By \( P_\kappa(x) \) there exist \( S_k \subseteq [\kappa]^n \) s.t. \( A_k \) belong to the \( \sigma \)-algebra generated by the family of cylinders:

\[
\{ \langle x_0, ..., x_n \rangle \in [\kappa]^{n+1} : x_0 < x_1 < ... < x_n \in S_k \} : j < n+1, i < \omega \}.
\]

For \( x_0 < ... < x_n \) define \( g(\langle x_0, ..., x_n \rangle) = \{ i < \omega : \langle x_0, ..., x_i \rangle \in S_i \} \). By \( n \mapsto (n+2)^\omega_1 \) there is a subset \( X \subseteq [\kappa]^{n+1} \) such that \( g(\langle x_0, ..., x_n \rangle) = X \) for all \( i < \omega \). Hence, \( f \) witnesses \( n \mapsto (n+2)^\omega_1 \) and so \( f \) is constant on \( X \).

Let \( x_0, ..., x_n \) be such that \( x_0 < ... < x_n \), \( x_0 < ... < x_n \), and

\[
f(\langle x_0, ..., x_n \rangle) = f(\langle y_0, ..., y_n \rangle).
\]

Hence there exists an integer \( k \) s.t. exactly one of the points \( \langle x_0, ..., x_n \rangle \) and \( \langle y_0, ..., y_n \rangle \) is an element of \( A_k \). So, there exist \( i < \omega \) and \( j < n+1 \) s.t. \( S_i \) distinguishes the points

\[
\langle x_0, ..., x_{j-1}, x_{j+1}, ..., x_n \rangle \quad \text{and} \quad \langle y_0, ..., y_{j-1}, y_{j+1}, ..., y_n \rangle,
\]

which contradicts the fact that \( g \) is constant on \( X \).

The above theorem and Corollary 1 give

COROLLARY 3. If GCH holds then for \( 1 < n < \omega \) \( P_n(x) \) if \( x \in \omega_n \).

It seems natural to ask whether \( P_\kappa(x) \) is a theorem of ZFC. The following generalization of a theorem due to Kunen [Ku1] provides a negative answer to this question.

THEOREM 3. If \( M \) is a model of ZFC and \( f \) is a Cohen generic function over \( M \) adding at least \( \omega_n \) reals \( (0 < n < \omega) \) then \( M[f] \models \neg \Gamma_\kappa(\alpha_n, \beta) \).

Proof. We shall show that for every model \( M \) of ZFC and every Cohen generic function \( f : \omega_1 \times X \to \mathbb{R} \), if \( \Gamma_\kappa = \{ x \in [\kappa]^{\omega_1} : f(x) = 0 \} \) then \( M[f] \models \neg \Gamma_\kappa(\alpha_n, \beta) \).

Let us assume that for some \( n \) the above statement is not true. Then there exists a minimal \( m < \omega \) (\( m > 0 \)) s.t. for some model \( M \) of ZFC and a Cohen generic function \( f : \omega_1 \times X \to \mathbb{R} \) over \( M \),

\[
M[f] \models \neg \Gamma_\kappa(\alpha_n, \beta) \cap (\theta(\alpha_n) \cap M).
\]
THEOREM 4. If $2^\omega > 2^n$, then $\neg Q_n((2^\omega)^+)$. 

Proof. Let $\lambda = (2^\omega)^+$. For each $\alpha < \lambda$ choose a one-to-one function $h_{\alpha}: \alpha \to \beta(\alpha)$ and define

$$S_\alpha = \{\langle \xi, \eta \rangle: \eta < \alpha \text{ and } \xi \in h_{\alpha}(\eta) \} \subseteq \kappa \times \lambda.$$ 

Now suppose that there is a countable family $\mathcal{B} = \{D_\alpha: \alpha < \omega_1\}$ of subsets of $\kappa \times \lambda$ s.t. $\{S_\alpha: \alpha < \lambda\} \subseteq [\beta(\alpha^+)] \cup [\mathcal{B}]$. Choose $X \subseteq \kappa$ s.t. $|X| = (2^\omega)^+$ and for each $\alpha < \omega_1$ choose $D_\alpha \cap (\kappa \times X) = D_\alpha \times X$ for some $D_\alpha \subseteq X$ (X is a subset of a counter-image of the point $\gamma: \lambda \to \kappa \times \omega_2$ defined by $g(\eta)(\xi, \eta) = 0$ if $\langle \xi, \eta \rangle \in D_\alpha$). Thus

$$\{S_\alpha \cap (\kappa \times X): \alpha < \lambda\} \subseteq [\beta(\alpha^+)].$$ 

Fix $\alpha < \lambda$ s.t. $X \subseteq \alpha$. Now, if $\eta_1, \eta_2 \in X$ and $\eta_1 \neq \eta_2$ then

$$h_{\alpha}(\eta_1) \neq h_{\alpha}(\eta_2),$$

i.e. $\langle \xi, \eta_1 \rangle \in S_\alpha \neq \langle \xi, \eta_2 \rangle \in S_\alpha \neq \{\langle \xi, \eta_1 \rangle \in S_\alpha \}$. Since $|X| > 2^n$, it is easy to see that $S_\alpha \cap (\kappa \times X) \notin [\beta(\alpha^+)]$, because otherwise each set $\{\langle \xi, \eta \rangle \in S_\alpha \}$ would be determined by some real number. This gives contradiction.

An easy corollary to this theorem is that $2^\omega = \omega_2$, $2^\omega = \omega_1$ and $2^{\omega_1} \geq \omega_1$ imply $\neg Q_n((2^\omega)^+)$. Another consequence is that $\neg Q_n((2^\omega)^+)$. Let us also notice that an easy modification of the proof of Theorem 4 (using the fact that $\neg P_{\alpha}(\omega_2)$) gives also that, for any $\alpha < \omega_1$, $Q_\alpha(\alpha)$ implies $\kappa < 2^\omega$. So the following problem might be mentioned in this context:

"does $Q_\alpha(\alpha)$ imply $\kappa < \omega_{\alpha + 1}$ for $2 < \kappa < \omega_1$?"

Let us finally note that in a model of ZFC obtained by adding at least $\omega_\alpha$ Cohen reals, for every $\kappa < \omega$ we have $Q_\alpha(\alpha)$ iff $\kappa < \omega_{\alpha + 1}$.

The proof is similar to that of Theorem 3.

References


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Terminal continua and the homogeneity *

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Abstract. In the paper we prove the following statements: (1) every hereditarily indecomposable and continuously homogeneous continuum is one-dimensional; (2) every proper terminal subcontinuum of a homogeneous curve is tree-like; (3) every homogeneous hereditary $\theta$-continuum is arc-like.

1. Terminal continua. Definitions which are not recalled here can be found in [13]. All spaces in this paper are metric.

A compact space $X$ has *Kelley's property* at $x \in X$ if for every continuum $Y \subseteq X$ containing $x$ and for every sequence $x_n$ of points of $X$ converging to $x$, there exists a sequence of continua $Y_n \subseteq X$ converging to $Y$ such that $x_n \in Y_n$. A space $X$ has Kelley's property if it has Kelley's property at each point (see [21]).

A space is said to be *homogeneous with respect to the class $M$ of mappings* if for every two points $p$ and $q$ of $X$, there exists a continuous surjection $f$ from $X$ onto itself such that $f \in M$ and $f(p) = q$. A continuum homogeneous with respect to homeomorphisms (continuous maps) will be simply called *homogeneous (continuously homogeneous)*.

Chatyrvinsk has observed in [2] that (1.1) Continua which are homogeneous with respect to open mappings have Kelley's property.

A subcontinuum $Q$ of $X$ is called terminal if $K \subseteq C(X)$ and $K \cap Q \neq \emptyset$ imply either $K \subseteq Q$ or $Q \subseteq K$, where $C(X)$ denotes as usually the space of all subcontinua of $X$ with the Hausdorff distance. We will denote the collection of all terminal subcontinua of $X$ by $T(X)$ and the collection of all indecomposable subcontinua of $X$ by $IN(X)$. The following proposition is an immediate consequence of above definitions.

(1.2) If a continuum $X$ has Kelley's property, then $T(X)$ is closed in $C(X)$.

We have (see [10])

(1.3) If $f$ is a continuous mapping from a continuum $X$ onto $Y$, $K \subseteq T(Y)$ and $C$ is a component of $f^{-1}(K)$, then $f(C) = K$.

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