# A minimum theorem for $n$-valued multifunctions

by

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**Abstract.** A multifunction is *$n$-valued* if all its point images consist of exactly $n$ points. Continuous $n$-valued multifunctions $\varphi: |K| \to |K|$ on a compact polyhedron $|K|$ possess a Nielsen number $N(\varphi)$ which is a lower bound for the number of fixed points of all continuous $n$-valued multifunctions homotopic to $\varphi$. Here it is shown that if $\varphi: M \to M$ is a continuous $n$-valued multifunction on a compact triangulable manifold (with or without boundary) of dimension at least three, then $N(\varphi)$ is a sharp lower bound, i.e. there exists a continuous $n$-valued multifunction $\varphi': M \to M$ which has precisely $N(\varphi)$ fixed points.

**1. Introduction.** An *$n$-valued multifunction* is a correspondence $\varphi: X \to Y$ from a topological space $X$ into a topological space $Y$ for which $\varphi(x)$ consists of exactly $n$ points for all $x \in X$. Such multifunctions have been studied in [8], [9], [10] and [11] under the additional assumption that they are continuous, i.e. both upper and lower semicontinuous. (See e.g. [2], Ch. VI, for definitions.) From the point of view of fixed point theory they behave in many ways like single-valued continuous functions.

The fact that they possess simplicial approximations [9], Theorem 4 and a generalization of the "Hopf construction" can be used to show that the fixed point set $\mathrm{Fix}\,\varphi = \{x \in |K| \mid x \in \varphi(x)\}$ of an $n$-valued continuous multifunction $\varphi: |K| \to |K|$ is generically finite if $|K|$ is a compact polyhedron. Lefschetz numbers $L(\varphi)$ with the property that $L(\varphi) \neq 0$ implies $\mathrm{Fix}\,\varphi \neq \emptyset$ have been defined by B. O'Neill [8] for a class of multifunctions which includes the $n$-valued continuous ones. Extensions of Brouwer's fixed point theorem and the Borsuk–Ulam theorem are contained in [10]. A fixed point index for $n$-valued continuous multifunctions on compact polyhedra was introduced in [11], as well as a Nielsen number which is a lower bound for the number of fixed points of $\varphi$ and invariant under homotopy. By a homotopy between two $n$-valued continuous multifunctions $\varphi_0, \varphi_1: X \to Y$ we mean in this paper always an $n$-valued continuous multifunction $\Phi: X \times I \to Y$, where $I = [0, 1]$, $\Phi(\cdot, 0) = \varphi_0$ and $\Phi(\cdot, 1) = \varphi_1$.

Here we prove a Minimum Theorem (Theorem 5.2) which shows that the Nielsen number $N(\varphi)$ of $\varphi: X \to X$ is in fact a sharp lower bound, i.e. that there exists an $n$-valued continuous multifunction $\varphi': X \to X$ homotopic to $\varphi$ which has precisely $N(\varphi)$ fixed points, if $X$ is a compact triangulable manifold (with or without boundary) of dimension at least three. For maps (i.e. single-valued continuous functions) such a minimum theorem is well known. It was proved in 1942 by F. Wecken [13] for a certain class of compact polyhedra which includes manifolds of dimension at least three, and extended to a wider class of polyhedra in 1966 by Gen-Hua Shi [12] (see also [3], Ch. VIII) and in 1980 by Boju Jiang [4]. Shi and especially Jiang also simplified Wecken's proof considerably, but it is still a complicated one. The assumption in our Minimum Theorem 5.2 that $M$ is at least three-dimensional cannot be weakened, as it is now known that the theorem is false for maps on surfaces. This fact, which was announced without proof by J. Weier [14] in 1956, was finally proved by Boju Jiang [6] in 1984.

Our proof of Theorem 5.2 uses ideas from Jiang's proof [4], as it extends his concepts of special homotopies of maps and of paths to special homotopies of $n$-valued continuous multifunctions and of $n$-paths. (By an *$n$-path* in $X$ we mean an $n$-valued continuous multifunction $\alpha\colon I\to X$.) But as we restrict our setting to triangulable manifolds, we can simplify many parts of Jiang's proof. We do not need the concept of a normal PL path [4], § 3, and can e.g. replace the sophisticated but lengthy proof of [4], Lemma 5.1 by an easy general position argument concerning coincidences of two maps to obtain our corresponding Lemma 4.3. The proof that two isolated fixed points of a proximity map on certain polyhedra can be united is a tricky one (see [12], Lemmas 1.2 and 1.3, or [3], Lemmas 2 and 3, Ch. VIII C, pp. 126–131), whereas the proof of our corresponding result is quick (see Lemmas 3.2 and 3.3). As an $n$-valued continuous multifunction is a map if $n=1$, this paper includes therefore a proof of the Minimum Theorem for selfmaps of compact triangulable manifolds which is considerably shorter than Jiang's proof for selfmaps of more general polyhedra. (See the final Remark 5.3.) But one should keep in mind that if the manifold is further assumed to be differentiable, then the Minimum Theorem can be proved very elegantly with the help of the Whitney trick [5].

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**2. Some tools: Coincidence Lemma and Splitting Lemma.** The proof of our theorem will use, in the proof of Lemma 4.3 and in the final part of the proof of Theorem 5.2, the Lemma 2.2 which concerns coincidences of two maps. Lemma 2.2 will only be needed in the special case where $P=I\times I$ is the unit square, but the proof of this case is not simpler than the one of the general form in which we have stated it. It is in the application of the Coincidence Lemma 2.2 that the assumption that $M$ is at least three-dimensional is crucial, as it is false if $P=I\times I$ and $M$ is a surface. We shorten the proof of the Coincidence Lemma 2.2 by separating a detail in the form of Lemma 2.1.

We need some notation. $\sigma^k$ stands for an open $k$-simplex, $\bar\sigma^k$ for the corresponding closed simplex and $\dot\sigma^k$ for its boundary. As usual we define $\sigma^{-1}=\emptyset$. The interior, closure and boundary of a space $X$ are denoted by $\operatorname{Int}X$, $\operatorname{Cl}X$ and $\operatorname{Bd}X$. We write $B^m(r)$ for the ball $\{x\in R^m \mid d_E(0,x)\leqslant r\}$ in Euclidean $m$-space $(R^m, d_E)$ with origin 0. If $d$ is a metric for $M$, then $\bar d(f,f')$ denotes the distance in the sup metric between the two maps $f,f'\colon X\to M$.



LEMMA 2.1. *Let $0<k<m$, $-1\leqslant l<k$ and $r>0$. If $\sigma^l$ is a face of $\sigma^k$, then every map*

$$v\colon (\dot\sigma^k, \dot\sigma^k-\bar\sigma^l, \bar\sigma^l)\to (B^m(r), B^m(r)-0, 0)$$

*has an extension to*

$$\bar v\colon (\bar\sigma^k, \bar\sigma^k-\bar\sigma^l, \bar\sigma^l)\to (B^m(r), B^m(r)-0, 0).$$

Proof. If $l=-1$, then $v$ is of the form $v\colon \dot\sigma^k\to B^m(r)-0$, and as $\pi_{k-1}(B^m(r)-0)=0$, the map $v$ extends to $\bar v\colon \bar\sigma^k\to B^m(r)-0$.

Now assume that $l\geqslant 0$. Then there exists a homeomorphism $h$ from the quotient space $\bar\sigma^k/\bar\sigma^l$ onto $B^k(1)$. We write $h(\{\bar\sigma^l\})=y_0$, and label the points $B^k(1)-y_0$ as $y_t$, with $0<t\leqslant 1$, so that $y_1$ ranges over $\operatorname{Bd}B^k(1)-y_0$ and $y_t$ is the point on the segment from $y_0$ to $y_1$ with $d_E(y_0, y_t)=td_E(y_0, y_1)$. Hence the points of $\bar\sigma^k/\bar\sigma^l$ can be labelled $x_t=h^{-1}(y_t)$. Let $\bar q\colon \bar\sigma^k\to\bar\sigma^k/\bar\sigma^l$ be the quotient map, let $q\colon \dot\sigma^k\to\dot\sigma^k/\bar\sigma^l$ be its restriction to $\dot\sigma^k$ and define a map

$$w\colon (\dot\sigma^k/\bar\sigma^l, (\dot\sigma^k-\dot\sigma^l)/\dot\sigma^l, \bar\sigma^l/\bar\sigma^l)\to (B^m(r), B^m(r)-0, 0)$$

by $w=v\circ q^{-1}$. If we extend $w$ to the map

$$\bar w\colon (\bar\sigma^k/\bar\sigma^l, (\bar\sigma^k-\bar\sigma^l)/\bar\sigma^l, \bar\sigma^l/\bar\sigma^l)\to (B^m(r), B^m(r)-0, 0)$$

given by

$$\bar w(x_t)=\begin{cases} tw(x_1) & \text{if } 0<t\leqslant 1,\\ 0 & \text{if } t=0,\end{cases}$$

then $\bar v=\bar w\circ\bar q$ extends $v$ as required.

LEMMA 2.2. (Coincidence Lemma). *Let $M$ be a compact manifold of dimension $m\geqslant 3$, let $P$ be a compact polyhedron of dimension $p<m$, and let $P_0\subset P_1$ be subpolyhedra of $P$. ($P_0$ and $P_1$ can be empty.) Given $\varepsilon>0$ and two maps $f, g\colon P\to \operatorname{Int}M$ so that*

$$f=g \text{ on } P_0, \quad f\neq g \text{ on } P_1-P_0,$$

*there exists a map $f'\colon P\to\operatorname{Int}M$ so that*

(i) $f'=f$ *on* $P_1$,
(ii) $f'\neq g$ *on* $P-P_0$,
(iii) $\bar d(f,f')<\varepsilon$.

Proof. The proof is essentially a general position argument, but we need some $\varepsilon$-tics to satisfy condition (iii).

As $f(P) \cup g(P)$ is compact, we can choose finite collections $\{V_j\}$, $\{U_j\}$ of open sets in $M$ so that $f(P) \cup g(P) \subset \bigcup V_j$, each $V_j \subset U_j \subset \operatorname{Cl} U_j \subset \operatorname{Int} M$, and so that for each index $j$ there exists a homeomorphism $h_j$: $(\operatorname{Cl} U_j, \operatorname{Cl} V_j) \to (B^m(1), B^m(\frac{1}{2}))$, where $m$ is the dimension of $M$. Let $\lambda > 0$ be the Lebesgue number of the cover $\{V_j\}$. As all $h_j$ are uniformly continuous, we can determine, successively,

$$\delta_p, \varepsilon_{p-1}, \delta_{p-1}, \ldots, \varepsilon_1, \delta_1, \varepsilon_0$$

with

$$0 < \delta_k < \tfrac{1}{2} \quad \text{for} \quad k = 1, 2, \ldots, p,$$
$$0 < \varepsilon_0 \leqslant \varepsilon_1 \leqslant \ldots \leqslant \varepsilon_p = \min(\varepsilon, \lambda/3)$$

so that for all $x, y \in \operatorname{Cl} U_j$, for all indices $j$ and for $k = 1, 2, \ldots, p$

$$d_E(h_j(x), h_j(y)) \leqslant \delta_k \quad \text{implies} \quad d(x, y) \leqslant \tfrac{1}{2}\varepsilon_k,$$
$$d(x, y) \leqslant 2\varepsilon_{k-1} \quad \text{implies} \quad d_E(h_j(x), h_j(y)) \leqslant \delta_k.$$

Using a subdivision, if necessary, we can assume that for each simplex $\sigma$ of $P$ the diameters $\operatorname{diam} f(\bar{\sigma}) < \varepsilon_0/4$ and $\operatorname{diam} g(\bar{\sigma}) < \varepsilon_0/4$ and that $P_0$ is full in $P$ (i.e. that $\bar{\sigma} \cap P_0$ is either empty or a face of $\sigma$; see e.g. [7], p. 52). We write

$$C = \{x \in P \mid f(x) = g(x)\}$$

for the coincidence set of $f$ and $g$, and proceed to define $f'$ inductively on all 0-, 1-, ..., $p$-simplexes of $P$.

If $\sigma^0$ is a 0-simplex of $(P - P_1) \cap C$, we choose $f'(\sigma^0) \in \bigcup V_j$ arbitrarily so that $0 < d(f'(\sigma^0), g(\sigma^0)) < \varepsilon_0$; otherwise we define $f'(\sigma^0) = f(\sigma^0)$. Then $f'$ satisfies (i), (ii) and $\bar{d}(f, f') < \varepsilon_0$ on the 0-skeleton of $P$.

Now we assume that $f'$ has been constructed on the $(k-1)$-skeleton of $P$, for $1 \leqslant k < p$, such that $f'$ satisfies (i), (ii) and $\bar{d}(f, f') < \varepsilon_{n-1}$. If $\sigma^k$ is a $k$-simplex of $P - P_1$ and $\bar{\sigma}^k \cap C \neq \emptyset$, then

$$\operatorname{diam}[f(\bar{\sigma}^k) \cup g(\bar{\sigma}^k) \cup f'(\dot{\sigma}^k)] < \tfrac{1}{2}\varepsilon_0 + 2\varepsilon_{k-1} < 3\varepsilon_{k-1} \leqslant \lambda,$$

so $f(\bar{\sigma}^k) \cup g(\bar{\sigma}^k) \cup f'(\dot{\sigma}^k) \subset V_j$ for some index $j$. We define a map $v$: $\dot{\sigma}^k \to B^m(1)$ by choosing $v(x)$, for each $x \in \dot{\sigma}^k$, as the tip of the vector $\overrightarrow{h_j(g(x)) h_j(f'(x))}$ attached to the origin of $R^m$. As

$$d(g(x), f'(x)) \leqslant d(g(x), f(x)) + d(f(x), f'(x)) < \tfrac{1}{2}\varepsilon_0 + \varepsilon_{k-1} < 2\varepsilon_{k-1}$$

and as $\dot{\sigma}^k \cap P_0$ is a face $\bar{\sigma}^l$ of $\sigma^k$ of dimension $-1 \leqslant l < k$, the map $v$ is actually of the form

$$v\colon (\dot{\sigma}^k, \dot{\sigma}^k - \bar{\sigma}^l, \bar{\sigma}^l) \to (B^m(\delta_k), B^m(\delta_k) - 0, 0),$$

and has according to Lemma 2.1 an extension to

$$\bar{v}\colon (\bar{\sigma}^k, \sigma^k \cup (\dot{\sigma}^k - \bar{\sigma}^l), \bar{\sigma}^l) \to (B^m(\delta_k), B^m(\delta_k) - 0, 0).$$

We define, for each $x \in \bar{\sigma}^k$, a point $f'(x) \in \operatorname{Cl} U_j$ by the condition that $\overrightarrow{0 h_j(f'(x))}$ is the sum of $\overrightarrow{0 h_j(g(x))}$ and $\overrightarrow{0 v(x)}$ in $B^m(1)$ under vector addition in $R^m$. Then



$d_E(h_j(g(x)), h_j(f'(x))) \leqslant \delta_k$ and hence

$$d(f(x), f'(x)) \leqslant d(f(x), g(x)) + d(g(x), f'(x)) < \tfrac{1}{2}\varepsilon_0 + \tfrac{1}{2}\varepsilon_k \leqslant \varepsilon_k$$

for all $x \in \bar{\sigma}^k$. We now complete the definition of $f'$ on all $k$-simplexes $\sigma^k$ of $P$ by putting $f' = f$ on $\bar{\sigma}^k$ if $\sigma^k \subset P_1$ or $\bar{\sigma}^k \cap C = \emptyset$, and so obtain a map $f'$ on the $k$-skeleton of $P$ which satisfies (i), (ii), and $\bar{d}(f, f') < \varepsilon_k$. At the $p$th stage we have constructed thus a map $f'$ with the desired properties.

Many results in [9], [10] and [11] concerning $n$-valued multifunctions have been obtained with the help of the "Splitting Lemma". We say that a multifunction $\varphi$: $X \to Y$ *splits into n distinct maps* if $\varphi(x) = \{f_1(x), f_2(x), \ldots, f_n(x)\}$ for all $x \in X$, where $f_i$: $X \to Y$ are maps with $f_i(x) \neq f_j(x)$ for all $x \in X$, $i, j \in \{1, 2, \ldots, n\}$ and $i \neq j$. We write $\varphi = \{f_1, f_2, \ldots, f_n\}$, and call it simply a *splitting* of $\varphi$, as splittings into nondistinct maps are not used in this paper. The following lemma was proved in [9], Lemma 1, but is essentially due to S. Banach and S. Mazur [1], Satz 1.

LEMMA 2.4. (Splitting Lemma). *Let $X$ and $Y$ be compact Hausdorff. If $X$ is path-connected and simply connected and $\varphi$: $X \to Y$ is $n$-valued and continuous, then $\varphi$ splits into $n$ distinct maps.*

The Splitting Lemma is used here e.g. in the case $X = I$ and $X = I \times I$ to show that $n$-paths $\alpha$: $I \to M$ and homotopies of $n$-paths have splittings.

**3. Special homotopies of $n$-valued multifunctions.** Let $\varphi_0, \varphi_1$: $A \to X$ be $n$-valued continuous multifunctions from a subspace $A$ of $X$ into $X$. We write $\varphi = \{\varphi_t\}_{t \in I}$ for a homotopy $\Phi$: $A \times I \to X$ from $\varphi_0$ to $\varphi_1$, and call $\Phi$ a *special homotopy* if $\operatorname{Fix} \varphi_t = \operatorname{Fix} \varphi_0$ for all $t \in I$ and $\varphi_t(x) = \varphi_0(x)$ for all $x \in \operatorname{Fix} \varphi_0$ and $t \in I$. Two $n$-valued continuous multifunctions $\varphi_0, \varphi_1$: $A \to X$ which have the same fixed point set are called *specially homotopic* if there exists a special homotopy from $\varphi_0$ to $\varphi_1$.

LEMMA 3.1. (Special homotopy extension property for $n$-valued multifunctions). *Let $A$ be a subspace of $X$ and let $A$ and $X$ be* ANR's. *If $\varphi_0$: $X \to X$ is an $n$-valued continuous multifunction and $\Phi_A$: $A \times I \to X$ is a special homotopy of $\varphi_0|A$, then $\Phi_A$ can be extended to a special homotopy $\Phi$: $X \times I \to X$ of $\varphi_0$.*

This lemma is equal to [4], Lemma 2.1 if $n = 1$, and its proof is analogous.

Special homotopies will be used in the proof of Lemma 3.3. If $n = 1$, Lemma 3.3 states the well-known fact that two isolated fixed points of a proximity map can be united. The proof of the case $n \geqslant 2$ needs a slightly sharpened version of the case $n = 1$ which is given in Lemma 3.2. Two maps $f_0, f_1$: $A \to X$ are $\gamma$-homotopic, for $\gamma > 0$, if there exists a homotopy $\{f_t\}_{t \in I}$ from $f_0$ to $f_1$ so that $\bar{d}(f_t, f_{t'}) < \gamma$ for all $t, t' \in I$.

LEMMA 3.2. *Let $M$ be a compact manifold of dimension $\geqslant 2$, let $Q$ be an arc in* $\operatorname{Int} M$ *from $x_1$ to $x_2$ and $N$ a closed neighbourhood of $Q$. Given $\gamma > 0$, there exists an $\varepsilon > 0$ such that if $f$: $N \to M$ is a map with the properties*

(i) $\operatorname{Fix} f \cap Q = \{x_1, x_2\}$,

(ii) $x_1$ *and* $x_2$ *are isolated fixed points of* $f$,

(iii) $d(x, f(x)) < \varepsilon$ *for all* $x \in Q$,

*then f is* $\gamma$-*homotopic relative* $N - \operatorname{Int} N$ *to a map* $f'\colon N \to M$ *with* $\operatorname{Fix} f' = \operatorname{Fix} f - \{x_1\}$.

Proof. We select $\delta > 0$ so that if $f, f'\colon N \to M$ are two maps with $\bar{d}(f, f') < \delta$ and $f = f'$ on a subset $A$ of $N$, then $f$ is $\gamma$-homotopic to $f'$ relative $A$. (See Corollary 4, Ch. III A, p. 40 in [3] and its proof.) We can also choose $\varepsilon > 0$ so that $d(x, f(x)) < \varepsilon$ for all $x \in Q$ implies that there exist closed tubular neighbourhoods $U$ and $V$ of $Q$ in $N$ with $f(U) \subset V$. We consider $U$ and $V$ as the cones of their boundaries from $x_2$, and define a map $g\colon (U, x_2) \to (V, x_2)$ by

$$g(tx_2 + (1-t)x) = tx_2 + (1-t)f(x) \quad \text{for all} \quad x \in \operatorname{Bd} U \text{ and } 0 \leqslant t \leqslant 1 .$$

Then $\operatorname{Fix} g = \{x_2\}$, so $g$ extends to the map $f'\colon N \to M$ with $\operatorname{Fix} f' = \operatorname{Fix} f - \{x_1\}$ given by

$$f'(x) = \begin{cases} g(x) & \text{for} \quad x \in U, \\ f(x) & \text{for} \quad x \in N - U. \end{cases}$$

It is easy to see that it is possible to choose $\varepsilon > 0$ and $U$ so small that $d(x, f(x)) < \varepsilon$ for all $x \in Q$ implies $\bar{d}(f, f') < \delta$, hence $f$ is $\gamma$-homotopic to $f'$ relative $N - U \supset N - \operatorname{Int} N$.

The proof of the next lemma uses the *gap* $\gamma(\varphi)$ of an $n$-valued continuous multifunction $\varphi\colon X \to M$. It was defined in [9], § 3 as

$$\gamma(\varphi) = \inf\{d(y_i, y_j) |\ y_i, y_j \in \varphi(x),\ x \in X,\ y_i \neq y_j\} .$$

Hence $\gamma(\varphi) > 0$ if $X$ is compact.

Lemma 3.3. *Let $M$ be a compact manifold of dimension* $\geqslant 2$, *let* $x_1, x_2 \in \operatorname{Int} M$ *be two isolated fixed points of an $n$-valued continuous multifunction* $\varphi\colon M \to M$ *and let $Q$ be an arc in* $\operatorname{Int} M$ *from* $x_1$ *to* $x_2$ *with* $\operatorname{Fix} \varphi \cap Q = \{x_1, x_2\}$. *Then there exists an* $\varepsilon > 0$ *so that if* $\varphi|Q$ *is specially homotopic to an $n$-valued continuous multifunction* $\psi = \{g_1, g_2, \ldots, g_n\}\colon Q \to M$ *with* $x_1, x_2 \in \operatorname{Fix} g_1$ *and* $d(x, g_1(x)) < \varepsilon$ *for all* $x \in Q$, *then $\varphi$ is homotopic to an $n$-valued continuous multifunction* $\varphi'\colon M \to M$ *with* $\operatorname{Fix} \varphi' = \operatorname{Fix} \varphi - \{x_1\}$.

Proof. It follows from Lemma 3.1 (with $X = M$ and $A = Q$) that $\varphi$ is specially homotopic to an $n$-valued continuous multifunction $\varphi''\colon M \to M$ with $\varphi''|Q = \psi$. Let $N$ be a closed tubular neighbourhood of $Q$ with $\operatorname{Fix} \varphi'' \cap N = \{x_1, x_2\}$. Then $\varphi''|N$ splits according to the Splitting Lemma 2.4. We index the splitting $\varphi''|N = \{f_1, f_2, \ldots, f_n\}$ so that $f_1|Q = g_1$, and use Lemma 3.2 to choose $\varepsilon > 0$ so that $f_1$ is $\gamma(\varphi'')$-homotopic relative $\operatorname{Bd} N$ to a map $f_1'\colon N \to M$ with $\operatorname{Fix} f_1' = \operatorname{Fix} f_1 - \{x_1\}$. It we define $\varphi'\colon M \to M$ by

$$\varphi'(x) = \begin{cases} \{f_1'(x), f_2(x), \ldots, f_n(x)\} & \text{for} \quad x \in N, \\ \varphi''(x) & \text{for} \quad x \in M - N, \end{cases}$$

then $\varphi'$ has the properties described in Lemma 3.3.



**4. Special homotopies of $n$-paths.** Let $Q = q(I)$, for $q\colon I \to M$, be an arc in $M$ and $\alpha\colon I \to M$ an $n$-valued continuous multifunction. We say that $\alpha$ is an *$n$-path from* $x_1$ *to* $x_2$ if $\alpha = \{a_1, a_2, \ldots, a_n\}$ can be indexed so that $a_1(0) = x_1$ and $a_1(1) = x_2$, and call $\alpha$ *special* (with respect to $q$) if $\alpha = \{a_1, a_2, \ldots, a_n\}$ can be indexed so that

$$a_1(0) = q(0), \quad a_1(1) = q(1)$$

and

$$a_i(s) \neq q(s) \quad \text{for} \quad i = 1, 0 < s < 1 \text{ and } i \neq 1, 0 \leqslant s \leqslant 1 .$$

Homotopy of $n$-paths is always understood to be relative to their endpoints, i.e. two $n$-paths $\alpha_0, \alpha_1\colon I \to M$ are called homotopic if there exists an $n$-valued continuous multifunction $\{\alpha_t\}_{t \in I}\colon I \times I \to M$ so that $\alpha_t(0) = \alpha_0(0)$ and $\alpha_t(1) = \alpha_1(1)$ for all $t \in I$. Two special $n$-paths $\alpha_0, \alpha_1\colon I \times I \to M$ are called *specially homotopic* if there exists a homotopy $\{\alpha_t\}_{t \in I}\colon I \to M$ so that every $n$-path $\alpha_t\colon I \to M$ is a special $n$-path. Hence an $n$-valued continuous multifunction $\{H_1, H_2, \ldots, H_n\}$ is a special homotopy if and only if it can be indexed so that

$$\begin{aligned} & H_1(0, t) = q(0),\ H_1(1, t) = q(1) \quad \text{for} \quad 0 \leqslant t \leqslant 1, \\ (4.1) \quad & H_i(s, t) \neq q(s) \quad \text{for} \quad i = 1, 0 < s < 1, 0 \leqslant t \leqslant 1 \text{ and } i \neq 1, 0 \leqslant s, t \leqslant 1, \\ & H_i(0, t) = H_i(0, 0),\ H_i(1, t) = H_i(1, 0) \quad \text{for} \quad i = 2, 3, \ldots, n . \end{aligned}$$

Special homotopies of $n$-valued multifunctions and of $n$-paths are related as follows.

Lemma 4.2. *Let* $Q = q(I)$ *be an arc in $M$ from* $x_1$ *to* $x_2$, *let* $\varphi\colon M \to M$ *be an $n$-valued continuous multifunction with* $\operatorname{Fix} \varphi \cap Q = \{x_1, x_2\}$ *and let* $\alpha\colon I \to M$ *be an $n$-path from* $x_1$ *to* $x_2$. *Then the $n$-valued continuous multifunction* $\varphi|Q\colon Q \to M$ *is specially homotopic to the $n$-valued continuous multifunction* $\alpha \circ q^{-1}\colon Q \to M$ *if and only if the $n$-path* $\alpha\colon I \to M$ *is specially homotopic to the $n$-path* $\varphi \circ q\colon I \to M$.

Proof. If $q \times 1\colon I \times I \to Q \times I$ is given by $(q \times 1)(s, t) = (q(s), t)$ and $\Phi\colon I \times I \to M$ is a special homotopy from $\varphi|Q$ to $\alpha \circ q^{-1}$, then $\Phi \circ (q \times 1)\colon I \times I \to M$ is a special homotopy from $\alpha$ to $\varphi \circ q$. The converse is proved similarly, using $(q \times 1)^{-1}\colon Q \times I \to I \times I$.

The next lemma, which is crucial for the proof of Theorem 5.2, is modelled on [4], Lemma 5.1, but its proof is much simpler for manifolds.

Lemma 4.3. *Let $M$ be a compact manifold of dimension* $\geqslant 3$. *If two $n$-paths* $\alpha_0, \alpha_1\colon I \to \operatorname{Int} M$ *are special with respect to an arc* $q\colon I \to \operatorname{Int} M$ *and are homotopic, then they are specially homotopic with respect to $q$.*

Proof. Let $\{\alpha_t\}_{t \in I} = \{H_1, H_2, \ldots, H_n\}$ be the given homotopy, indexed so that (4.1) is satisfied. Using a collaring argument if $\operatorname{Bd} M \neq \emptyset$ we can assume that $H_i(s, t) \in \operatorname{Int} M$ for all $i = 1, 2, \ldots, n$, $0 \leqslant s, t \leqslant 1$. We write $\gamma$ for the gap of $\{\alpha_t\}_{t \in I}$ and $\bar{q}\colon I \times I \to \operatorname{Int} M$ for the map defined by $\bar{q}(s, t) = q(s)$ for all $(s, t) \in I \times I$.

We can use the Coincidence Lemma 2.2, with $P = I \times I = \{(s, t) |\ 0 \leqslant s, t \leqslant 1\}$, $P_1 = \operatorname{Bd} P$, $P_0 = (\operatorname{Bd} I) \times I$ and with $H_1$ and $\bar{q}_1$ instead of $f$ and $g$, to change $H_1$ to $H_1'$ with $H_1'(s, t) = H_1(s, t)$ for all $(s, t) \in (\operatorname{Bd} I) \times I$, $H_1'(s, t) \neq \bar{q}(s, t)$ for all

$(s,t) \in (\mathrm{Int}\, I)\times I$ and $\bar{d}(H_1'; \bar{q}) < \frac{1}{2}\gamma$. Then we use the Coincidence Lemma 2.2 with $P = I\times I$, $P_1 = (\mathrm{Bd}\, I)\times I$, $P_0 = \emptyset$ and $H_i$ $(i \geqslant 2)$, $q$ instead of $f$, $g$ to change $H_i$ for $i \geqslant 2$ to $H_i'$ with $H_i'(s,t) = H_i(s,t)$ for all $(\mathrm{Bd}\, I)\times I$, $H_i'(s,t) \neq \bar{q}(s,t)$ for all $(s,t) \in I\times I$ and $\bar{d}(H_i', \bar{q}) < \frac{1}{2}\gamma$. The homotopy $\{\alpha_t'\}_{t\in I} = \{H_1', H_2', \ldots, H_n'\}$ is a special homotopy from $\alpha_0$ to $\alpha_1$.

**5. Proof of the Minimum Theorem.** The proof of the Minimum Theorem for maps needs three steps: an approximation by a fix-finite map, the uniting of two fixed points in the same fixed point class and (if necessary) the removal of an isolated fixed point of index zero. For $n$-valued continuous multifunctions, the first step was done in [9], and the second will be described in the proof of Theorem 5.2. We shall now deal with the third one. The index $\mathrm{ind}(\varphi, x)$ of an isolated fixed point $x$ of the $n$-valued continuous multifunction $\varphi\colon M \to M$, where $x$ lies in a maximal simplex of $M$, was defined in [11], § 3. If $\varphi|\bar{\sigma} = \{f_1, f_2, \ldots, f_n\}$ and $f_1(x) = x$, then $\mathrm{ind}(\varphi, x) = \mathrm{ind}(f_1, x)$, where $\mathrm{ind}(f_1, x)$ is the ordinary fixed point index of the map $f_1$ at $x$ [3], p. 122.

LEMMA 5.1. (Fixed points of index zero). *Let $\sigma$ be a maximal simplex of the compact polyhedron $|K|$ and let $x \in \sigma$ be an isolated fixed point of index zero of the $n$-valued continuous multifunction $\varphi\colon |K| \to |K|$. Then $\varphi$ is homotopic to an $n$-valued continuous multifunction $\varphi'\colon |K| \to |K|$ so that* $\mathrm{Fix}\,\varphi' = \mathrm{Fix}\,\varphi - \{x\}$.

Proof. Let $U \subset \sigma$ be a closed Euclidean neighbourhood of $x$ with $U \cap \mathrm{Fix}\,\varphi = \{x\}$. We index $\varphi|U = \{f_1, f_2, \ldots, f_n\}$ so that $f_1(x) = x$, hence $\mathrm{ind}(f_1, x) = 0$. Then we select $\delta > 0$ so that the two maps $f, f'\colon U \to K$ with $\bar{d}(f, f') < \gamma$ and $f = f'$ on $\mathrm{Bd}\, U$ are $\gamma(\varphi)$-homotopic relative $\mathrm{Bd}\, U$, where $\gamma(\varphi) > 0$ is the gap of $\varphi$. (See the proof of Corollary 4, Ch. III A, p. 40 in [3].) It follows from [3], Theorem 4, Ch. VIII B, p. 123, that there exists a map $f_1'\colon U \to K$ with $\bar{d}(f_1, f_1') < \delta$, $f_1' = f_1$ on $\mathrm{Bd}\, U$ and $\mathrm{Fix}\, f_1' \cap U = \emptyset$. If we define $\varphi'\colon |K| \to |K|$ by

$$\varphi'(x) = \begin{cases} \{f_1'(x), f_2(x), \ldots, f_n(x)\} & \text{for} \quad x \in U, \\ \varphi(x) & \text{for} \quad x \in |K| - U, \end{cases}$$

then the $\gamma(\varphi)$-homotopy from $f_1$ to $f_1'$ induces a homotopy from $\varphi$ to the $n$-valued continuous multifunction $\varphi'$ with $\mathrm{Fix}\,\varphi' = \mathrm{Fix}\,\varphi - \{x\}$.

We are finally ready to prove the Minimum Theorem for $n$-valued continuous multifunctions.

THEOREM 5.2. (Minimum Theorem). *Let $M$ be a compact triangulable manifold (with or without boundary) of dimension $\geqslant 3$. Then every $n$-valued continuous multifunction $\varphi\colon M \to M$ is homotopic to an $n$-valued continuous multifunction $\varphi'\colon M \to M$ which has $N(\varphi)$ fixed points.*

Proof. According to [9], Theorem 6 and [11], Lemma 4.1 we can assume that $\varphi$ is fix-finite and that all its fixed points are isolated and lie in maximal simplexes. If $\mathrm{Bd}\, M \neq \emptyset$, then a collaring argument allows us to assume that $\varphi(M) \subset \mathrm{Int}\, M$. In view of Lemma 5.1 it is therefore sufficient to show that if $x_1$ and $x_2$ are two isolated fixed points of an $n$-valued continuous multifunction $\varphi\colon M \to \mathrm{Int}\, M$ which belong to the same fixed point class, then $\varphi$ is homotopic to an $n$-valued continuous multifunction $\varphi'\colon M \to \mathrm{Int}\, M$ with $\mathrm{Fix}\,\varphi' = \mathrm{Fix}\,\varphi - \{x_1\}$.



So let $x_1$ and $x_2$ be two isolated fixed points in the same fixed point class of $\varphi\colon M \to \mathrm{Int}\, M$. As in [11], § 5 this means that there exists a path $q\colon I \to M$ from $x_1$ to $x_2$ so that $\varphi \circ q = \{a_1, a_2, \ldots, a_n\}$, where $a_1(0) = x_1$, $a_1(1) = x_2$ and $a_1$ is homotopic to $q$ relative $\{0, 1\}$, and according to [11], Lemma 6.1 we can assume that $q(I) = Q$ is an arc in $\mathrm{Int}\, M$ with $\mathrm{Fix}\,\varphi \cap Q = \{x_1, x_2\}$. Let $\eta > 0$ be chosen so that $\bar{d}(q, a_i) > \eta$ for $i = 2, 3, \ldots, n$, and let $\gamma(\varphi) > 0$ be the gap of $\varphi$.

We determine $\varepsilon > 0$ as in Lemma 3.3, and define a path $p_\varepsilon\colon I \to \mathrm{Int}\, M$ by

$$p_\varepsilon(s) = q(s - \delta \sin \pi s), \quad 0 \leqslant s \leqslant 1,$$

where $\delta = \delta(\varepsilon) > 0$ is selected so that $\bar{d}(p_\varepsilon, q) < \min(\varepsilon, \eta)$. Then $\bar{d}(p_\varepsilon, a_i) > 0$ for $i = 2, 3, \ldots, n$, and therefore $\alpha_\varepsilon = \{p_\varepsilon, a_2, \ldots, a_n\}\colon I \to \mathrm{Int}\, M$ is an $n$-path.

As $a_1$ is homotopic to $q$, it is homotopic to $p_\varepsilon$ relative $\{0, 1\}$, and we can assume that this homotopy is of the form $H_1\colon I\times I \to \mathrm{Int}\, M$. We define $H_i\colon I\times I \to \mathrm{Int}\, M$, for $i = 2, 3, \ldots, n$, by $H_i(s,t) = a_i(s)$ for all $(s,t) \in I\times I$. As $H_1(s,t) \neq H_i(s,t)$ for $s = 0, 1$ or $t = 0$, we can use the Coincidence Lemma 2.2, with $P = I\times I = \{(s,t)|\ 0 \leqslant s, t \leqslant 1\}$, $P_1 = \{(s,t)|\ s = 0, 1 \text{ or } t = 0\}$, $P_0 = \emptyset$ and with $H_i$, $H_1$ instead of $f$ and $g$, to change $H_i$ to $H_i'\colon I\times I \to \mathrm{Int}\, M$ so that for $i = 2, 3, \ldots, n$

$$H_i'(s,t) = H_i(s,t) \quad \text{for } s = 0, 1 \text{ or } t = 0,$$

$$H_i'(s,t) \neq H_1(s,t) \quad \text{for all } (s,t) \in I\times I,$$

$$\bar{d}(H_i, H_i') < \min(\tfrac{1}{2}\gamma(\varphi), \eta).$$

If $i, j = 2, 3, \ldots, n$ and $i \neq j$, then

$$\bar{d}(H_i', H_j') \geqslant \bar{d}(H_i, H_j) - \bar{d}(H_i, H_i') - \bar{d}(H_j, H_j') > 0,$$

so $\psi = \{H_1, H_2', \ldots, H_n'\}$ is $n$-valued. If we define $a_i'\colon I \to \mathrm{Int}\, M$ by $a_i'(s) = H_i'(s, 1)$, then $\psi$ is a homotopy from the $n$-path $\varphi \circ q$ to the $n$-path $\alpha_\varepsilon' = \{p_\varepsilon, a_2', \ldots, a_n'\}$.

As the paths $\varphi \circ q$ and $\alpha_\varepsilon'$ are special, it follows from Lemma 4.3 that they are specially homotopic, hence Lemma 4.2 shows that $\varphi|Q$ is specially homotopic to $\psi = \alpha_\varepsilon' \circ q^{-1}\colon Q \to M$. But $\bar{d}(p_\varepsilon, q) < \varepsilon$, so we see from Lemma 3.3 that $\varphi$ is homotopic to an $n$-valued continuous multifunction $\varphi''\colon M \to M$ with $\mathrm{Fix}\,\varphi'' = \mathrm{Fix}\,\varphi - \{x_1\}$. Using a collaring argument we can obtain $\varphi'$ as desired.

Remark 5.3. An inspection shows how the arguments leading to the proof of the Minimum Theorem 5.2 simplify if $n = 1$, i.e. if $\varphi$ is a map. In this case Lemma 5.1 is well known and Lemmas 3.1 and 4.2 are contained in [4], Lemma 2.1 and beginning of § 4. Therefore all that is required to prove the Minimum Theorem for selfmaps of compact triangulable manifolds are Lemmas 2.1 and 2.2, slightly shortened proofs of Lemmas 3.2 and 3.3, the first half of the proof of Lemma 4.3 and the beginning and end of the proof of Theorem 5.2.

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# Stratégies gagnantes dans certains jeux topologiques

par

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**Abstract.** We prove that on an $\alpha$-favorable space for the Banach–Mazur game, there exists always an $\alpha$-winning strategy depending only on $\alpha$ and $\beta$ last moves. We give an example of a completely regular $\alpha$-favorable space on which the player $\alpha$ has no winning strategy depending only on $\beta$ last move.

**Introduction.** Rappelons que le *jeu de Banach–Mazur* sur un espace topologique $(X, \mathcal{T})$ est un jeu infini où deux joueurs $\alpha$ et $\beta$ choisissent alternativement à chaque coup, un ouvert non vide contenu dans l'ouvert choisi par l'autre joueur au coup précédent; c'est le joueur $\beta$ qui commence à jouer. Ainsi au cours d'une partie les joueurs $\alpha$ et $\beta$ construisent deux suites d'ouverts non vides $(V_n)_{n\in N}$ et $(U_n)_{n\in N}$ respectivement, avec $V_n \supset U_n \supset V_{n+1}$; le joueur $\alpha$ gagne la partie si $\bigcap_{n\in N} U_n = \bigcap_{n\in N} V_n \neq \emptyset$. Le jeu (ou l'espace $X$) est dit $\alpha$-*favorable* si le joueur $\alpha$ possède une stratégie gagnante. L'interêt des espaces $\alpha$-favorables tient au fait qu'ils forment une large classe d'espaces de Baire stable par produit et qui contient tous les cas classiques.

La notion de stratégie est utilisée ici au sens des jeux à information parfaite, c'est-à-dire qu'à chaque coup les joueurs sont informés de tous les coups précédemment joués et un joueur peut tenir compte de ces informations dans la construction d'une statégie. Le but de ce travail est d'étudier pour un jeu $\alpha$-favorable donné, l'existence de stratégies simples: plus précisément on s'intéressera à trois types de stratégies:

(I) Les stratégies $\sigma$ dépendant seulement du dernier coup joué (par le joueur $\beta$), c'est-à-dire de la forme $\sigma(V_0, U_0, V_1, \ldots, U_{n-1}, V_n) = \tau(V_n)$.

(II) Les stratégies $\sigma$ dépendant seulement des deux derniers coups joués (les derniers coups joués par les joueurs $\alpha$ et $\beta$ respectivement), c'est-à-dire de la forme: $\sigma(V_0, U_0, V_1, \ldots, U_{n-1}, V_n) = \tau(U_{n-1}, V_n)$.

(III) Les stratégies $\sigma$ dépendant seulement des deux derniers coups joués par le joueur $\beta$, c'est-à-dire de la forme $\sigma(V_0, U_0, V_1, \ldots, V_{n-1}, U_{n-1}, V_n) = \tau(V_{n-1}, V_n)$.