A note on the Mac Dowell-Specker theorem

by

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Dedicated to G. Takeuti on his 60th birthday

Abstract. By using formalized recursion theoretic arguments, here reminiscent of a finite-injury priority argument, one can remove the countability assumption in the Kirby–Paris refinement of the Mac Dowell-Specker theorem on end extensions of models of arithmetic.

The Mac Dowell-Specker Theorem states that any model of Peano arithmetic has a proper elementary end extension. In [KPa], L. Kirby and J. Paris refined this result obtaining a correlation between subsystems of Peano arithmetic and the existence of proper end extensions which are elementary with respect to $S_3$ and $H_3$ formulas. Their result is:

Theorem 1. For any countable model $M$ of $S_3$ and $n \geq 2$ $M \vDash B_n$ iff $M$ admits a proper $n$-elementary end extension $K$ which satisfies $S_3$.

The Kirby–Paris construction used very strongly the countability of the model. In view of the cardinality-free statement of the Mac Dowell-Specker Theorem, we might expect the conclusion of Theorem 1 to hold for models of any cardinality. Such a possibility was first suggested by A. Wilkie. By using formalized recursion theoretic arguments (in a manner reminiscent of a simple priority argument mixed with G. Kreisel’s proof of Gödel’s second independence theorem), we obtain the desired result, thus answering Question 2 of [C1]. Since the early work of L. Kirby and J. Paris, many results in models of arithmetic have been obtained for countable models (consider also the notion of recursive saturation in the case of countable vs. uncountable models). G. Müller has mentioned the desire to extend results in models of arithmetic into the uncountable, so as to make precise those concepts and theorems which rely on cardinality considerations and those which do not. R. Kossak has established several results in this direction and the present note should be seen as a very minor contribution to this program.

* Work done while the author was visiting the Department of Computer Science of the University of Toronto in spring 1984 and partially supported by the NSERC. I would like to express my most hearty thanks to Professor S. A. Cook.
§ 1. Notation and Definitions. Our notation is standard, as in [Pa] or [C1]. The language of arithmetic is \( \{ +, 0, 1, < \} \). A formula is \( \Sigma_0 = \Pi_0 = \Delta_0 \) if all its quantifiers are bounded: \( \forall x < y, \exists x < y \). A formula is \( \Sigma_{n+1} \) if it contains a block of unbounded existential quantifiers followed by a \( \Pi_n \) matrix; a formula is \( \Pi_{n+1} \) if its negation is \( \Sigma_{n+1} \); a formula is \( \Delta_n \) in a model \( M \) if it is equivalent to \( M \) to both a \( \Sigma_n \) and \( \Pi_n \) formula. We write
\[
M \models \varphi
\]
to mean that \( K \) is a proper \( n \)-elementary end extension of \( M \): that is, that \( M \not\models \varphi \)
\[
M \models \varphi(\vec{m}) \iff K \models \varphi(\vec{m})
\]
for all \( \Sigma_n \) formulas \( \varphi \) and parameters \( \vec{m} \in M \) and that \( K \) adds no new element below \( M \).

Recall the notion of complete \( \Delta_0 \)-ultrafilter, due to [Ka] and [Kr] independently.

Definition. A collection \( \U \) of \( \Delta_0 \) definable (with parameters) subsets of \( M \) is a complete \( \Delta_0 \)-ultrafilter on \( M \) if
\[
\begin{align*}
(1) & \quad M \models \varphi, \varphi \not\models \varphi; \\
(2) & \quad \text{if } A, B \in \U \text{ then } A \cap B \in \U; \\
(3) & \quad \text{if } A \in \U \text{ and } B \text{ is a } \Delta_0(M) \text{ superset of } A \text{ then } B \in \U; \\
(4) & \quad \text{if } A \in \U \text{ then } \langle A, A \rangle := M - A \text{ belongs to } \U; \\
(5) & \quad \text{if } X \subseteq M \times M \text{ is } \Delta_0 \text{ definable, } a \in M \text{ and } \forall \iota < \alpha \langle X, \iota, a \rangle \in \U \text{ then} \\

\bigwedge_{i < \alpha} \langle X_i, i, a \rangle \in \U
\end{align*}
\]

where the \( i \)-th section \( X_i \) of \( X \) is defined as \( \{ x : (i, x) \in X \} \).

A collection \( \U \) of \( \Sigma_1(M) \) subsets of \( M \) is called a complete \( \Sigma_1 \)-ultrafilter on \( M \) if \( \U \) satisfies
\[
\begin{align*}
(4') & \quad \text{if } A \in \U \text{ then either } A \in \U \text{ or there exists a } B \in \U \text{ with } B \subseteq A \\

\text{as well as (1), (2), (3) and (5') where (3') and (5') are obtained by replacing } & \Delta_0 \text{ by } \Sigma_0 \text{ in (3), (5). (Often one explicitly states } \Delta_0 \text{ resp. } \Sigma_0 \text{ complete rather than complete.)}
\end{align*}
\]

We consider subsystems of Peano arithmetic: \( \text{PA} \) consists of the usual finite axiomatization of some "minimal" arithmetic (see [Pa]). \( \text{IB} \) is the subsystem of \( \text{PA} \) plus \( \Sigma_1 \)-induction with parameters and \( \text{BES} \) the subsystem of \( \text{PA} \) with \( \Sigma_1 \)-collection with parameters (see [Pa]). We let \( \text{ID}_\Delta \) denote \( \text{PA} \) plus the scheme
\[
\forall x \forall \langle \psi(x, \bar{u}) \rightarrow \neg \theta(x, \bar{u}) \rangle \land \psi(0, \bar{u}) \land \forall x \forall y \langle \psi(x, \bar{u}) \rightarrow \psi(x + 1, \bar{u}) \rangle \rightarrow \forall x \forall \bar{u} \psi(x, \bar{u})
\]
where \( \psi, \theta \) are \( \Sigma_1 \) formulas.

§ 2. Principal Results.

Theorem 2. For any model \( M \) of \( \text{BES} \), there exists a complete \( \Sigma_1 \)-ultrafilter.

Proof. Let \( <, > \) be a recursive pairing function with recursive projections \( \langle \rangle_0, \langle \rangle_1 \), so that \( \langle \langle 0 \rangle_0, \langle 1 \rangle_1 \rangle = i \). Let \( \psi_i(x) \) mean \( \text{Sat}(\langle 0 \rangle_0, x, \langle 1 \rangle_1) \), where \( \text{Sat}(m, x, j) \) is the usual \( \Sigma_1 \) definable satisfaction whose meaning is that \( \Phi(x, \langle 0 \rangle_1) \) is true, where the Gödel number of \( \Phi \) is \( \langle 0 \rangle_1 \). Let \( \text{Seq} \) designate the usual \( A_1 \) predicate identifying sequence number and for \( \sigma \) a sequence number, let \( \text{lh}(\sigma) \) be the length or domain of \( \sigma \). Thus \( \text{lh}(\sigma) = \{ 0, ..., \text{lh}(\sigma) - 1 \} \).

Let \( 2^M \) abbreviate the set
\[
\sigma : M \models \text{Seq}(\sigma) \land \forall \iota < \text{lh}(\sigma) \langle \sigma(\iota) = 0 \text{ or } \sigma(\iota) = 1 \rangle.
\]

If \( \psi(x) \) is (equivalent in \( M \) to) a \( \Sigma_1 \) formula having only \( x \) as free variable, then let
\[
\hat{\psi}(x)^{\mathfrak{M}} = \sigma : M \models \psi(\sigma).
\]

By \( \langle A_i : i \in M \rangle \), we mean \( \langle \hat{\psi}(x)^{\mathfrak{M}} : i \in M \rangle \), so that \( \langle A_i : i \in M \rangle \) is a \( \Sigma_1 \) enumeration of all \( \Sigma_1 \) definable (with parameters) subsets of \( M \).

Before giving the proof, we give first an approximate but incorrect idea for the construction of the complete \( \Sigma_1 \)-ultrafilter \( \U \), so that the correct intuitive idea for the construction, and finally proceed by a series of claims to verify that the construction works.

Approximate idea. Let \( \langle A_i : i \in M \rangle \) be a \( \Sigma_1 \) enumeration of all \( \Sigma_1 \) definable (with parameters) subsets of \( M \). If \( A \) is a subset of \( M \), then let \( A^\alpha = A \) and \( A^1 = M - A \). Consider the tree
\[
T = \{ \sigma : 2^M : \text{there exist } i \in M \text{ many } x \in \bigwedge \langle A_i^{\mathfrak{M}} : i < \text{lh}(\sigma) \rangle \}
\]

Then the tree \( T = \Sigma_1 \) definable (without parameters). Clearly any unbounded branch \( f \) in \( T \) yields a \( \Sigma_1 \)-complete ultrafilter \( \U \) defined by
\[
A \in \U \iff f(i) = 0 \text{ where } i \text{ is a } \Sigma_1 \text{ index of the set } A \text{ (hence } A = A_i) \text{.}
\]

However, by the technique of [C1], one would presumably need \( \text{ID}_\Delta \) to obtain a definable unbounded branch and \( \text{ID}_\Delta \) to obtain a piecewise definable (but perhaps unbounded) unbounded branch. Thus this argument would appear to require \( \text{BES} \), which is too strong an assumption.

Idea. Again, let \( \langle A_i : i \in M \rangle \) be a \( \Sigma_1 \) enumeration of all \( \Sigma_1 \) definable (with parameters) subsets of \( M \). Consider the tree
\[
T = \{ \sigma : 2^M : \text{there exist } i \in M \text{ many } x \in \bigwedge \langle A_i^{\mathfrak{M}} : i < \text{lh}(\sigma) \rangle = 0 \}
\]

In other words, \( T \) is forced to "witness" \( 1(h) \) many elements in the intersection of those \( A_i \) which \( \sigma \) claims to be in \( \U \), yet says nothing about the complements \( M - A_i \). There is no constraint about \( \bigwedge \langle A_i^{\mathfrak{M}} : i < \text{lh}(\sigma) \rangle = 0 \). In particular, if \( \sigma(0) = 1 \) for all \( i \) in the domain of \( \sigma \), then \( \sigma \) belongs to \( T \). For a \( \Sigma_1 \) definable (with parameters) subset \( A \) of \( M \), let
\[
A \in \U \iff \text{there exists a } \Sigma_1 \text{ index } i \text{ for } A, \text{ hence } A = A_i, \text{ for which } \sigma_0(i) = 0 \text{, except for bounded many } n, \text{ where } \sigma_n \text{ is the leftmost node of } T \text{ at level } n.
\]
It will be shown that $\mathcal{B}_T$ is sufficient to show that there is always a leftmost node at each level, although perhaps no unbounded branch passes through any $\sigma_x$. As well, it will be shown that:

(a) The definition of $\mathcal{W}$ is independent of the choice of the $\Sigma_1$ indices (see Fact 3).

(b) Eventually the leftmost nodes $\sigma_x$ agree on small initial segments, so that $\mathcal{W}$ is well defined in the part of its definition requiring all but boundedly many of the $\sigma_x$ to "claim" that $A_T$ belongs to $\mathcal{W}$ (see Claim 1).

(c) $\mathcal{W}$ satisfies the properties $(1), (2), (3'), (4')$, and $(5')$ and thus is a complete $\Sigma_1$-ultrafilter.

Details of the formal proof.

For $\sigma$ in $2^{<\omega}$, let

$$T_\sigma = \{ \nu \in 2^{<\omega} : \exists \alpha \leq \nu \text{ or } (\sigma < \nu \& \text{there exist} \ 1b(\tau) \text{ many } x \text{ with } W < 1b(\tau) [\tau(0) = 0 \rightarrow \psi_A(\alpha)] \}.$$ 

Clearly, for any $\sigma$ in $2^{<\omega}$, $T_\sigma$ is a $\Sigma_1$ definable (without parameters) tree, hence $A_2$ definable. Furthermore, if $\sigma$ satisfies

$$(\dagger) \quad \bigotimes \{\psi(x)^M : i < 1b(\sigma) \text{ and } \sigma(i) = 0 \} \text{ is unbounded in } M,$$

then $T_\sigma$ is unbounded in $M$, since $\sigma^{-1} \cdots \sigma^{-1}$ belongs to $T$ for all $m$ in $M$.

Let $T$ denote $T_\sigma$. Note well that

$$\bigotimes \{\psi(x)^M : i < 1b(\sigma) \text{ and } \sigma(i) = 0 \} = \bigotimes \{\psi(x)^M : i < 1b(\sigma) \text{ and } \sigma(i) = 0 \rightarrow \psi_A(\alpha)\}^M$$

so the expression in $(\dagger)$ is equivalent to a $\Sigma_1$ formula.

Now for each $n$ belonging to $\mathcal{M}$, let $\sigma_n$ denote the leftmost node or lexigraphic least node in $T$ of length $n$. This uses $I\Delta_2$ which, by [Cl] Lemma 12 (also independently by H. Friedman), is implies by $\mathcal{B}_T$. The proof of the theorem follows from a series of claims.

CLAIM 1. "Things settle down". $M \models \forall n \exists m > n \forall k \exists m (\sigma_m \upharpoonright n = \sigma_k \upharpoonright n)$.

Proof. Suppose not: let $n$ belong to $M$ and

$$M \models \forall n \exists m > n \exists k > m (\sigma_m \upharpoonright n \neq \sigma_k \upharpoonright n).$$

A trivial consequence of the fact that $\sigma_n$ is the leftmost node in $T$ having length $m$ is that: $i < n \leq k < m$ and $\sigma_m(i) \neq \sigma_n(i)$ and $\sigma_n(\uparrow i) = \sigma_k(\uparrow i)$ then

(1) $\sigma_m(i) = 0$ and $\sigma_n(i) = 1$

(2) for all $r > k$, if $\sigma_r(\uparrow i) = \sigma_m(\uparrow i)$ then $\sigma_m(i) = 1$.

(This is reminiscent of G. Kreisel’s proof of Gödel’s second independence theorem.)

Now let

$$h: m \leq \text{least } k > m [\sigma_n \upharpoonright n \neq \sigma_k \upharpoonright n].$$

Again by $I\Delta_2$, it follows that $h$ is a total $A_2(M)$ function. Using $I\Delta_2$ (or $A_2$ weak comprehension), there is a sequence

$$x = (\sigma_{m_1},..., \sigma_{m_2}, \ldots)$$

belonging to $M$, where $m_i = h(\uparrow 0(m))$ for $i < 2^{n+1}$. But then

$$\sigma_{m_1} \upharpoonright n = \sigma_{m_2} \upharpoonright n,$$

and $n$ is a contradiction. Q.E.D. Claim.

Note the correspondence

$$h: n \rightarrow \text{least } m > n [\forall k > m (\sigma_m \upharpoonright n \neq \sigma_k \upharpoonright n)]$$

requires $I\Delta_2$, too strong an assumption, so although "things settle down", the model $M$ doesn’t necessarily know "where things settle down".

Let

$$\mathcal{W} = \{ A \subseteq M : A \text{ is } \Sigma_1(M) \text{ definable and there exists } i \text{ belonging to } M \text{ such that } A = (\psi_A(x))^M \text{ and }$$

$$M \models \exists n_0 \forall n > n_0 \sigma_i(n) = 0 \}.$$ 

Then $\mathcal{W}$ is simply the collection of $\Sigma_1(M)$ subsets of $M$ which are claimed to be in the ultrafilter by almost all of the leftmost nodes of the tree $T$. Note that given a $\Sigma_1(M)$ subset $A$ of $M$, there may be no least index $i$ for which $A = (\psi_A(x))^M$, since $M$ is not necessarily a model of $I\Delta_2$. We now verify that $\mathcal{W}$ is a complete $\Sigma_1$-ultrafilter. Condition (1) obviously holds.

CLAIM 2. If $A \in \mathcal{W}$ and $n \in M$ then $\{ a \in A : a \upharpoonright n \in \mathcal{W} \}$.

Proof. If not, then let $i_0$ be an index of $A$ such that

$$M \models \exists n_0 \forall n > n_0 \sigma_i(n) = 0$$

and let $i_0 > i_0$ be any index of $\{ a \in A : a \upharpoonright n \in \mathcal{W} \}$. Let $n_0, n_1$ be respectively stages where decisions concerning $\psi_i, i < i_0$ resp. $\psi_i, i < i_0$ have settled down. By definition of the tree $T$, clearly

$$(\dagger) \quad \bigotimes \{\psi(x)^M : \sigma_m(n) = 0 \}$$

is unbounded in $M$. Let $\sigma = \sigma_m \upharpoonright \uparrow 0$ and consider $T_\sigma$ as previously defined — plainly $T_\sigma \supseteq T$, and $T$ is unbounded in $M$. Again by $I\Delta_2$, let $\tau_n$ denote the leftmost node in $T_\sigma$ of length $n$. Then $\tau_n$ belongs to $T$ and is to the left of $\sigma_m$, contradicting the definition of $\sigma_m$. Q.E.D. Claim.

Using the techniques in the proof of the above claim, it is easy to obtain the following

FACT 3. If $A \in \mathcal{W}$ then for every index $i$ of $A$,

$$M \models \exists n_0 \forall n > n_0 \sigma_i(n) = 0.$$ 

CLAIM 4. If $n_0$ is a stage where decisions concerning $\psi_i, i \leq i_0$ have settled down then letting

$$A = (\bigcap \{\psi(x)^M : \sigma_i(n) = 0\})$$

we have that $A \in \mathcal{W}$.
Proof. Let \( t_i \) be an index for \( \bigcap_{c \in \mathcal{C}} \{ \psi(x)^M : \sigma_n(i) = 0 \} \), and let \( n_1 \) be a stage where decisions concerning \( \psi_i, i \in t_i \), have settled down. Supposing that \( A \not\in \mathcal{F} \) in order to obtain a contradiction, let \( \sigma = \sigma_n \) \( t_i \) 0 and let \( T_a \) be as previously defined — then \( T_a \not\subseteq T \). By the definition of the tree \( T \), it is clear that \( A \not\subseteq T \). Using IDA, let \( \tau \) be the leftmost node in \( T_a \) of length \( n \). Then \( \tau_n \) belongs to \( T \) and is to the left of \( \sigma_m \), a contradiction. 

Now suppose that \( X \subseteq M \times M \) is \( \Sigma_1(M) \) definable, \( a \in M \) and that
\[
\forall i < a \ (X)_i \in \mathcal{F}.
\]

If \( X = \{ (b, c) : M \vdash \theta(b, c) \} \), where \( \theta \) is \( \Sigma_1(M) \), then clearly there is a \( d_i \) function
\[
h : 1 \to \text{Goedel number of } \theta(i, x) \text{ in the enumeration}
\]
\[
\{ \psi_n : a \in M \}.
\]
Since \( M \) satisfies \( \mathbf{BS}_n \), \( h'(0, \ldots, a) \) is bounded in \( M \). So by the previous Fact 3,
\[
(\forall i < a \ \psi_{k_0}(i)) \in \mathcal{F}
\]
and by the previous claim, it follows that
\[
\bigcap_{i < a} (X)_i \in \mathcal{F}.
\]
Thus condition (3) is satisfied, and thus condition (2) holds.

Claim 5. If \( A \vdash \Sigma_1(M) \) definable and \( A \not\in \mathcal{F} \) then there exists \( B \subseteq A \) with \( B \in \mathcal{F} \).

Proof. Let \( t_0 \) be some index of \( A \) and let \( n_0, n_1 \) be respectively stages on which decisions concerning \( \psi_i, i < t_0 \), resp. \( \psi_i, i < t_0 \), have settled down. By definition of the tree \( T \), clearly
\[
C = \bigcap_{i < t} \{ \psi(x)^M : \sigma_m(i) = 0 \}
\]
is unbounded in \( M \).

Subclaim. \( A \cap C \) is bounded in \( M \).

Proof. If not, then let \( \sigma = \sigma_m \) \( t_0 \) 0 and consider \( T_a \) as previously defined — plainly \( T_a \not\subseteq T \) and \( T_a \) is unbounded in \( M \) since we are supposing that \( A \cap C \) is unbounded in \( M \). Using IDA, let \( \tau \) denote the leftmost node in \( T_a \) of length \( n \). Then \( \tau_n \) belongs to \( T \) and is to the left of \( \sigma_m \), a contradiction. 

Thus \( A \cap C \) is bounded above in \( M \) by some ordinal \( m \). Letting
\[
B = C \cap \{ a \in M : a \geq m \}
\]
it follows that \( B \subseteq A \) and \( B \) is of the form \( \psi(x)^M \) where \( \psi \) is \( \Sigma_1(M) \).

By the Claims 4 and 2, \( B \in \mathcal{F} \), Q.E.D. 

Thus condition (4) holds and thus condition (3) holds. This concludes the proof of Theorem 2. 

Clearly, a similar argument yields

**Theorem 3.** For any model \( M \) of \( \mathbf{BS}_{n+1} \) with \( n \geq 1 \), there exists a complete \( \Sigma_n \)-ultrafilter. 

Recall the following.

**Fact [(C1) — in proof of Proposition 1].** If there exists a complete \( \Sigma_n \) ultrafilter \( \mathcal{F} \) on a countable model \( M \) of \( \mathbf{IE}_n \), then \( M \) admits a proper \( n \)-elementary end extension \( K \) which is a model of \( \mathcal{F} \).

**Sketch.** One first defines
\[
K = \{ (f : f \vdash \Sigma_n \text{ partial function from } M \text{ into } M \text{ with } \text{dom}(f) \in \mathcal{F}) \} \mathcal{F}
\]
and verifies that \( K \) is a proper \( (n+1) \)-elementary extension of \( M \) satisfying \( "M = K = B \mathcal{F}_n" \), which is the scheme
\[
\forall x < a \ \exists y \ \psi(x, y, m) \rightarrow \exists b \ \forall x < a \ \exists y < b \ \psi(x, y, m)
\]
where \( \psi \) is \( \Sigma_n, a \in M \), and \( m \in K \). Now form a \( \Sigma_{n+1} \) ultrafilter \( \mathcal{F} \) with
1. every element of \( \mathcal{F} \) is an unbounded \( \Sigma_{n+1}(K) \) definable subset of \( K \)
2. if \( f \) is a \( \Sigma_{n+1}(K) \) partial function with \( \text{dom}(f) \in \mathcal{F} \) and \( \text{rg}(f) \subseteq \{ 0, \ldots, m \} \) for some \( m \in M \), then there exists \( B \in \mathcal{F} \) with \( B \subseteq \text{dom}(f) \) and \( f \) constant on \( B \).

Now let
\[
L = \{ (f : f \vdash \Sigma_{n+1}(K) \text{ partial function from } K \text{ into } K) \} \mathcal{F}
\]
with \( \text{dom}(f) \in \mathcal{F} \). 

Then one verifies that \( L \vdash \Sigma_{n+2} \) and that
\[
M \subseteq L
\]
where \( K \) is the cofinal closure of \( K \) in \( L \). Details can be found in [C1]. 

This yields 

**Corollary 4.** For any countable model \( M \) of \( \mathbf{BS}_{n+1} \) with \( n \geq 1 \), there exists a proper \( n \)-elementary end extension \( K \) of \( M \) which satisfies \( \mathbf{BS}_n \). Thus there exist \( K_1, \ldots, K_n \) such that
\[
M \subseteq K_1 \subseteq K_2 \subseteq \ldots \subseteq K_n.
\]

**Remarks.** (i) Using Proposition 3 of [C1] which states that if a model \( M \) of \( \mathbf{IE}_n \) admits a proper \( (n+1) \)-elementary end extension which satisfies \( \mathbf{IE}_{n+1} \), then \( M \) satisfies \( \mathbf{IE}_{n+1} \), it is easy to prove the converse of Corollary 4.

(2) This provides certain insight on the arithmetical version of a question due to M. Kaufmann in [K3] p. 102: whether for \( 1 \leq i \) any countable model \( M \) of \( \mathbf{BS}_{n+1} \) admits a proper \( (n+1) \)-elementary end extension which satisfies \( \mathbf{BS}_n \).

We now have the principal result

**Theorem 5.** For any model \( M \) of \( \mathbf{IE}_n \) and \( n \geq 2 \),
\[
M \vdash \mathbf{BS}_n \text{ if } M \text{ admits a proper } (n+1) \text{-elementary end extension } K \text{ which satisfies } \mathbf{IE}_n.
\]
Proof. (⇐) By the proof of Theorem 3, we obtain a complete $\mathcal{A}_n$ ultrafilter on $M$. Now (following the technique of [Ka] and [Kr], independently), let
$$K = \{ f : f \text{ is a } \mathcal{A}_n(\mathcal{M}) \text{ function from } M \text{ into } M \}/\mathcal{R}.$$  
It is easy to check that $K$ satisfies the conclusion on the right-hand side.

(⇒) Due to [Ki-Pa].

A final remark concerning collection schemes in arithmetical and tree properties.

In [Pa] a “mild refinement” of the arithmetized completeness theorem was given: (a weak statement of this result is as follows) if $M$ is a model of $P + \text{BE}_3$ and there is a $\Delta_4$ definition of a theory $T$ such that $M + \text{Con}(T)$ then there is a $\Delta_4$ definition of a Henkin model $K$ for $T$. Using $\text{BE}_3$, we can actually produce a $\Delta_4$ definition of the least unbounded branch in the associated Henkin tree $T$

$$\text{Tr} = \{ \langle i, e, \mathcal{M} : M \prec \mathcal{M} \text{ } \exists \text{ } \varphi(n) : \text{ } i < \text{lh}(\sigma), \varphi \upharpoonright i \text{ } \sigma \text{ } \text{ is a } \mathcal{A}_4 \text{ number} \}$$

and where $\{ \langle i, e, \mathcal{M} : M \prec \mathcal{M} \text{ } \exists \text{ } \varphi(n) : \text{ } i < \text{lh}(\sigma), \varphi \upharpoonright i \text{ } \sigma \text{ } \text{ is a } \mathcal{A}_4 \text{ number} \}$ is a $\Delta_4$ enumeration of all sentences in the Henkinized language $\{ +, \cdot, 0, 1, \} \cup \{ c_n : n \in \mathcal{M} \}$. Now Kreisel’s argument, as given in [Sm], yields an easy model theoretic argument that

$$T \vdash \text{Con}(T)$$

for any recursively axiomatizable theory $T$ of arithmetic containing $\text{BE}_2$. (The only potentially non-obvious verification is Smoryński’s Lemma 6.2.3 — here we use $\Delta_2$)

References


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Received 1 October 1984;
in revised form 16 September 1985

BOOKS PUBLISHED BY THE POLISH ACADEMY OF SCIENCES, INSTITUTE OF MATHEMATICS


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