On completely Ramsey sets

by

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Abstract. We consider completely Ramsey sets. Our main result says that the union of less than $\varkappa(N^*)$ of completely Ramsey sets is completely Ramsey and the union of $\varkappa(N^*)$ of completely Ramsey sets can be a set which is not completely Ramsey.

1. Introduction. The Ramsey theorem says that if the set of all k-element subsets of natural numbers, N, is partitioned into two parts, then there is an infinite subset of N such that all its k-element subsets are contained in one of those parts. The infinite version of Ramsey theorem is false. This permits us to define Ramsey sets.

A family S of infinite subsets of natural numbers is called *Ramsey* if there is an infinite subset of natural numbers such that either all its infinite subsets are contained in S or all its infinite subsets are contained in the complement of S.

A family S of infinite subsets of natural numbers is called completely Ramsey, a CR set, if for every infinite subset $V \subset N$ and every finite subset $x \subset N$ there is an infinite subset $W \subset V$ such that

$$\langle x, W \cup x\rangle \subset S \quad \text{or} \quad \langle x, W \cup x\rangle \cap S = \emptyset,$$

where $\langle x, W \cup x\rangle$ denotes the set of all infinite subsets $A \subset N$ such that $x \subset A \subset W \cup x$.

If we take the empty set instead of a finite set x and the set of natural numbers instead of an infinite set V in the above definition, then we conclude that any CR set is Ramsey.

Identifying subsets of natural numbers with theirs characteristic functions, one can see that infinite subsets of natural numbers form a G_δ dense subset of the Cantor set 2^N which is nowhere compact, thus being the same as irrationals with the order topology. We call this topology the natural topology. From this point of view it turns out that analytic sets are Ramsey. This result is due to Mathias [3] and Silver [6]. It has a few application in functional analysis, for instance in Farahat's proof of Rosenthal's Dichotomy. For this and other applications see Diestel [2]. Galvin and Prikry [4] showed that Borel sets are completely Ramsey.

The Ellentuck topology is the coarsest topology on the set of all infinite subset of natural numbers such that all sets of the form $\langle x, V\rangle$, where x is a finite subset of N and V is an infinite subset of N, are open. Since a family $\{\langle x, N\setminus y\rangle\colon x, y \in H\}$ (we denote the set of finite subsets of N by H and the set of all infinite subsets of N by T) is a base for the natural topology, the Ellentuck topology is finer than the natural topology. It turns out, Ellentuck [3], that CR sets coincide with sets having the Baire property in the Ellentuck topology. We call this fact the Ellentuck theorem. Thus CR sets form a σ-field which contains all analytic subsets, because a σ-field of sets having the Baire property is invariant under the Souslin operation [3].

An almost-disjoint partition of natural numbers is a maximal family consisting of infinite pairwise almost-disjoint subsets of natural numbers. A family of almost-disjoint partitions of natural numbers is called a matrix. A matrix U is called *shattering* if for each infinite subset $V \subset N$ there is a family $B \in U$ such that V meets at least two members of B in an infinite set. Let $\varkappa(N^*)$ be the least cardinal of cardinalities of shattering matrixes. This cardinal was introduced and studied by Balcar, Pelant and Simon [1]. They showed that $\varkappa(N^*)$ is an uncountable regular cardinal not greater than the continuum and that the value of $\varkappa(N^*)$ depends on the axioms of set theory. For other details about $\varkappa(N^*)$ see [1].

We consider coverings of the set of all infinite subsets of N by nowhere dense sets in the sense of the Ellentuck topology. The estimation on the size of such families allows us to show that the union of less than $\varkappa(N^*)$ completely Ramsey sets is completely Ramsey and the union of $\varkappa(N^*)$ completely Ramsey sets may be a set which is not completely Ramsey. To date the following result is known: the union of countable many CR sets is completely Ramsey, Galvin and Prikry [4], see also Lemma 3 in Silver [6].

Our definition of a CR set is different from the definitions in Ellentuck [3], Galvin and Prikry [4] or Silver [6]. In Ellentuck [3] sets of the form (for $x \in H$ and $V \in T$)

$$(x, V)^{\omega} = \{M \in T\colon x \subset M \subset V \cup x \text{ and if } t \in M,\ \sup x \geqslant t, \text{ then } t \in x\}$$

were used instead of sets of the form $\langle x, V\rangle$. However, the two definitions of CR sets are obviously equivalent.

The Ellentuck topology is an important tool in the investigations of CR sets. Accordingly, we describe some facts about this topology in the last section.

2. Additivity of the σ-field of CR sets.

LEMMA 1. *If $S \subset T$ is a dense and open subset in the Ellentuck topology, then for each infinite subset V of natural numbers and for all finite subsets x, y of natural numbers there is an infinite set $M \in \langle x, V \cup x\rangle$ such that $\langle y, M \cup y\rangle \subset S$.*

Proof. Let $f_0, f_1, \ldots, f_n$ be the sequence of all subsets of the union $x \cup y$. There is an infinite set $M_0 \subset V$ such that

$$\langle f_0, M_0 \cup f_0\rangle \subset S.$$

Assume inductively, that sets

$$M_k \subset \ldots \subset M_0 \subset V$$

have been defined. Let $M_{k+1} \subset M_k$ be an infinite subset such that

$$\langle f_{k+1}, M_{k+1} \cup f_{k+1}\rangle \subset S;$$

this is possible since S is a dense CR set. Let $M = M_n \cup x$. To finish the proof we show that

$$\langle y, M \cup y\rangle \subset S.$$

Let $P \in \langle y, M \cup y\rangle$. Take $f_k = P \cap (x \cup y)$. We have

$$f_k \subset P \subset M \cup y \subset M_k \cup x \cup y.$$

Therefore

$$f_k \subset P = P \cap (M_k \cup x \cup y) = P \cap M_k \cup P \cap (x \cup y) = P \cap M_k \cup f_k,$$

i.e. $P \in \langle f_k, M_k \cup f_k\rangle \subset S$. ■

LEMMA 2. *If $S \subset T$ is a dense and open subset in the Ellentuck topology, then for each infinite subset V of natural numbers there is an infinite subset $M \subset V$ such that for every finite subset x of natural numbers $\langle \emptyset, M \cup x\rangle \subset S$.*

Proof. Let $f_1, f_2, \ldots$ be the sequence of all finite subsets of natural numbers. Let $M_1 \subset V$ be an infinite subset such that

$$\langle f_1, M_1 \cup f_1\rangle \subset S.$$

Take $y_1 \in M_1$. Assume, inductively, that we have defined sets

$$\{y_1, \ldots, y_n\} \subset M_n \subset M_{n-1} \subset \ldots \subset M_1 \subset V.$$

Let M_{n+1} be an infinite set such that

$$\{y_1, \ldots, y_n\} \subset M_{n+1} \subset M_n$$

and

$$\langle f_{n+1}, M_{n+1} \cup f_{n+1}\rangle \subset S;$$

this is possible by Lemma 1. Take

$$y_{n+1} \in M_{n+1}\setminus\{y_1, \ldots, y_n\}$$

and

$$M = \{y_1, y_2, \ldots\}.$$

If $x \in H$ and $P \in \langle \emptyset, M \cup x\rangle$, then setting $f_n = P\setminus M$, we have

$$P \in \langle f_n, M \cup f_n\rangle \subset \langle f_n, M_n \cup f_n\rangle \subset S.$$

Since P was chosen arbitrarily, we have $\langle \emptyset, M \cup x\rangle \subset S$. ■

A family $S \subset T$ is called *nowhere Ramsey*, an NR set, if it is nowhere dense in the Ellentuck topology. In virtue of the Ellentuck theorem NR sets form a σ-ideal consisting of CR sets.

LEMMA 3. *If $S \subset T$ is a* NR *set, then there is an almost-disjoint partition of natural numbers U such that*

$$S \cap \bigcup \{\langle \emptyset, V \cup x\rangle:\ x \in H \text{ and } V \in U\} = \emptyset .$$

Proof. By virtue of Lemma 2 we conclude that the set

$$S_* = \{V \in T:\ \langle \emptyset, V \cup x\rangle \cap S = \emptyset \text{ for every } x \in H\}$$

is dense and open in the Ellentuck topology. Assuming $U \subset S_*$ to be an almost-disjoint partition of natural numbers, we see that it satisfies the conclusion. ■

THEOREM 1. *The union of less than $\varkappa(N^*)$ nowhere Ramsey sets is nowhere Ramsey.*

Proof. Let λ be a cardinal number less than $\varkappa(N^*)$ and let $\{S_\alpha:\ \alpha < \lambda\}$ be a family of NR sets. In virtue of Lemma 3 for each set S_α there is an almost-disjoint partition of natural numbers R_α such that

$$\emptyset = S_\alpha \cap \bigcup \{\langle \emptyset, V \cup x\rangle:\ V \in R_\alpha \text{ and } x \in H\} .$$

Since the matrix $\{R_\alpha:\ \alpha < \lambda\}$ cannot be shattering relative to any infinite subset of natural numbers, the set

$$\bigcap \{\bigcup \{\langle \emptyset, V \cup x\rangle:\ V \in R_\alpha \text{ and } x \in H\}:\ \alpha < \lambda\}$$

is dense and open in the Ellentuck topology. Obviously it is disjoint with each set S_α. ■

THEOREM 2. *The union of less than $\varkappa(N^*)$ completely Ramsey sets is completely Ramsey.*

Proof. Let λ be a cardinal number less than $\varkappa(N^*)$ and let $\{S_\alpha:\ \alpha < \lambda\}$ be a family of CR sets. In virtue of the Ellentuck theorem we can assume that each S_α is of the form $B_\alpha \cup V_\alpha$, where V_α is an open set in the Ellentuck topology and B_α is an NR set. We have

$$\bigcup \{S_\alpha:\ \alpha < \lambda\} = \bigcup \{V_\alpha:\ \alpha < \lambda\} \cup \bigcup \{B_\alpha:\ \alpha < \lambda\} .$$

But $\bigcup \{B_\alpha:\ \alpha < \lambda\}$ is an NR set by Theorem 1 and applying the Ellentuck theorem, we find that $\bigcup \{S_\alpha:\ \alpha < \lambda\}$ is a CR set. ■

THEOREM 3. *The family of all infinite subsets of natural numbers is the union of $\varkappa(N^*)$ nowhere Ramsey sets.*

Proof. Let $\{R_\alpha:\ \alpha < \varkappa(N^*)\}$ be a shattering matrix. Each set

$$P_\alpha = \bigcup \{\langle \emptyset, V \cup x\rangle:\ V \in R_\alpha \text{ and } x \in H\}$$

is dense and open in the Ellentuck topology. Let us observe that

$$\bigcap \{P_\alpha:\ \alpha < \varkappa(N^*)\} = \emptyset .$$

Using de Morgan's law, we are done. ■

THEOREM 4. *The union of $\varkappa(N^*)$ completely Ramsey sets can be a set which is not completely Ramsey.*

Proof. Let $S \subset T$ be a non CR set, for instance a Berstein set in the sense of natural topology. Let $\{P_\alpha:\ \alpha < \varkappa(N^*)\}$ be a covering consisting of NR sets, of T. Thus $P_\alpha \cap S$ is an NR set and

$$S = \bigcup \{P_\alpha \cap S:\ \alpha < \varkappa(N^*)\} . \ ■$$

3. The Ellentuck topology.

PROPOSITION 1. *The family $U = \{\langle x, V\rangle:\ x \in H$ and $V \in T\}$ forms a base for the Ellentuck topology which consists of closed-open sets in this topology.*

Proof. Sets $\langle x, V\rangle$ are closed-open in the Ellentuck topology since

$$\langle x, V\rangle = T \setminus \bigcup \{\langle \{t\}, N\rangle:\ t \in N \setminus V\} \cup \bigcup \{\langle \emptyset, V \setminus \{s\}\rangle:\ s \in x\} .$$

The family U is a base for the Ellentuck topology because

$$\langle x, V\rangle \cap \langle y, W\rangle = \langle x \cup y, V \cap W\rangle . \ ■$$

PROPOSITION 2. *There exists a family of infinite subsets of natural numbers which is dense and open in the Ellentuck topology and of first category in the natural topology.*

Proof. Let V be the set of even numbers. The family

$$U = \bigcup \{\langle \emptyset, V \cup x\rangle:\ x \in H\} \cup \bigcup \{\langle \emptyset, (N \setminus V) \cup x\rangle:\ x \in H\}$$

is as we desired. ■

Let P be the set of reals consisting of real numbers of the form

$$x = \sum_{n=0}^{\infty} \frac{2x_n}{3^{n+1}} ,$$

where $x_n = 1$ infinitely many times and $x_n = 0$ for other n. The function

$$f(x) = \{n \in N:\ x_n = 1\}$$

is a one-to-one mapping of P onto T. It is a homeomorphism if P inherits the order topology from reals and T is equipped with the natural topology.

PROPOSITION 3. *If A is an infinite subset of natural numbers, then the image of the right closed segment $(0, f^{-1}(A)]$ under the function f is open in the Ellentuck topology.*

Proof. We have

$$f^{-1}(A) \in f^{-1}(\langle \emptyset, A\rangle) \subset (0, f^{-1}(A)]$$

and therefore

$$f\big((0, f^{-1}(A)]\big) = f\big((0, f^{-1}(A))\big) \cup \langle \emptyset, A\rangle .$$

Thus we are done because the set $f\big((0, f^{-1}(A))\big)$ is open in the natural topology, whence it is open in the Ellentuck topology; let us recall that the family $\{\langle x, N \setminus y\rangle:\ x, y \in H\}$ is a base for the natural topology and therefore sets open in the natural topology are open in the Ellentuck topology. ■

PROPOSITION 4. *The space of all infinite subsets of natural numbers with the Ellentuck topology contains a closed and separable subspace which contains a closed and discrete subset of the cardinality continuum.*

Proof. Let V be the set of odd numbers and let

$$F = \{M \in T\colon\ V \subset M\} = T \setminus \bigcup \{\langle \emptyset, N \setminus \{t\}\rangle\colon\ t \in V\}.$$

The subspace F is obviously closed. It is separable because the set $\{V \cup x\colon\ x \in H\}$ is dense in F. Let h be a one-to-one mapping of the Cartesian product of natural numbers by itself onto the set of even numbers. We set

$$A^* = \{z = h(t, s)\colon\ t \in A \text{ and } s \notin A\}$$

and

$$U = \{V \cup A^*\colon\ \emptyset \neq A \subsetneq N\}.$$

The set U has the cardinality continuum and it is contained in F. Since

$$T \setminus U = \bigcup \{\langle \{h(t, s), h(r, t)\}, N\rangle\colon\ s, r, t \in N\} \cup \bigcup \{\langle \emptyset, V \cup (A^* \setminus \{z\})\rangle\colon\ A \subset N \text{ and } z \in A^* \cup V\}$$

the set U is closed. It is discrete in the Ellentuck topology because

$$U \cap \langle \emptyset, V \cup A^*\rangle = \{V \cup A^*\}. \ ■$$

Proposition 4 implies Keesling's result [5], which says, in our terms, that the Ellentuck topology is not normal. To see this it is enough to note that the subspace F from the above proposition cannot be normal. Note also that, since there is a closed discrete subset of the cardinality continuum, there is one of arbitrary cardinality less than the continuum.

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Squares with diamonds and Souslin trees with special squares

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Abstract. The squares and the diamonds are useful set-theoretic axioms used in construction of infinite objects. Here we introduce and study different versions of such combinatorial principles on successor of singular cardinals. We prove some implications (in ZFC), inquire the situation in L, and give an application.

Introduction. One feature of the work of Jensen and Johnsbråten [J&J] is the construction in L of a Souslin tree T such that its square — minus the diagonal, of course — is a special tree (that is, embeddable into the rationals). We present in § 4 a generalization of this result to higher cardinals. In L:

For any cardinal $\varkappa$ there is a Souslin tree of height $\varkappa^+$ such that its square — minus the diagonal — is special.

The proof of [J&J] can be generalized to successors of regular cardinals — but successor of singulars seem to require a different approach. A new kind of a diamond sequence is used to construct the trees; it is called a "square sequence with built-in diamond". In fact there are several kinds of square sequences with diamonds. Such a sequence was first presented by C. Gray in his thesis [G]. We present here, essentially three other forms which are discussed in §§ 1–3. The forms in §§ 2 and 3 hold in L and require the fine structure for their proof (the proof of § 2 is simpler than that of § 3); the form in § 1 seems weaker than that of § 2 but it holds in a very general setting — in fact it is a consequence of GCH + usual kind of squares. (So reading of § 1 does not require knowledge of fine structure.) Each section can be read independently of the others (the construction in § 4 uses the square sequence of § 1 but the reader can see that the form of § 2 yields a slightly simpler proof).

In § 1 ideas of K. Kunen (the proof that $\diamondsuit^* \to \diamondsuit$), and of J. Gregory [Gr] and [S] are used.

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