Reduction and irreducibility for words and tree-words

by

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Abstract. In a semigroup of words modulo a certain equivalence relation (inspired by the notion of a piecewise linear map from an arc to an arc) we show that equivalence classes have unique irreducible representatives. Further in the paper we develop some results with applications in continua theory. In particular we investigate when different reduction paths for a word commute with respect to certain letters. We also generalize the notion of a word to a tree-word and give sufficient conditions for reducibility of tree-words to (chain-) words. These results have applications to chainability of tree-like continua.

1. Introduction. This paper contains a collection of combinatorial results on words. In particular we are interested in irreducible words modulo a certain class of reductions and when these words are unique. Many of the results in sections 2, 3 and 4 can be phrased in the language of semigroups of words (see e.g. Corollary 3.7) and we hope they may be of some independent interest to algebraists. However, they were proved for the purpose of establishing the existence of chain covers for certain tree-like continua (see [2]). In Section 5 we develop machinery more specifically directed to this end.

The results in Section 4 were first obtained by Peter Minc. We wish to express our sincere thanks to Professor Minc for several valuable conversations leading to the use of this material in the paper.

2. Throughout this paper $\mathscr{A}$ will be a fixed set called the *alphabet*. If n and m are natural numbers with $n<m$, then $[n,m]$ will denote the set $\{n,n+1,\dots,m\}$ of all natural numbers between n and m (inclusive). A *word* is a finite sequence of elements of $\mathscr{A}$, i.e., a function $w\colon [1,n]\to\mathscr{A}$ for some natural number n. n is called the *length* of the word and is denoted $||w||$. Let n and m be natural numbers. A function r from $[1,n]$ onto (1) $[1,m]$ is called a *reduction function* if for every $i,j\in[1,n]$, $|r(i)-r(j)|\leqslant 1$ whenever $|i-j|\leqslant 1$. (In other words r "sends adjacents to adjacents". r may also be thought of as a piecewise linear map from the linear graph with vertices $1,2,\dots,n$ onto the linear graph with vertices $1,2,\dots,m$.) Let

* The second author was supported in part by N. S. F. grant number MCS-8104866.

(1) In this section and the next, we will consider only surjective reduction functions. In Sections 4 and 5 we will drop the requirement that r be onto.

$w_1\colon [1, n] \to \mathscr{A}$ and $w_2\colon [1, m] \to \mathscr{A}$ be words. We say that w_1 is *reducible to* w_2 if there is a reduction function $r\colon [1, n] \twoheadrightarrow [1, m]$ such that the following diagram commutes:

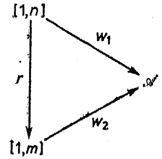

Fig. 1

We will write $r\colon w_1 \to w_2$ to denote the fact that r is a reduction function as above. Note that in this situation $\|w_1\| \geqslant \|w_2\|$. As a concrete example, we may note that the word *ABCBABCD* is reducible to the word *ABCD* and the graph of the reduction looks like this:

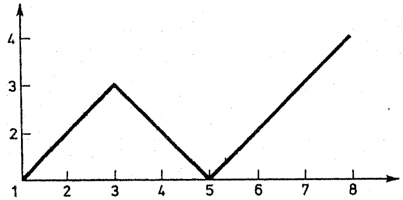

Fig. 2

This reduction function is an example of a *simple fold*. It has been shown that every reduction function is a composition of simple folds and monotone maps; so our next job is to describe these reduction functions. The simple folds are of three types.

DEFINITION 2.1. A reduction function $r\colon [1, n] \twoheadrightarrow [1, m]$ is called an *interior fold* if there are integers b and c such that $1 < b < c < n$, r is one-to-one on the sets $[1, b]$, $[b, c]$ and $[c, n]$ and $r([b, c]) = r([1, b]) \cap r([c, n])$.

The graph of an interior fold looks like one of the following two figures.

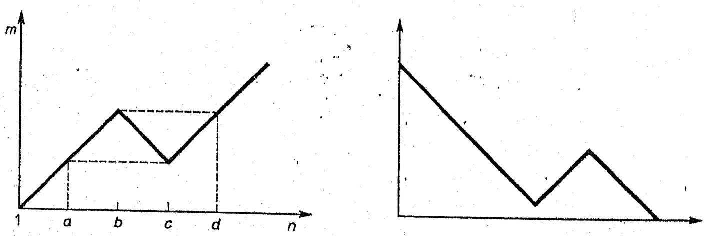

Fig. 3

Note that by the third condition there will be integers a and d such that $1 \leqslant a < b$ and $c < d \leqslant n$, $b-a = c-b = d-c$, and $r([a, b]) = r([b, c]) = r([c, d])$. The sets $[a, b]$, $[b, c]$ and $[c, d]$ will be denoted D_1^r, D_2^r and D_3^r respectively and their union, the interval $[a, d]$, will be denoted D^r.

DEFINITION 2.2. A reduction function $r\colon [1, n] \twoheadrightarrow [1, m]$ is called an *end fold* if r is not one-to-one, but there is an integer b, $1 < b < n$, such that r is one-to-one on the intervals $[1, b]$ and $[b, n]$.

The graph of an end fold looks like one of the following four figures. In all

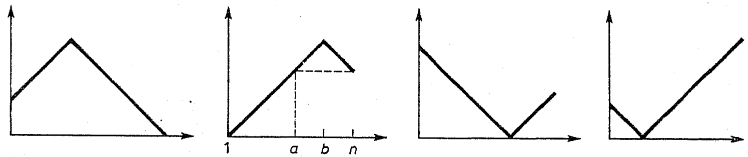

Fig. 4

cases there will be an integer a such that $1 \leqslant a < b$ (or $b < a \leqslant n$), $b-a = n-b$ (or $a-b = b-1$) and $r([a, b]) = r([b, n])$ (or $r([b, a]) = r([1, b])$). The sets $[a, b]$ and $[b, n]$ (or $[1, b]$ and $[b, a]$) will be called D_1^r and D_2^r respectively, and their union will be denoted D^r.

DEFINITION 2.3. A reduction function which is one-to-one will be called a *trivial fold* (or simply *trivial*). Trivial folds which reverse orientation will be called *flips*.

DEFINITION 2.4. *A simple fold* is either an interior fold, an end fold or a trivial fold. For nontrivial folds, the number $|a-b|+1$ will be called the *length* of the fold. The points b and c will be called *pivot points*.

DEFINITION 2.5. A reduction function $r\colon [1, n] \twoheadrightarrow [1, m]$ is said to be *monotone* if for every $t \in [1, m]$ there are integers $a_t, b_t \in [1, n]$ such that $r^{-1}(t) = [a_t, b_t]$. Below are sketches of the graphs of two typical monotone reduction functions.

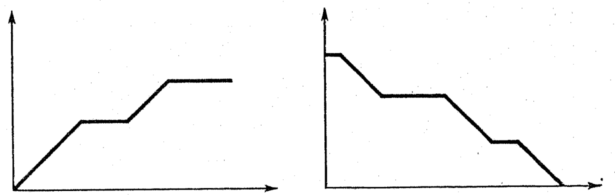

Fig. 5

DEFINITION 2.6. A reduction function $r\colon [1, n] \twoheadrightarrow [1, m]$ is said to be *light* if for every $i \in [1, n-1]$, $r(i) \neq r(i+1)$.

The following lemma was proved in [3] and [5].

LEMMA 2.7. *Let* $r\colon [1,n] \twoheadrightarrow [1,m]$ *be a nontrivial reduction function. Then there is a sequence of reduction functions* $r_0, r_1, \ldots, r_k$ *such that*

(1) r_0 *is monotone*

(2) r_i *is a nontrivial simple fold for all* $i \in [1,k]$

(3) $r = r_k \circ r_{k-1} \circ \ldots \circ r_1 \circ r_0$.

Moreover, if r *is light, then* r_0 *is a trivial function.* (²)

3. The Main Theorem for Chain Words. Before proving the main theorem we need a preliminary lemma. We will then prove the theorem for a special case. Finally, we will prove the main theorem by induction and using Lemma 2.7.

DEFINITION 3.1. Let w_1 and w_2 be words and let $s\colon w_1 \to w_2$ be (nontrivial) simple fold. s is said to be *minimal* if it has minimal length among all (nontrivial) simple folds with domain w_1.

LEMMA 3.2. *Let* w *be a word, let* $s\colon w \to w_1$ *be a simple fold and let* $m\colon w \to w_2$ *be a minimal fold. Suppose* $D_j^m \cap D_i^s \cap D_{i+1}^s \neq \emptyset$ *for some* i *and* j *(note that* $D_i^s \cap D_{i+1}^s$ *is always a singleton set). Then either* $D_j^m \subset D_i^s$ *or* $D_j^m \subset D_{i+1}^s$.

Proof. The proof is by contradiction. Let $X = D_j^m \cap D_i^s$ and $Y = D_j^m \cap D_{i-1}^s$. Since m is minimal $X \cup Y = D_j^m$. Now suppose $D_j^m \not\subset D_i^s$ and $D_j^m \not\subset D_{i+1}^s$. Then neither X nor Y is a singleton set. Suppose further (case 1) that $j \geqslant 2$. Let $D_{j-1}^m \cap D_j^m = \{a\}$ and $D_i^s \cap D_{i+1}^s = \{b\}$. Let k be the cardinality of X. Then k is strictly less than the length of the fold m (= the cardinality of the set D_j^m). By the symmetry of the word w about the pivot points a and b, we have for all $l = 1, 2, \ldots, k-1$, $w(a-l) = w(a+l)$ and $w(b-l) = w(b+l)$. From this, it is straightforward to construct an interior fold on w with length k, contradicting the minimality of m. (See the next paragraph for a visual description of what is happening here.) The case $j = 1$ is similar and is omitted. ∎

We may picture what is happening in the above proof as follows. Note that D_j^m, D_{j-1}^m, D_i^s, D_{i+1}^s, X and Y may all be considered subwords of the word w (by restricting the function w to any of these sets). We have indicated the relationships of these words to one another in the figure below. X^{-1} represents the word X "written backwards".

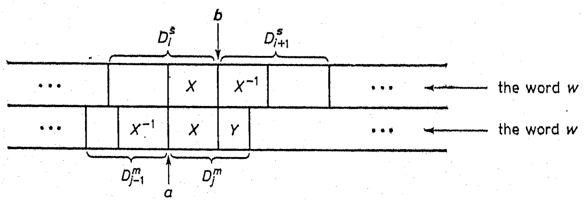

Fig. 6

(²) If r is a nontrivial, light reduction function, we may take r_0 to be the identity by putting a flip in r_1 if necessary.

The contradictory interior fold is obtained from the sequence of subwords $X^{-1}XX^{-1}$. In the case $j = 1$, a contradictory interior fold $Y^{-1}YY^{-1}$ will be obtained. Diagrams like the above will be used in the proof of the following lemma.

LEMMA 3.3 (special case of the main theorem). *Let* w, w_1 *and* w_2 *be words, let* $s\colon w \to w_1$ *be a nontrivial simple fold and let* $m\colon w \to w_2$ *be a minimal fold. Then there exists a "common reduction" of* w_1 *and* w_2, *i.e., a word* v *and reduction functions* $r_1\colon w_1 \to v$ *and* $r_2\colon w_2 \to v$.

Proof. The proof falls into four main cases, depending on whether s and/or m is an interior fold or an end fold. Each main case falls into as many as five subcases, determined by the way D^m and D^s overlap. Lemma 3.2 gives some control over these cases. We enumerate six subcases, of which one is trivial.

Case 1: $D^m \cap D^s = \emptyset$. Case 2: $D^m \subset D_j^s$ for some j. Case 3: $D^m \subset D_j^s \cup D_{j+1}^s$ for some j, but not case 2. Case 4: $D^m \subset D^s$, but not case 2 or case 3. (By Lemma 3.2 and the minimality of m, this case can only occur if m and s are the same interior fold, in which case the lemma is trivial.) Case 5: $D^m \not\subset D^s$, but $D^m \cap D^s \subset D_j^s$ for $j = 1$ or 3. Case 6: $D^m \not\subset D^s$, $D^m \cap D^s \subset D_j^s \cup D_{j+1}^s$ for some j, but not Case 5. (By Lemma 3.2 and the minimality of m, Cases 1, 5 and 6 exhaust the possibilities when $D^m \not\subset D^s$.)

All of the twenty remaining cases are easy. Several are trivial. We will consider only one in detail. Suppose m and s are both interior folds and Case 3 holds. Then by Lemma 3.2, either $D_1^m \subset D_1^s$ and $D_2^m \cup D_3^m \subset D_2^s$, or $D_1^m \cup D_2^m \subset D_1^s$ and $D_3^m \subset D_2^s$, or $D_1^m \subset D_2^s$ and $D_2^m \cup D_3^m \subset D_3^s$, or $D_1^m \cup D_2^m \subset D_2^s$ and $D_3^m \subset D_3^s$. These possibilities are all symmetric to one another. We will consider only the first one. The following diagram indicates how r_1 and r_2 can be constructed.

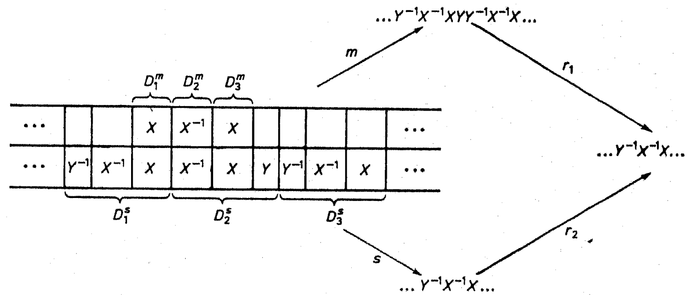

Fig. 7

DEFINITION 3.4. A word w is said to be *irreducible* if every reduction function with domain w is trivial.

THEOREM 3.5. *Let* w *be a word and let* $r_1\colon w \to w_1$ *and* $r_2\colon w \to w_2$ *be reduction functions. Then* w_1 *and* w_2 *admit a common reduction.*

Proof. Without loss of generality, we may suppose that w_1 and w_2 are irreducible. (If they are not, reduce them until they are. Any common reduction of the resulting words will be a common reduction of w_1 and w_2 by composition.) Now factor the reduction functions r_1 and r_2 as in Lemma 1.1: $r_1 = r_1^n \circ r_1^{n-1} \circ \ldots \circ r_1^0$ and $r_2 = r_2^m \circ r_2^{m-1} \circ \ldots \circ r_2^0$. Now the mappings r_i^0 $(i = 1, 2)$ collapse any two adjacent letters in w which are the same letter and are collapsed by r_i (they are constructed like the monotone parts in the monotone-light decomposition theorem). Moreover, since w_1 and w_2 are irreducible, they do not contain any adjacent letters which are the same letter. Since r_1 and r_2 send adjacents to adjacents, they must then collapse any two adjacent letters in w which are the same letter; and r_1^0 and r_2^0 must always do this collapsing because the remaining functions whose compositions make up r_1 and r_2 are light. But this means that r_1^0 and r_2^0 are the same map. (3) Therefore, they have the same range $\hat{w}$, which is the domain of both r_1^1 and r_2^1. And since the remaining maps are light, so is their composition. So we may suppose for the rest of the proof (by starting from $\hat{w}$ if necessary) that the maps r_1 and r_2 are light.

Now let m be a minimal fold on $\hat{w}$. Then by Lemma 3.3 we can form the following diagram. v_4 and v_5 are given by the lemma. Next let r be a reduction function on v_3 whose range is an irreducible word v_6. We proceed by induction on the length

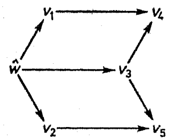

Fig. 8

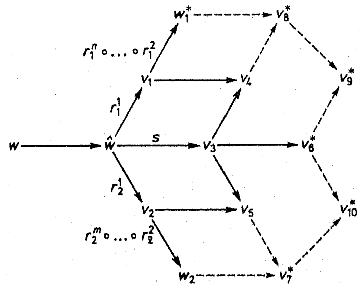

Fig. 9

(3) r_1^0 and r_2^0 may actually differ by a flip. However, we may make them into the same map by taking the flip out of r_1^0 and composing it with r_1^1. The result will be a new r_1^1 which is still a simple fold.

of the word w. Since all the words in question have length strictly less than $||w||$, by the induction hypothesis, we can fill in the dotted arrows to obtain the diagram in Fig. 9. The words marked with asterisks must all be irreducible words and are therefore the same word modulo a flip. The theorem now follows easily. ■

COROLLARY 3.6. *Let w be a word and let $r_1: w \to w_1$ and $r_2: w \to w_2$ be reduction functions. If w_1 and w_2 are irreducible words, then $w_1 = w_2$ modulo a flip.*

Proof. By the main theorem w_1 and w_2 admit a common reduction v. Since w_1 and w_2 are irreducible, the reduction functions onto v must be trivial. ■

COROLLARY 3.7. *Let $\mathscr{W}$ be the set of all words in the alphabet $\mathscr{A}$. Let $\mathscr{E}$ be the smallest equivalence relation on $\mathscr{W}$ such that if w_1 and w_2 are words in $\mathscr{W}$ and there is a reduction function from w_1 onto w_2, then $(w_1, w_2) \in \mathscr{E}$. Then each equivalence class in $\mathscr{W}$ has a unique irreducible representative (modulo a flip).*

Proof. Suppose that w_1 and w_2 are equivalent and that both are irreducible. Then w_1 and w_2 can be connected as in the following diagram by a sequence of

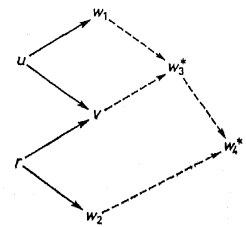

Fig. 10

alternately reversing arrows representing reduction functions. Applying the main theorem repeatedly to obtain the successive common reductions $v_1, v_2, \ldots, v_n$, we may connect w_1 and w_2 by a sequence of trivial reduction functions. ■

4. Commutativity on first, middle and last letters. Irreducibility of subwords. In this section we generalize our notation somewhat. If w_1 and w_2 are words, the notation $\alpha: w_1 \to w_2$ will denote the fact that α is a reduction function of w_1 *into* w_2 (so that w_1 is reducible to a *subword* of w_2). If α reduces w_1 *onto* w_2, we will write $\alpha: w_1 \twoheadrightarrow w_2$. If w is a word with first letter f and last letter l, we will write $w = [f, l]$. Words marked with asterisks will always be irreducible.

LEMMA. 4.1. *Let $\alpha, \beta: w_1 \to w_2^*$. If α is onto, then so is β.*

Proof. Let $\beta(w_1)$ denote the subword of w_2^* which is the "range" of β. Then $||\beta(w_1)|| \leqslant ||w_2^*||$. By Theorem 3.5 $\beta(w_1)$ and w_2^* admit a common reduction. Let w_3^* be an irreducible common reduction of $\beta(w_1)$ and w_2^*. Then $||w_2^*|| = ||w_3^*||$.

But since $\beta(w_1)$ is reducible to w_3^*, we also have $||w_3^*|| \leqslant ||\beta(w_1)|| \leqslant ||w_2^*||$. Therefore $||\beta(w_1)|| = ||w_2^*||$. ■

Definition 4.2. Let $w_1 = [f_1, l_1]$ and let $\alpha, \beta\colon w \to w_1$. If $\alpha^{-1}(f_1) = \beta^{-1}(f_1)$ and $\alpha^{-1}(l_1) = \beta^{-1}(l_1)$, we will say that α and β *commute with respect to endpoints* and will write $\alpha \underset{\llcorner}{e} \beta$. Note that we are treating α and β here as if they map the letters of w to the letters of w_1. Strictly speaking, this is not true (see the introduction to Section 2); however, the meaning is clear and we will continue to use the convention throughout the rest of the paper.

Theorem 4.3. *Let $\alpha\colon w_1 \to w^*$, where $w_1 = [f_1, l_1]$ and $w^* = [f^*, l^*]$, be a reduction function with $\alpha(f_1) = f^*$. Then $\alpha(w_1)$ admits a reduction function $r\colon \alpha(w_1) \twoheadrightarrow w_2^*$, where w_2^* is an (irreducible) initial subword of w^* (i.e., $f^* \in w_2^*$). Moreover, $r^{-1} \circ r(f^*) = f^*$.*

Proof. If $\alpha(w_1)$ is an irreducible word, we are done. If not, then any nontrivial function on $\alpha(w_1)$ can be realized as the composition of a sequence of simple folds (see 2.7). Moreover, since $\alpha(w_1)$ is an initial subword of the irreducible word w^*, any such fold must be an end fold on the "right" end of $\alpha(w_1)$. ($\alpha(w_1)$ cannot admit any nontrivial monotone reductions by the same reasoning.) As each such endfold is performed, the result is another initial subword of w^*, admitting only endfolds on its right end. Eventually, no more such folds are possible, and the resulting word is the desired w_2^*. r is the composition of the right endfolds. If any of the end folds ever failed to be one-to-one on f^*, then w^* would admit a "left" end fold, violating its irreducibility. Thus $r^{-1} \circ r(f^*) = f^*$. ■

Theorem 4.4. *Let $\alpha, \beta\colon w_1 \to w^*$, where $w_1 = [f_1, l_1]$ and $w^* = [f^*, l^*]$. Suppose further that either α is onto or $\alpha(f_1) = \beta(f_1) = f^*$. Then $\alpha \underset{\llcorner}{e} \beta$.*

Proof. We consider first the following

Special Case. Suppose α is onto and moreover α maps no proper subword of w_1 onto w^*. Then any other reduction $\gamma\colon w_1 \to w^*$ must have this same property. (By 4.1 γ must be onto and if $\gamma|w_1'$ is onto then by 4.1 $\alpha|w_1'$ is also onto.) Thus $\beta^{-1}(\{f^*, l^*\}) = \{f_1, l_1\} = \alpha^{-1}(\{f^*, l^*\})$. So the only way α and β can fail to commute with respect to endpoints is if one of the two maps "flips endpoints" and the other does not. Suppose that this happens. Without loss of generality, assume that $\alpha(f_1) = f^* = \beta(l_1)$ and $\alpha(l_1) = l^* = \beta(f_1)$. Let x be a letter which is not in w^*, and consider the new words xw_1, xw^* and w^*x and new reduction functions $\bar{\alpha}\colon xw_1 \twoheadrightarrow xw^*$ and $\bar{\beta}\colon xw_1 \twoheadrightarrow w^*x$ given by $\bar{\alpha}(x) = x$, $\bar{\alpha}|w_1 = \alpha$, $\bar{\beta}(x) = x$ and $\bar{\beta}|w_1 = \beta$. Then since xw^* and w^*x are both irreducible, 3.6 implies that there is a flip carrying xw^* onto w^*x. But this implies that w^* possesses enough symmetry to admit an endfold. This contradiction establishes the special case.

We now prove the general statement by induction on $n = ||w^*||$. If $n = 1$ (or 2 or 3) the theorem is trivial. So assume the theorem true for all $n < n_0$ and let $||w^*|| = n_0$. Let $X = \alpha^{-1}(\{f^*, l^*\}) \cup \beta^{-1}(\{f^*, l^*\})$. We must show that $\alpha(x) = \beta(x)$ for all $x \in X$. We first claim that $\alpha(x) = \beta(x)$ for *some* $x \in X$. If neither α nor β is onto, this follows immediately by hypothesis, since we must have $\alpha(f_1) = \beta(f_1) = f^*$. So we may assume that one of the maps is onto and it follows from 4.1 that both are onto. Now let $w_2 = [f_2, l_2]$ be a subword of w_1 such that $\alpha|w_2$ is onto and α maps no proper subword of w_2 onto w^*. Then by the special case $\alpha|w_2 \underset{\llcorner}{e} \beta|w_2$ and we have $\alpha(f_2) = \beta(f_2) \in \{f^*, l^*\}$.

Now let $y_0 = \min\{x \in X\colon \alpha(x) = \beta(x)\}$ and $z_0 = \max\{z \in X|$ for all $x \in X$ where $y_0 \leqslant x \leqslant z$, $\alpha(x) = \beta(x)\}$. Suppose that for some $y \in X$, $\alpha(y) \neq \beta(y)$. Then either $y < y_0$ or $y > z_0$. Without loss of generality, we assume that $y > z_0$ and let $z_1 = \min\{z \in X\colon z > z_0\}$. Then z_1 is well defined and $\alpha(z_1) \neq \beta(z_1)$. Assume further (without loss of generality) that $\alpha(z_0) = \beta(z_0) = f^*$. There are then two possibilities:

Case 1. Either $\alpha(z_1) = l^*$ or $\beta(z_1) = l^*$ (but not both). Then by the special case $\alpha|[z_0, z_1] \underset{\llcorner}{e} \beta|[z_0, z_1]$ and we have $\alpha(z_1) = \beta(z_1) = l^*$, a contradiction.

Case 2. Either $\alpha(z_1) = f^*$ or $\beta(z_1) = f^*$ (but not both) and $\alpha(z_1) \neq l^* \neq \beta(z_1)$. Assume (without loss of generality) that $\alpha(z_1) = f^*$. Let $Z = [z_0, z_1]$. By 4.3 there are reduction functions $r_1\colon \alpha(Z) \twoheadrightarrow w_3^*$ and $r_2\colon \beta(z) \twoheadrightarrow w_4^*$ where w_3^* and w_4^* are (irreducible) initial subwords of w^* and $r_1^{-1}(f^*) = f^* = r_2^{-1}(f^*)$. Either $w_3^* \subset w_4^*$ or $w_4^* \subset w_3^*$, and by applying 4.1 to the maps $r_1 \circ \alpha|Z$ and $r_2 \circ \beta|Z$, we conclude that $w_3^* = w_4^*$. Moreover, $||w_3^*|| \leqslant ||\alpha(Z)|| < n_0$, so the induction hypothesis applies and we have $r_1 \circ \alpha(z_1) = f^* = r_2 \circ \beta(z_1)$. But since $r_2^{-1}(f^*) = f^*$, this implies that $\beta(z_1) = f^*$, a contradiction. ■

Corollary 4.5. *Let $\alpha\colon w_1 \to w_2^*$ and $\beta\colon w_1 \twoheadrightarrow w_3$, where $w_1 = [f_1, l_1]$, $w_2^* = [f_2, l_2]$ and $w_3 = [f_3, l_3]$. Suppose further that $\alpha(f_1) = f_2$ and $\beta(f_1) = f_3$. Then there is a reduction function $r\colon w_3 \to w_2^*$ such that $r(f_3) = f_2$.*

Proof. Let $r_1\colon w_1 \twoheadrightarrow w_4^*$ be a reduction function where w_4^* is an initial subword of w_2^*, as in 4.3. It follows from 3.6 that there is a reduction function $r\colon w_3 \to w_4^*$. By 4.1 $r \circ b$ is onto. By 4.4 $r_1 \circ \alpha \underset{\llcorner}{e} r \circ \beta$. ■

Definition 4.6. Let $w = [f, l]$ and let m_1 and m_2 be letters in w with $f < m_1 < m_2 < l$. Then we write $w = [f, m_1][m_1, m_2][m_2, l]$.

Lemma 4.7. *Let $w = [f, m_1][m_1, m_2][m_2, l]$ and suppose there exist reduction functions $\alpha\colon [m_1, m_2] \to [f, m_1]$ such that $\alpha(m_1) = m_1$ and $\beta\colon [m_1, m_2] \to [m_2, l]$ such that $\beta(m_2) = m_2$. Then w admits an interior fold. In particular w is not irreducible.*

Proof. If $[m_1, m_2]$ is irreducible, then α and β are one-to-one and the lemma follows easily. So let s be a minimal fold on $[m_1, m_2]$. If s is an interior fold, we are done. So suppose s is an end fold $A^{-1}A$ (where A is an irreducible word. See the text preceding figure 6 for an explanation of this notation). Without loss of generality, assume that $A^{-1}A$ sits on the "left end" of $[m_1, m_2]$. Then α must carry A^{-1} one-to-one onto the word A, which must consequently sit at the right end of $[f, m_1]$. Consequently, w admits the interior fold $AA^{-1}A$. ■

Theorem 4.8. *Let $w^* = [f^*, l^*]$ be an irreducible word with $||w^*|| \geqslant 3$. Then there exists a letter m in w^* with $f^* < m < l^*$ such that $[f^*, m]$ and $[m, l^*]$ are irreducible.*

Proof. Let z be the letter in w^* immediately following f^*. Since w^* is irreducible, z cannot be the same letter as f^*. Therefore, $[f^*, z]$ is an irreducible word. Let $m_1 = \max\{x \in w^*: f^* < x < l^* \text{ and } [f^*, x] \text{ is irreducible}\}$. We claim that $[m_1, l^*]$ is also irreducible.

Suppose not. Then by 4.3 $[m_1, l^*]$ admits a reduction function r_1 onto the irreducible word $[m_2, l^*]$ where $m_1 < m_2$ and $r_1^{-1}(l^*) = l^*$. Thus $m_2 < l^*$. r_1 must be the identity function on $[m_2, l^*]$. Thus if we let $\beta = r_1|[m_1, m_2]$, we have β: $[m_1, m_2] \to [m_2, l^*]$ with $\beta(m_2) = m_2$. Since $m_1 < m_2 < l^*$, $[f^*, m_2]$ is not irreducible by definition of m_1. So by 4.3, there is a reduction function r_2: $[f^*, m_2] \to$ $\to [f^*, x]$ where $[f^*, x]$ is an irreducible word. Since r_2 must be the identity on the irreducible word $[f^*, m_1]$, it is not difficult to see that we must have $x = m_1$. Thus if we set $\alpha = r_2|[m_1, m_2]$, we have α: $[m_1, m_2] \to [f^*, m_1]$ with $\alpha(m_1) = m_1$. By 4.7 w^* then fails to be irreducible. ■

Lemma 4.9. *Let $w_1 = [f_1, l_1]$ and $w_2 = [f_2, l_2]$ be words. Let m be a letter in w_2 such that $f_2 < m < l_2$ and let α, β: $w_1 \to w_2$ such that $\alpha(w_1) \subset [f_2, m]$, $\beta(w_1) \subset [m, l_2]$ and $\alpha(f_1) = \beta(f_1) = m$. Suppose further that either* (1) *$\alpha(l_1) = f_1$ or* (2) *$\alpha(l_1) = m$ and $\beta(l_1) \neq m$. Then either w_2, $[f_2, m]$ or $[m, l_2]$ fails to be irreducible.*

Proof. Suppose $[f_2, m]$ and $[m, l_2]$ are irreducible. We will show that w_2 admits an endfold. We claim that we may take $\alpha(w_1)$ and $\beta(w_1)$ to be irreducible also. If they are not, then reduce them "toward m" with reduction functions r_1 and r_2, using 4.3. It is not difficult to verify that the reduction functions $r_1 \circ \alpha$ and $r_2 \circ \beta$ still satisfy the hypotheses (note that if (1) holds then $\alpha(w_1)$ is automatically irreducible).

So assume that $\alpha(w_1)$ and $\beta(w_1)$ are irreducible. Then by 3.6, there is a 1—1 reduction γ: $\beta(w_1) \twoheadrightarrow \alpha(w_1)$. By 4.4 $\alpha \underset{\text{c}}{\text{e}} \gamma \circ \beta$. Thus $\gamma(m) = \gamma(\beta(f_1)) = \alpha(f_1) = m$. It follows that (2) cannot hold. For if $\alpha(l_1) = m$, then $\gamma(\beta(l_1)) = m$. But this implies that $\beta(l_1) = m$, since γ is one-to-one. So we must have $\alpha(l_1) = f_2$. Thus $\alpha(w_1) = [f_2, m]$. It follows easily (using γ^{-1}) that w_2 admits an endfold. ■

Theorem 4.10. *Let $w^* = [f^*, l^*]$ and let m be a letter in w^* such that $f^* < m < l^*$ and $[f^*, m]$ and $[m, l^*]$ are irreducible. Let α, β: $w_1 \to w^*$ be reduction functions such that either* (1) *α is onto or* (2) *$\alpha(f_1) = \beta(f_1) = f^*$. Then α and β commute on the set $\{f^*, m, l^*\}$; i.e., for all $y \in \{f^*, m, l^*\}$, $\alpha^{-1}(y) = \beta^{-1}(y)$.*

Proof. Let $X = \alpha^{-1}(\{f^*, m, l^*\}) \cup \beta^{-1}(\{f^*, m, l^*\})$. We must show that $\alpha(x) = \beta(x)$ for all $x \in X$. By 4.4 $\alpha \underset{\text{c}}{\text{e}} \beta$, so there is an $x_0 \in X$ such that $\alpha(x_0) = \beta(x_0)$. Suppose there is an $x \in X$ such that $\alpha(x) \neq \beta(x)$. Assume, without loss of generality that $x_0 < x$. Then let

$$x_1 = \max\{y \in X: \text{ for all } z \in X \text{ with } x_0 \leqslant z \leqslant y,\ \alpha(z) = \beta(z)\}.$$

Let x_2 be the "immediate successor" of x_1 in X. Then by our assumption on x, x_2 is well defined and $\alpha(x_2) \neq \beta(x_2)$.

Case 1. $\alpha(x_1) = f^*$ $(= \beta(x_1))$ or $\alpha(x_1) = l^*$ $(= \beta(x_1))$. Then by the definition of x_2, we have $\alpha([x_1, x_2]) \cup \beta([x_1, x_2]) \subset [f^*, m]$ or $\alpha([x_1, x_2]) \cup \beta([x_1, x_2]) \subset [f^*, m]$ or $\alpha([x_1, x_2]) \cup \beta([x_1, x_2]) \subset [m, l^*]$. It follows from 4.4 that $\alpha(x_2) = \beta(x_2)$, a contradiction.

Case 2. $\alpha(x_1) = m$ $(= \beta(x_1))$. By the definition of x_2, α maps $[x_1, x_2]$ into either $[f^*, m]$ or $[m, l^*]$ and similarly for β. If α and β both map $[x_1, x_2]$ into the "same side" of w^*, then 4.4 implies once again that $\alpha(x_2) = \beta(x_2)$. So α and β must map $[x_1, x_2]$ into opposite sides. But then 4.9 applies to yield a contradiction. ■

5. Tree words and graph words. In this section we generalize the notion of word and obtain some results with applications in continua theory. Recall that a word is a function on an initial set X of natural numbers into an alphabet. We may think of X as being embedded in an order preserving way in a topological arc I. Conversely, any triple (w, X, I), where w is a function on the finite subset X of the arc I, may be viewed as a word. We will use this language to generalize the concept of a word. The main result of the section is 7.7, which is somewhat analogous to 3.5. It states that under certain strong conditions, if π_1: $w \to w_1$ and π_2: $w \to w_2$ are reduction functions, where w is a graph- word, w_1 is a tree-word, and w_2 is a word, then there is a reduction function from w_1 onto w_2. This result will be used in [2] to show that certain hereditarily indecomposable continua of span 0 (which are necessarily tree-like) are chainable. (It will follow that continua of span 0 which are the image of P under an induced map are chainable). We would be very interested to know if the set B can be dropped from the hypotheses of Theorems 5.6 and 5.7. If so, it will follow that the hypothesis of hereditarily indecomposability is not needed.

Definition 5.1. A *tree-word* is a triple (w, X, T) where T is a tree (a topological graph with no cycles), X is a finite subset of T, and w is a function from X to some alphabet. A *graph-word* may be defined similarly, where T is replaced by any topological graph G. In cases where no confusion is likely, we will simply use the letter w to represent the tree-word or graph-word (w, X, T) or (w, X, G). Ordinary words (w, X, I) will sometimes be called *chain-words*. Note that every chain word is a tree-word and every tree-word is a graph-word. One may think of a graph-word as a graph with "letters" $w(x)$ attached to it at the points $x \in X$. As in some of the arguments in the previous section, we will occasionally identify the points x and the letters $w(x)$.

Definition 5.2. Let (w, X, G) be a graph word and let x_1 and x_2 be (distinct) points in X. We say that x_1 and x_2 are *adjacent* if there is an arc $A \subset G$ such that $A \cap X = \{x_1, x_2\}$.

Definition 5.3. Let (w_1, X_1, G_1) and (w_2, X_2, G_2) be graph-words. A *reduction function* from w_1 to w_2 is a function r: $X_1 \to X_2$ carrying adjacent points of X_1 to adjacent points of X_2 and such that $w_1(x) = w_2(r(x))$ for each $x \in X_1$. If r is a reduction function from w_1 to w_2, we will write r: $w_1 \to w_2$. If r maps X_1 onto X_2, we will call r an *onto* reduction function and will write r: $w_1 \twoheadrightarrow w_2$. Note that for chain-words this notion of reduction coincides with the previous definition (see Section 2).

DEFINITION 5.4. Let (w_1, X_1, G_1) and (w_2, X_2, G_2) be graph-words and let $f\colon G_1 \to G_2$ be a (continuous) function. f is called a *complete reduction (function)* if $f|X_1$ is a reduction function and in addition $f^{-1}(f(G_1) \cap X_2) = X_1$.

DEFINITION 5.5. Let (w_1, X_1, G_1), (w_2, X_2, G_2) and (w_3, X_3, G_3) be graph-words. We say that (w_3, X_3, G_3) is *completely embedded* in $(w_1, X_1, G_1) \times (w_2, X_2, G_2)$ and write $(w_3, X_3, G_3) \subset (w_1, X_1, G_1) \times (w_2, X_2, G_2)$ or simply $w_3 \subset w_1 \times w_2$ if G_3 is (topologically) embedded in $G_1 \times G_2$ and the projection functions $\pi_1\colon G_3 \to G_1$ and $\pi_2\colon G_2 \to G_1$ are complete reduction functions.

THEOREM 5.6. *Let $(w, X, G) \subset (w_1, X_1, T) \times (w_2, X_2, I)$ where G is a graph, T a tree and I an arc. Let B and S be graphs whose union is G. Let $w_s = w|S$, $w_b = w|B$, $X_s = X \cap S$ and $X_b = X \cap B$. (It is not difficult to show that (w_s, X_s, S) and (w_b, X_b, B) are then completely embedded in $(w_1, X_1, T) \times (w_2, X_2, I)$.) Let $E(T) = \{e_0, e_1, \ldots, e_n\}$ be the set of endpoints of T and $E(I) = \{0, 1\}$ the set of endpoints of I. For each $j = 1, 2, \ldots, n$ let $S_j = \pi_1^{-1}([e_0, e_j]) \cap S$ where $[e_0, e_j]$ is the unique arc in T with endpoints e_0 and e_j. Suppose that $|X_2| \geqslant 3$ and*

(0) *w_2 is an irreducible word,*

(1) *$(e_0, 0) \in X_s$,*

(2) *S separates $T \times \{0\}$ from $T \times \{1\}$,*

(3) *$\forall j = 1, 2, \ldots, n\ \exists B_j \subset \pi_1^{-1}([e_0, e_j])$ such that $B_j \cup S_j$ is arcwise connected.*

Moreover, $B = \bigcup_{j=1}^{n} B_j$.

Let m be any letter in $w_2 = [f_2, l_2]$ such that $[f_2, m]$ and $[m, l_2]$ are irreducible. Let $S_m = S \cap \pi_2^{-1}(m)$ and $B_m = B \cap \pi_2^{-1}(m)$. Then

(i) *For each $t \in \pi_1(S_m \cup B_m)$, $\pi_1^{-1}(t) \cap G$ is a single point.*

Let $\{T(1), \ldots, T(k)\}$ be the set of closures of the components of $T - \pi_1(S_m)$. For each $i = 1, 2, \ldots, k$ let e_0^i be the unique endpoint of $T(i)$ which separates all other endpoints of $T(i)$ from e_0. Then

(ii) *For each $i = 1, 2, \ldots, k$ there exists $S(i) \subset S$, $B(i) \subset B$ and $I(i) \subset I$ such that*

(a) *$I(i)$ is either the topological arc $[0, m]$ or the topological arc $[m, 1]$,*

(b) *(0)–(3) above are satisfied, where e_0^i replaces e_0, m replaces 0 or 1, and w_s, w_b, w_1 and w_2 are restricted to $S(i)$, $B(i)$, $T(i)$ and $I(i)$ respectively.*

Proof of (i). Suppose (i) fails. Then there are x_1 and x_2 in $S \cap B$ such that $m \in \pi_2(\{x_1, x_2\})$, $\pi_2(x_1) \neq \pi_2(x_2)$ and $\pi_1(x_1) = \pi_1(x_2)$. We consider three cases.

Case 1. $x_1, x_2 \in S$. Suppose without loss of generality that $\pi_2(x_1) = m$ and $\pi_2(x_2) \neq m$. Choose $j \in \{1, 2, \ldots, n\}$ such that $\pi_1(x_1) \in [e_0, e_j]$. By (3) there are arcs I_1 and I_2 in $S_j \cup B_j$ joining $(e_0, 0)$ to x_1 and $(e_0, 0)$ to x_2 respectively. These arcs determine chain-words v_1 and v_2 by intersecting I_1 and I_2 with the set X. Orient the words so that they "start" at $(e_0, 0)$. Now let v be the chain-word $v_1 \overleftarrow{v}_1 v_2$ ($\overleftarrow{v}_1$ represents v_1 "written backwards") and let γ be the natural endfold mapping v to $\overleftarrow{v}_1 v_2$. Let $u = w_1|[e_0, e_j] \cap X$. We claim that there is a reduction function $r\colon u \to w_2$ such that $r(e_0) = 0$. Since S separates $T \times \{0\}$ from $T \times \{1\}$, there must be a point x of S_j in $\{e_j\} \times I$. Since $S_j \cup B_j$ is arcwise connected, there is an arc $A \subset S_j \cup B_j$ joining $(e_0, 0)$ and x. Note that $\pi_1(A) = [e_0, e_j]$. Let p be the chain word obtained by restricting w to $A \cap X$. Then $\pi_1\colon p \twoheadrightarrow u$ and $\pi_2\colon p \to w_2$ are reduction functions "carrying first letters to first letters". By 4.3 p admits a reduction function r_3 onto an irreducible subword w_3^* of w_2. By 3.5 u must then admit a reduction function r_4 onto w_3^*. By 4.4 $r_3 \underset{\smile}{e} r_4 \circ \pi_1$. It follows that $r_4(e_0) = 0$, completing the proof of the claim. Let $\alpha = r_4 \circ \pi_1 \circ \gamma\colon w \to w_2$ and $\beta = \pi_2 \circ \gamma\colon w \to w_2$. These reduction functions carry first letters to first letters, so by 4.10, they commute with respect to m. But $\alpha(x_1) = \alpha(x_2)$ and $\beta(x_1) = m \neq \beta(x_2)$. This contradiction completes the proof of case 1.

Case 2. $x_1 \in S$ and $x_2 \in B$. Choose $j \in \{1, 2, \ldots, n\}$ such that $x_2 \in B_j$ and repeat the argument for case 1.

Case 3. $x_1, x_2 \in B$. Suppose, without loss of generality, that $\pi_2(x_1) = m$. Since S separates $T \times \{0\}$ from $T \times \{1\}$, there must be an $x \in S$ such that $\pi_1(x) = \pi_1(x_1)$. By case 2 we must have $x = x_1$. But then case 2 applies to x_1 and x_2. This completes the proof of (i).

Proof of (ii). By (i) and the fact that S separates, we have $B_m \subset S_m$. Let $i \in \{1, 2, \ldots, k\}$ and let $E(T(i)) = \{e_0^i, e_1^i, \ldots, e_{n_i}^1\}$ be the set of endpoints of $T(i)$, where e_0^i is defined as above. Let $S(i, 1) = S \cap (T(i) \times [0, m])$ and $S(i, 2) = S \cap \cap (T(i) \times [m, 1])$. ($[0, m]$ and $[m, 1]$ are topological arcs.) We claim that either $S(i, 1)$ separates $T(i) \times \{0\}$ from $T(i) \times \{m\}$ or $S(i, 2)$ separates $T(i) \times \{m\}$ from $T(i) \times \{1\}$. For suppose not. Then let A_1 and A_2 be arcs in $(T(i) \times [0, m]) - S$ and $(T(i) \times [m, 1]) - S$ respectively such that A_1 is irreducible between $T(i) \times \{0\}$ and $T(i) \times \{m\}$ and A_2 is irreducible between $T(i) \times \{m\}$ and $T(i) \times \{1\}$. Since S meets the tree $T(i) \times m$ only in its endpoints (points of S_m), the arcs A_1 and A_2 meet $T(i) \times m$ in points a_1 and a_2 which may be joined by an arc A_3 in $T(i) \times m$ which misses S. But then $A_1 \cup A_2 \cup A_3$ is an arc in $(T \times I) - S$ joining $T \times \{0\}$ and $T \times \{1\}$. This contradiction establishes the claim.

Suppose (without loss of generality) that $S(i, 1)$ separates $T(i) \times \{0\}$ from $T(i) \times \{m\}$. Since $T(i) \times [0, m]$ is unicoherent and locally connected, some component of $S(i, 1)$, call it $S(i)$, separates $T(i) \times \{0\}$ from $T(i) \times \{m\}$ (see [1], p. 438). Let $I(i) = [0, m]$. It follows from (i) that $(e_0^i, m) \in S(i)$. Thus (1) and (2) are satisfied. It remains to show (3).

Let $j \in \{1, 2, \ldots, n_i\}$ and choose $l \in \{1, 2, \ldots, n\}$ such that $[e_0^i, e_j^i] \subset [e_0, e_l]$. Note that $(e_0^i, m) \in S$. If also $(e_j^i, m) \in S$, let $B(i)_j = C_1 \cup C_2$ where C_1 is the component of $(S_l \cup B_l) \cap (T(i) \times I(i))$ containing (e_0^i, m) and C_2 is the component of $(S_l \cup B_l) \cap (T(i) \times I(i))$ containing (e_j^i, m). If $(e_j^i, m) \notin S$, let $B(i)_j = C_1$. Note that in this case we must have $e_j^i = e \in E(T)$. Let $S(i)_j = S(i) \cap ([e_0^i, e_j^i] \times I(i))$. We claim that $B(i)_j \cup S(i)_j$ is arcwise connected.

Let $p, q \in B(i)_j \cup S(i)_j$. By (3) there is an arc $J \subset B_l \cup S_l$ connecting p and q.

Let J_p = the component of $J \cap ([e_0^i, e_j^i] \times I(i))$ containing p and J_q = the component of $J \cap ([e_0^i, e_j^i] \times I(i))$ containing q. If either J_p or J_q is J, we are done. If not, then J_p and J_q must be arcs joining p and q to either of the "corner points" (e_0^i, m) or (e_j^i, m). This is so because (i) and the relationship of $T(i)$ to S_m (see the definition of $T(i)$) imply that these points are the only two places where J can "run out of the box" $[e_0^i, e_j^i] \times I(i)$. If $(e_j^i, m) \notin S$ (second case above), then J_p and J_q must both contain the point (e_0^i, m) and $J_p \cup J_q$ is the desired arc. If $(e_j^i, m) \in S$, let A be a component of $S(i)_j$ which separates $[e_0^i, e_j^i] \times \{0\}$ from $[e_0^i, e_j^i] \times \{m\}$ in $[e_0^i, e_j^i] \times I(i)$. (Such a component must exist by the unicoherence of $[e_0^i, e_j^i] \times I(i)$ and the fact that $S(i)$ separates $T(i) \times \{0\}$ and $T(i) \times \{m\}$.) Then (i) implies that A must contain the points (e_0^i, m) and (e_j^i, m). $J_p \cup A \cup J_q$ thus contains the desired arc. Finally, construct the set $B(i)_j$ for every $j \in \{1, 2, \ldots, n_i\}$ and let $B(i) = \bigcup_{j=1}^{n_i} B(i)_j$. ■

THEOREM 5.7. *Let everything be as in the hypotheses of Theorem 5.6. Then there is a reduction function $r\colon w_1 \to w_2$.*

Proof. We will show that there is a set $S^* \subset S$ which separates $T \times \{0\}$ from $T \times \{1\}$ and such that the reduction function $\pi_1^* = \pi_1 | S^*$ is one-to-one and onto w_2. The desired reduction function will then be $r = \pi_2 \circ \pi_1^{*-1}$. We prove the existence of S^* by induction on $n = ||w_2||$. First note that if $n = 1$, 2 or 3, then we can take $S = S^*$. This is because w_2 will have to consist of distinct letters and π_1 and π_2, being reduction functions, must map "same letters to same letters" (π_1 must be onto because S is a separator).

So assume that the theorem is true for all $n < n_0$ and let $||w_2|| = n_0$. Choose m in w_2 such that $f_2 \neq m \neq l_2$ and $[f_2, m]$ and $[m, l_2]$ are irreducible (see 4.8). Let $T(i)$, $S(i)$, $B(i)$ and $I(i)$, $i = 1, 2, \ldots, k$ be defined as in the statement of 5.6. Then by 5.6 the induction hypothesis applies and we obtain separators $S(i)^*$ such that $\pi_1 | S(i)^*$ is one-to-one and onto for each i and $S(i)^*$ separates $T(i) \times \{m\}$ from $T(i) \times \{0\}$ or $T(i) \times \{1\}$. Let $S = \bigcup_{i=1}^{k} S(i)^*$. First note that the reduction function $\pi_1^* = \pi_1 | S^*$ (or more properly, $\pi_1 | X_s \cap S^*$) is one-to-one, because π_1^* is one-to-one on each of the sets $T(i) \times P(i)$ and the $T(i) \times P(i)$'s meet one another only along arcs $\{\pi_1(s)\} \times I(i)$, where $s \in S_m$. Moreover, X_s contains exactly one point in any of these arcs by (i) of 5.6. It remains to show that S^* separates $T \times \{0\}$ from $T \times \{1\}$.

Suppose not. Then there is a polygonal arc $A \subset T \times I - S^*$ running from $T \times \{0\}$ to $T \times \{1\}$. Consider the behavior of a point p moving along A from $T \times \{0\}$ to $T \times \{1\}$. Note that as p passes from one of the sets $T(i) \times I$ to another, it must pass through an arc $\{x\} \times I$, where $x \in \pi_1(S_m)$. Moreover, p cannot pass through the point (x, m) because this point is necessarily in S^* (see 5.6, (i)). It follows that p must cross the set $T \times \{m\}$ in the interior of some set $T(i) \times I$. That is, there must be an $i \in \{1, 2, \ldots, k\}$ and a subarc A' of A such that $A' \subset T(i) \times I$ and A' contains points p_1, p_2 and p_3 such that $p_1 = (x, y)$ where either $y = 1$ or

$y > m$ and $x \in \pi_1(S_m)$; $p_2 = (x, y)$ where $y = m$ and $p_3 = (x, y)$ where either $y = 0$ or $y < m$ and $x \in \pi_1(S_m)$. Suppose, without loss of generality, that $S(i)^*$ separates $T(i) \times \{m\}$ from $T(i) \times \{1\}$. Then A' contains a subarc A'' contained in $T(i) \times [m, 1]$ passing from a point q_1 to a point q_2, where $q_1 = (x, y)$ with $x \in \pi_1(S_m)$ and $y > m$ and $q_2 = (x, m)$ for some $x \in T(i)$ (we cannot have $y = 1$ since $S(i)^*$ separates $T(i) \times \{m\}$ from $T(i) \times \{1\}$). But by 5.6, (i) the arc $\{x\} \times [y, 1]$ misses $S(i)^*$. Thus $A'' \cup (\{x\} \times [y, 1])$ is a connected set joining $T(i) \times \{m\}$ and $T(i) \times \{1\}$, contradicting the fact that $S(i)^*$ separates.

QUESTION 5.8. Does Theorem 5.7 remain true if the set B is dropped, i.e., if $S = G$ and hypothesis (3) is deleted?

Added in proof. This question has been recently answered in the affirmative by the second author. The proof will appear in a forthcoming paper tentatively entitled "On reduction of tree-words to (chain-) words".

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Received 7 May 1984