an element of the sense set B. There is some $\mu < \varkappa^+$ such that $\mathrm{otp}(C_\mu) = \tau$ and $C_\mu \subseteq D$ and, for $\gamma \in C'_\mu$ $(A, B|\gamma) = S_\gamma$. Now, in the construction of S_μ, case 3.14 (1) always hold. Hence any $[x]$ for $x \in A$ is above a member of B and we have our orderer.

4.21. Claim. *T is a Souslin tree.*

Proof. Let $B \subseteq T$ be dense. Let $D \subseteq \varkappa^+$ be a closed unbounded set of ordinals greater than $\varkappa$ closed under Gödel's pairing function such that for $\alpha < \alpha' \in D$ any small set $A \subseteq T$ has an orderer $T_\alpha \to T_\mu$ as in the previous Lemma for $\mu < \alpha'$. Observe that if $\gamma = \alpha' + 1$ then an orderer d: $T_\alpha \to T_\gamma$ can be found which sends A above B and moreover satisfies 4.19 (ii) for $[x, y] \in [T_\alpha]^2$. To see this pick first any orderer$^+$ $T_\alpha \to T_{\alpha+1}$; look at the image of A; apply an orderer as in Lemma 4.20 and then an orderer*. (See Lemma 4.4 (4).)

Now we can find $\mu < \varkappa^+$ of cofinality $\mathrm{cf}(\varkappa)$ such that $C_\mu \subseteq D$ and for $\gamma \in C'_\mu$, $S_\gamma = (T|\gamma, B \cap \gamma)$. It follows now that any $[x] \in T_\mu$ extends some member of B.

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BEN GURION UNIVERSITY OF THE NEGEV
Beer Sheva
Israel
THE HEBREW UNIVERSITY
Jerusalem
Israel
UNIVERSITY OF CALIFORNIA
Berkeley, CA
U.S.A.

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A note on the Mac Dowell-Specker theorem

by

Peter G. Clote * (Boston)

Dedicated to G. Takeuti on his 60th birthday

Abstract. By using formalized recursion theoretic arguments, here reminiscent of a finite-injury priority argument, one can remove the countability assumption in the Kirby–Paris refinement of the Mac Dowell-Specker theorem on end extensions of models of arithmetic.

The Mac Dowell–Specker Theorem states that any model of Peano arithmetic has a proper elementary end extension. In [KiPa], L. Kirby and J. Paris refined this result obtaining a correlation between subsystems of Peano arithmetic and the existence of proper end extensions which are elementary with respect to Σ_n and Π_n formulas. Their result is.

Theorem 1. *For any countable model M of $I\Sigma_0$ and $n \geq 2$ $M \models B\Sigma_n$ iff M admits a proper n-elementary end extension K which satisfies $I\Sigma_0$.*

The Kirby–Paris construction used very strongly the countability of the model. In view of the cardinality-free statement of the Mac Dowell–Specker Theorem, we might expect the conclusion of Theorem 1 to hold for models of any cardinality. Such a possibility was first suggested by A. Wilkie. By using formalized recursion theoretic arguments (in a manner reminiscent of a simple priority argument mixed with G. Kreisel's proof of Gödel's second independence theorem), we obtain the desired result, thus answering Question 2 of [C1]. Since the early work of L. Kirby and J. Paris, many results in models of arithmetic have been obtained for *countable* models (consider also the notion of recursive saturation in the case of countable vs. uncountable models). G. Müller has mentioned the desire to extend results in models of arithmetic into the uncountable, so as to make precise those concepts and theorems which rely on cardinality considerations and those which do not. R. Kossak has established several results in this direction and the present note should be seen as a very minor contribution to this program.

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§ 1. Notation and Definitions. Our notation is standard, as in [Pa] or [Cl]. The language of arithmetic is $\{+, \cdot, 0, 1, <\}$. A formula is $\Sigma_0 = \Pi_0 = \Delta_0$ if all its quantifiers are *bounded*: $\forall x < y$, $\exists x < y$. A formula is Σ_{n+1} if it contains a block of unbounded existential quantifiers followed by a Π_n matrix; a formula is Π_{n+1} if its negation is Σ_{n+1}; a formula is Δ_n in a model M if it is equivalent to M to both a Σ_n and Π_n formula. We write

$$M \underset{n,e}{\preccurlyeq} K$$

to mean that K is a proper n-elementary end extension of M: that is, that $M \subsetneq K$,

$$M \models \theta(\vec{m}) \quad \text{iff} \quad K \models \theta(\vec{m})$$

for all Σ_n formulas θ and parameters $\vec{m} \in M$ and that K adds no new element below M. Recall the notion of *complete Δ_n-ultrafilter*, due to [Ka] and [Kr] independently.

DEFINITION. A collection $\mathcal{U}$ of Δ_n definable (with parameters) subsets of M is a *complete Δ_n ultrafilter on M* if

(1) $M \in \mathcal{U}$, $\varnothing \notin \mathcal{U}$;

(2) if $A, B \in \mathcal{U}$ then $A \cap B \in \mathcal{U}$;

(3) if $A \in \mathcal{U}$ and B is a $\Delta_n(M)$ superset of A, then $B \in \mathcal{U}$;

(4) if A is $\Delta_n(M)$ then either A or $\bar{A} := M - A$ belongs to $\mathcal{U}$;

(5) if $X \subseteq M \times M$ is Δ_n definable, $a \in M$ and $\forall i < a\ (X)_i \in \mathcal{U}$ then

$$\bigcap_{i<a} (X)_i \in \mathcal{U}$$

where the i-th section $(X)_i \underset{\text{df}}{=} \{x: (i, x) \in X\}$.

A collection $\mathcal{U}$ of $\Sigma_n(M)$ subsets of M is called a *complete Σ_n ultrafilter on M* if $\mathcal{U}$ satisfies

(4′) if A is $\Sigma_n(M)$, then either $A \in \mathcal{U}$ or there exists a $B \in \mathcal{U}$ with $B \subseteq \bar{A}$

as well as (1), (2), (3′) and (5′) where (3′) and (5′) are obtained by replacing Δ_n by Σ_n in (3), (5). (Often one explicitly states Δ_n complete resp. Σ_n complete rather than complete.)

We consider subsystems of Peano arithmetic: P^- consists of the usual finite axiomatization of some "minimal" arithmetic (see [Pa]). $I\Sigma_n$ is the subsystem of P^- plus Σ_n-induction with parameters and $B\Sigma_n$ the subsystem of P^- with Σ_n-collection with parameters (see [Pa]). We let $I\Delta_n$ denote P^- plus the scheme

$$[\forall x\ (\psi(x, \vec{u}) \leftrightarrow \neg\theta(x, \vec{u})) \wedge \psi(0, \vec{u}) \wedge \forall x\ (\psi(x, \vec{u}) \rightarrow \psi(x+1, \vec{u}))] \rightarrow \forall x\ \psi(x, \vec{u})$$

where ψ, θ are Σ_n formulas.

§ 2. Principal Results.

THEOREM 2. *For any model M of $B\Sigma_2$, there exists a complete Σ_1-ultrafilter.*

Proof. Let $<, >$ be a recursive pairing function with recursive projections $(\ )_0$, $(\ )_1$ so that $\langle (i)_0, (i)_1 \rangle = i$. Let $\psi_1(x)$ mean $\mathrm{Sat}((i)_0, x, (i)_1)$, where $\mathrm{Sat}(m, x, y)$ is the usual Σ_1 definable satisfaction whose meaning is that "$\Phi(x, (i)_1)$ is true, where the Gödel number of Φ is $(i)_0$". Let Seq designate the usual Δ_1 predicate identifying sequence numbers and for σ a sequence number, let $\mathrm{lh}(\sigma)$ be the length or domain of σ. Thus $\mathrm{lh}(\sigma) = \{0, \ldots, \mathrm{lh}(\sigma)-1\}$. Let $2^{<M}$ abbreviate the set

$$\{\sigma: M \models \mathrm{Seq}(\sigma)\ \&\ \forall i < \mathrm{lh}(\sigma)\ [\sigma(i) = 0 \text{ or } \sigma(i) = 1]\}.$$

If $\psi(x)$ is (equivalent in M to) a Σ_1 formula having only x as free variable, then let

$$\psi(x)^M = \{a \in M: M \models \psi(a)\}.$$

By $\langle A_i: i \in M \rangle$, we mean $\langle \psi_i(x)^M: i \in M \rangle$, so that $\langle A_i: i \in M \rangle$ is a Σ_1 enumeration of all Σ_1 definable (with parameters) subsets of M.

Before giving the proof, we give first an approximate but incorrect idea for the construction of the complete Σ_1-ultrafilter $\mathcal{U}$, secondly the correct intuitive idea for the construction, and finally proceed by a series of claims to verify that the construction works.

Approximate idea. Let $\langle A_i: i \in M \rangle$ be a Σ_1 enumeration of all Σ_1 definable (with parameters) subsets of M. If A is a subset of M, then let $A^0 = A$ and $A^1 = M - A$. Consider the tree

$$T = \{\sigma \in 2^{<M}: \text{there exist } \mathrm{lh}(\sigma) \text{ many } x \text{ in } \bigcap \{A_i^{\sigma(i)}: i < \mathrm{lh}(\sigma)\}\}.$$

Then the tree T is Σ_2 definable (without parameters). Clearly any unbounded branch f in T yields a Σ_1-complete ultrafilter $\mathcal{U}$ defined by

$$A \in \mathcal{U} \leftrightarrow f(i) = 0 \text{ where } i \text{ is a } \Sigma_1 \text{ index of the set } A \text{ (hence } A = A_i).$$

However, by the technique of [Cl], one would presumably need $I\Delta_4$ to obtain a *definable* unbounded branch and $I\Delta_3$ to obtain a *piecewise definable* (but perhaps undefinable) unbounded branch. Thus this argument would appear to require $B\Sigma_3$, which is too strong an assumption.

Idea. Again, let $\langle A_i: i \in M \rangle$ be a Σ_1 enumeration of all Σ_1 definable (with parameters) subsets of M. Consider the tree

$$T = \{\sigma \in 2^{<M}: \text{there exist } \mathrm{lh}(\sigma) \text{ many } x \text{ in } \bigcap \{A_i: i < \mathrm{lh}(\sigma) \text{ and } \sigma(i) = 0\}\}.$$

In other words, T is forced to "witness" $\mathrm{lh}(\sigma)$ many elements in the intersection of those A_i which σ claims to be in $\mathcal{U}$, yet says nothing about the complements $M - A_i$. There is no constraint about $\bigcap \{A_i^{\sigma(i)}: i < \mathrm{lh}(\sigma)\}$. In particular, if $\sigma(i) = 1$ for all i in the domain of σ, then σ belongs to T. For a Σ_1 definable (with parameters) subset A of M, let

$A \in \mathcal{U} \leftrightarrow$ there is a Σ_1 index i for A, hence $A = A_i$, for which $\sigma_n(i) = 0$, except for boundedly many n, where σ_n is the leftmost node of T at level n.

It will be shown that $B\Sigma_2$ is sufficient to show that there is always a leftmost node at each level, although perhaps no unbounded branch passes through any σ_n. As well, it will be shown that:

(a) The definition of $\mathscr{U}$ is independent of the choice of the Σ_1 indices (see Fact 3).

(b) Eventually the leftmost nodes σ_n agree on small initial segments, so that $\mathscr{U}$ is well defined in the part of its definition requiring all but boundedly many of the σ_n to "claim" that A_i belongs to $\mathscr{U}$ (see Claim 1).

(c) $\mathscr{U}$ satisfies the properties (1), (2), (3′), (4′), and (5′) and thus is a complete Σ_1-ultrafilter.

Details of the formal proof.

For σ in $2^{<M}$, let

$$T_\sigma = \{\nu \in 2^{<M}\colon \nu \precsim \sigma \text{ or } (\sigma \precsim \nu \,\&\, \text{there exist } \mathrm{lh}(\nu) \text{ many } x \text{ with } \forall i < \mathrm{lh}(\nu)\ [\nu(i) = 0 \to \psi_i(x)])\}\,.$$

Clearly, for any σ in $2^{<M}$, T_σ is a Σ_1 definable (without parameters) tree, hence Δ_2 definable. Furthermore, if σ satisfies

$$(*) \qquad \bigcap \{\psi_i(x)^M\colon i < \mathrm{lh}(\sigma) \text{ and } \sigma(i) = 0\} \text{ is unbounded in } M\,,$$

then T_σ is unbounded in M, since $\sigma^\frown\underbrace{1\ldots 1}_{m \text{ times}}$ belongs to T for all m in M.

Let T denote $T_\emptyset$. Note as well that

$$\bigcap \{\psi_i(x)^M\colon i < \mathrm{lh}(\sigma) \,\&\, \sigma(i) = 0\} = (\forall i < \mathrm{lh}(\sigma)[\sigma(i) = 0 \to \psi_i(x)])^M$$

so the expression in $(*)$ is equivalent to a Σ_1 formula.

Now for each n belonging to M, let σ_n denote the leftmost or lexicographic least node in T of length n. This uses $I\Delta_2$ which, by [C1] Lemma 12 (also independently by H. Friedman), is implies by $B\Sigma_2$. The proof of the theorem follows from a series of claims.

Claim 1. *"Things settle down".* $M \models \forall n\ \exists m > n\ \forall k \geqslant m\ (\sigma_m \restriction n = \sigma_k \restriction n)$.

Proof. Suppose not: let n belong to M and

$$M \models \forall m > n\ \exists k > m\ (\sigma_m \restriction n \neq \sigma_k \restriction n)\,.$$

A trivial consequence of the fact that σ_m is the leftmost node in T having length m is that: if $i \leqslant n < m < k$ and $\sigma_m(i) \neq \sigma_k(i)$ and $\sigma_m \restriction i = \sigma_k \restriction i$ then

(1) $\sigma_m(i) = 0$ and $\sigma_k(i) = 1$

(2) for all $r > k$, if $\sigma_r \restriction i = \sigma_m \restriction i$ then $\sigma_r(i) = 1$.

(This is reminiscent of G. Kreisel's proof of Gödel's second independence theorem.)

Now let

$$h\colon m \mapsto \text{least } k > m\ [\sigma_m \restriction n \neq \sigma_k \restriction n]\,.$$

Again by $I\Delta_2$, it follows that h is a total $\Delta_2(M)$ function. Using $I\Delta_2$ (or Δ_2 weak comprehension), there is a sequence

$$s = \langle m_0, \ldots, m_{2^n+1}\rangle$$

belonging to M, where $m_i = h^{(i)}(m)$ for $i \leqslant 2^n+1$. But then $\sigma_{m_{2^n}} \restriction n = \sigma_{m_{2^n+1}} \restriction n$, a contradiction. Q.E.D. Claim. ■

Note that the correspondence

$$h\colon n \to \text{least } m > n\ [\forall k > m(\sigma_m \restriction n \neq \sigma_k \restriction n)]$$

requires $I\Sigma_2$, too strong an assumption, so although "things settle down", the model M doesn't necessarily know "where things settle down".

Let

$$\mathscr{U} = \{A \subseteq M\colon A \text{ is } \Sigma_1(M) \text{ definable and there exists } i \text{ belonging to } M \text{ such that } A = (\psi_i(x))^M \text{ and } M \models \exists n_0\ \forall n \geqslant n_0\ \sigma_n(i) = 0\}\,.$$

Thus $\mathscr{U}$ is simply the collection of $\Sigma_1(M)$ subsets of M which are claimed to be in the ultrafilter by almost all of the leftmost nodes of the tree T. Note that given a $\Sigma_1(M)$ subset A of M, there may be no least index i for which $A = \psi_i(x)^M$, since M is not necessarily a model of $I\Sigma_2$. We now verify that $\mathscr{U}$ is a complete Σ_1-ultrafilter. Condition (1) obviously holds.

Claim 2. *If $A \in \mathscr{U}$ and $n \in M$ then $\{a \in A\colon a \geqslant n\} \in \mathscr{U}$.*

Proof. If not, then let i_0 be an index of A such that

$$M \models \exists m \forall n \geqslant m\, \sigma_n(i_0) = 0$$

and let $i_1 > i_0$ be any index of $\{a \in A\colon a \geqslant n\}$. Let n_0, n_1 be respectively stages where decisions concerning ψ_i, $i < i_1$ resp. ψ_i, $i \leqslant i_1$ have settled down. By definition of the tree T, clearly

$$\bigcap_{i < i_1} \{\psi_i(x)^M\colon \sigma_{n_0}(i) = 0\}$$

is unbounded in M. Let $\sigma = \sigma_{n_0} \restriction i_1^\frown 0$ and consider T_σ as previously defined — plainly $T_\sigma \subseteq T$, and T is unbounded in M. Again by $I\Delta_2$, let τ_n denote the leftmost node in T_σ of length n. Then τ_{n_1} belongs to T and is to the left of σ_{n_1}, contradicting the definition of σ_{n_1}. Q.E.D. Claim. ■

Using the techniques in the proof of the above claim, it is easy to obtain the following

Fact 3. *If $A \in \mathscr{U}$ then for every index i of A,*

$$M \models \exists n_0\ \forall n \geqslant n_0\ \sigma_n(i) = 0\,. \ ■$$

Claim 4. *If n_0 is a stage where decisions concerning ψ_i, $i \leqslant i_0$ have settled down then letting*

$$A \underset{\mathrm{df}}{=} \bigcap_{i \leqslant i_0} \{\psi_i(x)^M\colon \sigma_{n_0}(i) = 0\}$$

we have that $A \in \mathscr{U}$.

Proof. Let i_1 be an index for $\bigcap_{i\leqslant i_0}\{\psi_i(x)^M:\ \sigma_{n_0}(i)=0\}$, and let n_1 be a stage where decisions concerning ψ_i, $i\leqslant i_1$ have settled down. Supposing that $A\notin\mathcal{U}$ in order to obtain a contradiction, let $\sigma=\sigma_{n_1}\upharpoonright i_1{}^\frown 0$ and let T_σ be as previously defined — then $T_\sigma\subseteq T$. By the definition of the tree T, it is clear that A is unbounded and so T_σ is unbounded in M. Using $I\Delta_2$, let τ_n be the leftmost node in T_σ of length n. Then τ_{n_1} belongs to T and is to the left of σ_{n_1}, a contradiction. ■

Now suppose that $X\subseteq M\times M$ is $\Sigma_1(M)$ definable, $a\in M$ and that

$$\forall i<a\ (X)_i\in\mathcal{U}.$$

If $X=\{(b,c):\ M\models\theta(b,c)\}$ where θ is $\Sigma_1(M)$, then clearly there is a Δ_1 function

$$h:\ i\to\text{Gödel number of } \theta(i,x) \text{ in the enumeration}$$

$$\{\psi_n:\ n\in M\}.$$

Since M satisfies $B\Sigma_2$, $h''\{0,\ldots,a-1\}$ is bounded in M. So by the previous Fact 3,

$$(\forall i<a\ \psi_{h(i)}(x))^M\in\mathcal{U}$$

and by the previous claim, it follows that

$$\bigcap_{i<a}(X)_i\in\mathcal{U}.$$

Thus condition (5′) is satisfied, and thus condition (2) holds.

Claim 5. *If A is $\Sigma_1(M)$ definable and $A\notin\mathcal{U}$ then there exists $B\subseteq\bar{A}$ with $B\in\mathcal{U}$.*

Proof. Let i_0 be some index of A and let n_0, n_1 be respectively stages where decisions concerning ψ_i, $i<i_0$ resp. ψ_i, $i\leqslant i_0$ have settled down. By definition of the tree T, clearly

$$C \underset{\mathrm{df}}{=} \bigcap_{i<i_0}\{\psi_i(x)^M:\ \sigma_{n_0}(i)=0\}$$

is unbounded in M.

Subclaim. *$A\cap C$ is bounded in M.*

Proof. If not, then let $\sigma=\sigma_{n_0}\upharpoonright i_0{}^\frown 0$ and consider T_σ as previously defined — plainly $T_\sigma\subseteq T$ and T_σ is unbounded in M since we are supposing that $A\cap C$ is unbounded in M. Using $I\Delta_2$, let τ_n denote the leftmost node in T_σ of length n. Then τ_{n_1} belongs to T and is to the left of σ_{n_1}, a contradiction. ■

Thus $A\cap C$ is bounded above in M by some integer m. Letting

$$B\underset{\mathrm{df}}{=}C\cap\{a\in M:\ a\geqslant m\}$$

it follows that $B\subseteq\bar{A}$ and B is of the form $\psi(x)^M$ where ψ is $\Sigma_1(M)$.

By the Claims 4 and 2. $B\in\mathcal{U}$. Q.E.D. Claim. ■

Thus condition (4′) holds and thus condition (3′) holds. This concludes the proof of Theorem 2. ■

Clearly, a similar argument yields

Theorem 3. *For any model M of $B\Sigma_{n+1}$ with $n\geqslant 1$, there exists a complete Σ_n-ultrafilter.* ■

Recall the following.

Fact ([Cl] — in proof of Proposition 7). *If there exists a complete Σ_n ultrafilter $\mathcal{U}$ on a countable model M of $I\Sigma_0$, then M admits a proper n-elementary end extension $\bar{K}$ which is a model of $B\Sigma_n$.*

Sketch. One first defines

$$K=\{f:\ f \text{ is a } \Sigma_n \textit{ partial} \text{ function from } M \text{ into } M \text{ with } \mathrm{dom}(f)\in\mathcal{U}\}/\mathcal{U}$$

and verifies that K is a proper $(n+1)$-elementary extension of M satisfying "$M-B\Sigma_n$", which is the scheme

$$\forall x<a\ \exists y\ \psi(x,y,\vec{m})\to\exists b\ \forall x<a\ \exists y<b\ \psi(x,y,\vec{m})$$

where ψ is Σ_n, $a\in M$, and $\vec{m}\in K$. Now form a Σ_{n-1} ultrafilter $\mathcal{U}^*$ with

(1) every element of $\mathcal{U}^*$ is an unbounded $\Sigma_{n-1}(K)$ definable subset of K

(2) if f is a $\Sigma_{n-1}(K)$ partial function with $\mathrm{dom}(f)\in\mathcal{U}^*$ and $\mathrm{rg}(f)\subseteq\{0,\ldots,m\}$ for some $m\in M$, then there exists $B\in\mathcal{U}^*$ with $B\subseteq\mathrm{dom}(f)$ and f constant on B.

Now let

$$L=\{f:\ f \text{ is a } \Sigma_{n-1}(K) \textit{ partial} \text{ function from } K \text{ into } K \text{ with } \mathrm{dom}(f)\in\mathcal{U}^*\}/\mathcal{U}^*$$

Then one verifies that $L\models I\Sigma_{n-2}$ and that

$$M\underset{n,e}{\preccurlyeq}\bar{K}\quad\text{and}\quad\bar{K}\models B\Sigma_n$$

where $\bar{K}$ is the cofinal closure of K in L. Details can be found in [Cl]. ■

This yields

Corollary 4. *For any countable model M of $B\Sigma_{n+1}$, with $n\geqslant 1$, there exists a proper n-elementary end extension K of M which satisfies $B\Sigma_n$. Thus there exist $K_1,\ldots,K_n$ such that*

$$M\underset{n,e}{\precneqq}K_1\underset{n-1,e}{\precneqq}K_2\underset{n-2,e}{\precneqq}\cdots\underset{1,e}{\precneqq}K_n.$$

Remarks. (i) Using Proposition 3 of [Cl] which states that if a model M of $I\Sigma_0$ admits a proper $(n+1)$-elementary end extension which satisfies $I\Sigma_n$, then M satisfies $B\Sigma_{n+2}$, it is easy to prove the converse of Corollary 4.

(2) This provides certain insight on the arithmetical version of a question due to M. Kaufmann in [Ka] p. 102: whether for $1\leqslant n$ any countable model M of $B\Sigma_{n+1}$ admits a proper $(n+1)$-elementary end extension K which satisfies $B\Sigma_n$.

We now have the principal result

Theorem 5. *For any model M of $I\Sigma_0$ and $n\geqslant 2$,*

$M\models B\Sigma_n$ iff M admits a proper n-elementary end extension K which satisfies $I\Sigma_0$.

Proof. (⇒) By the proof of Theorem 3, we obtain a complete Δ_{n-1} ultrafilter on M. Now (following the technique of [Ka] and [Kr], independently), let

$$K = \{f\colon f \text{ is a } \Delta_{n-1}(M) \text{ function from } M \text{ into } M\}/\mathcal{U}\,.$$

It is easy to check that K satisfies the conclusion on the right-hand side.

(⇐) Due to [Ki–Pa]. ■

A final remark concerning collection schemes in arithmetic and tree properties. In [Pa] a "mild refinement" of the arithmetized completeness theorem was given: (a weak statement of this result is as follows) if M is a model of $P+B\Sigma_2$ and there is a Δ_1 definition of a theory T such that $M \models \mathrm{Con}(T)$ then there is a Δ_2 definition of a Henkin model K for T. Using $B\Sigma_2$, we can actually produce a Δ_2 definition of the *lefmost* unbounded branch in the associated Henkin tree Tr where

$$\mathrm{Tr} = \{\sigma \in 2^{<M}\colon M \models \text{"there is no proof of } 0=1 \text{ of Gödel number} \leqslant \mathrm{lh}(\sigma) \text{ admitting the } \psi_i^{\sigma(i)},\ i<\mathrm{lh}(\sigma), \text{ as axioms}\}$$

and where $\{\psi_i\colon i<M\}$ is a Δ_1 enumeration of all sentences in the Henkinized language $\{+, \cdot, 0, 1, <\} \cup \{c_n\colon a \in M\}$. Now Kreisel's argument, as given in [Sm], yields an easy model theoretic argument that

$$T \nvdash \mathrm{Con}(T)$$

for any recursively axiomatizable theory T of arithmetic containing $B\Sigma_2$. (The only potentially non-obvious verification is Smorynski's Lemma 6.2.3 — here we use $I\Delta_2$.)

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DEPT. OF COMPUTER SCIENCE
BOSTON COLLEGE
Chestnut Hill Massachusetts 02167
U.S.A.

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