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Received 22 December 1983;
in revised form 19 April 1984

On rest points of dynamical systems

by

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Abstract. For a dynamical system π in an ENR-space X we define the index of rest points $I(\pi, U)$ in a relatively compact open set U as $\lim_{t\to 0^+} \operatorname{ind}(\pi_t, U)$. We prove, that if K is a compact set such that any positive semitrajectory intersects K then $I(\pi, \operatorname{int} K)$ is equal to the Euler characteristics $\chi(X)$. If X is a separable 2-manifold having the Betti numbers finite and any trajectory intersects K then $I(\pi, \operatorname{int} K) \geqslant \chi(X)$.

In the present paper we apply the theory of isolated invariant sets and the fixed point theory to prove some results concerning rest points of dynamical systems. The theory of isolated invariant sets was introduced and developed by C. C. Conley, R. W. Easton, R. C. Churchill and others, but the main idea comes from paper [13] of T. Ważewski. We base ourselves on paper [2], in which there are references and historical remarks concerning this theory. We use the fixed point index constructed by A. Dold and presented in [3].

In Section 1 we present the basic facts in the theory of dynamical systems and isolated invariant sets. Section 2 is devoted to asymptotically stable sets. The main result of this section is Theorem 2.4. It presents a sufficient condition for the existence of an asymptotically stable set in a space X, such that X is its region of attraction. In Section 3 we present the notion of an ENR and we prove that for dynamical systems in 2-manifolds there are blocks which are ENR's (Theorem 3.3). The index of a rest point is defined in Section 4. We compute the index in the interior of a block (Theorem 4.4). In Section 5 we assume that a dynamical system is generated by a C^1 vector field v and we compare the index with the degree of v. In Section 6 we apply the properties of the index to prove some results concerning the existence of trajectories and rest points in a given set.

The author expresses his gratitude to Professors A. Pelczar and S. Sędziwy for their valuable remarks during the preparation of this paper.

1. Preliminaries. In this paper X denotes a topological locally compact space satisfying the second axiom of countability. We say that

$$\pi\colon \boldsymbol{R}\times X \to X$$

is a dynamical system in X if π is continuous, $\pi(0,x)=x$ and $\pi(s+t,x) = \pi(s,\pi(t,x))$ for any $x\in X$ and $s,t\in\mathbf{R}$. Sometimes we will write $\pi_t(x)$ instead of $\pi(t,x)$.

For $A\subset X$ and $J\subset\mathbf{R}$ we define

$$\pi(J,A)=\{\pi(t,x):\ t\in J,\ x\in A\}.$$

We use the notation:

$$\pi(A)=\pi(\mathbf{R},A),$$
$$\pi^+(A)=\pi([0,\infty),A),$$
$$\pi^-(A)=\pi((-\infty,0],A).$$

If $A=\{x\}$ we omit the brackets, i.e., $\pi(x)=\pi(\{x\})$, etc.

A point $x\in X$ is called a rest point if $\pi(x)=\{x\}$.

If π is a dynamical system in X, $\hat{X}=X\cup\{\infty\}$ is the one-point compactification of X. By $\hat{\pi}$ we will denote the extension of π defined by:

$$\hat{\pi}=\pi \text{ in } \mathbf{R}\times X,\qquad \hat{\pi}(t,\infty)=\infty \text{ for any } t\in\mathbf{R}.$$

$\hat{\pi}$ is a dynamical system in $\hat{X}$.

Set

$$A^+(x)=\{y\in X:\ \exists t_n,\ t_n\to\infty, \text{ such that } \pi(t_n,x)\to y\},$$

$$A^-(x)=\{y\in X:\ \exists t_n,\ t_n\to-\infty \text{ such that } \pi(t_n,x)\to y\}.$$

A set S is called invariant if $\pi(S)=S$. An invariant set is called isolated if it is compact and is the maximal invariant set in some neighbourhood of itself.

We say that $\Sigma\subseteq X$ is a section if there exists a $\delta>0$ such that $\pi|_{(-\delta,\delta)\times\Sigma}$ is a homeomorphism with an open range.

Let B be a compact set and let Σ^+ and Σ^- be sections with disjoint closures. Let $\delta>0$ be such that

$$\pi((-\delta,\delta),\Sigma^+)\cap\pi((-\delta,\delta),\Sigma^-)=\emptyset.$$

B is called a block if

(a) $(\mathrm{cl}\Sigma^\pm\setminus\Sigma^\pm)\cap B=\emptyset$,

(b) $\pi((-\delta,\delta),\Sigma^+)\cap B=\pi([0,\delta),\Sigma^+\cap B)$,
$\pi((-\delta,\delta),\Sigma^-)\cap B=\pi((-\delta,0],\Sigma^-\cap B)$,

(c) for each $x\in\partial B\setminus(\Sigma^+\cup\Sigma^-)$ there are real numbers $\varepsilon_1,\varepsilon_2$, $\varepsilon_1<0<\varepsilon_2$ such that $\pi(\varepsilon_1,x)\in\Sigma^+$, $\pi(\varepsilon_2,x)\in\Sigma^-$ and

$$\pi([\varepsilon_1,\varepsilon_2],x)\subset\partial B.$$

Sections $\Sigma^\pm$ which fulfil the conditions from the definition of the block B are called associated with B.

Let B be a block. Set

$$b^+=\Sigma^+\cap\partial B,\quad b^-=\Sigma^-\cap\partial B,$$
$$A^+=\{x\in B:\ \pi^+(x)\subset B\},\quad A^-=\{x\in B:\ \pi^-(x)\subset B\},$$
$$a^+=A^+\cap b^+,\quad a^-=A^-\cap b^-,$$
$$A=A^+\cup A^-.$$

The set $S(B)=A^+\cap A^-$ is the maximal invariant set in the interior of B, and so it is an isolated invariant set.

The proofs of the following two propositions are in [2], pp. 336–338.

Proposition 1.1. *The sets $b^\pm$, $A^\pm$, $a^\pm$ are compact and*

$$a^\pm\subset\mathrm{int}\,b^\pm(\mathrm{rel}\,\Sigma^\pm).$$

For $x\in B$ define

$$\sigma^+(x)=\begin{cases}\inf\{t\leqslant 0:\ \pi([t,0],x)\cap\Sigma^+=\emptyset\} & \text{for } x\in B\setminus\Sigma^+,\\ 0 \quad \text{for } x\in\Sigma^+,\end{cases}$$

$$\sigma^-(x)=\begin{cases}\sup\{t\geqslant 0:\ \pi([0,t],x)\cap\Sigma^-=\emptyset\} & \text{for } x\in B\setminus\Sigma^-,\\ 0 \quad \text{for } x\in\Sigma^-.\end{cases}$$

Proposition 1.2. *The functions*

$$\sigma^+:\ B\to[-\infty,0],\qquad \sigma^-:\ B\to[0,\infty]$$

are continuous.

Proposition 1.3. *$x\in\partial b^+$ (rel Σ^+) if and only if*

$$\pi([0,\sigma^-(x)],x)\subset\partial B.$$

The same is valid if we interchange the signs $+$ and $-$.

Proof. The implication from the left to the right is proved in [2] (Prop. 3.7(c), p. 338). Assume that $x\in\mathrm{int}\,b^+(\mathrm{rel}\,\Sigma^+)$. Then $\pi(\delta/2,x)$ belongs to the open set $\pi((0,\delta),\mathrm{int}\,b^+(\mathrm{rel}\,\Sigma^+))\subset B$ (since Σ^+ is the local section and (b) is valid). Thus $\pi(\delta/2,x)\in\mathrm{int}\,B$, contradicting the assumption.

In the sequel we adopt the following convention. If we deal with several blocks B_j, $j=1,2,\ldots$, the corresponding sets and functions will be denoted $b_j^\pm$, $a_j^\pm$, $\sigma_j^\pm$, etc.

Proposition 1.4. *Let B be a block and $\Sigma^\pm$ be sections associated with B. Assume that $d\subset b^+$ is compact and $a^+\subset\mathrm{int}\,d(\mathrm{rel}\,\Sigma^+)$. Then*

$$B_1=B\setminus\bigcup_{x\in b^+\setminus d}\pi([0,\sigma^-(x)],x)$$

is also a block, $b_1^+=d$ and $S(B_1)=S(B)$.

Proof. Since the set $b^+\setminus\mathrm{int}\,d(\mathrm{rel}\,\Sigma^+)$ is compact and separated from a^+, the function σ^- restricted to this set is bounded from above by a finite constant Q.

The set $\{x \in B: \sigma^-(x) - \sigma^+(x) > Q\}$ is open, contains A and is contained in B_1. This implies $A \subset \operatorname{int} B_1(\operatorname{rel} B)$. The set B_1 is compact. Let $\{x_n\} \subset B_1$ and let $\lim_{n\to\infty} x_n = x_0$. We can assume that $x_0 \notin A$. Thus $\pi(\sigma^+(x_n), x_n) \in B_1 \cap b^+ = d$, $\pi(\sigma^+(x_0), x_0) \in d$ and $x_0 \in B_1$. It suffices to prove that the sections $\Sigma^\pm$ fulfil (a), (b), (c) for the set B_1. Conditions (a) and (b) follow simply from the analogous conditions for B. In order to prove (c) assume that $x \in \partial B_1 \setminus (\Sigma^+ \cup \Sigma^-)$. Observe that $\sigma^+(x) > -\infty$ and $\sigma^-(x) < \infty$ since $A \subset \operatorname{int} B_1(\operatorname{rel} B)$. We prove that

$$\pi([\sigma^+(x), \sigma^-(x)], x) \subset \partial B_1 .$$

If $x \in \partial B$, the assertion is obvious. Assume that there exists a sequence $\{x_n\} \subset B \setminus B_1$ such that $\lim_{n\to\infty} x_n = x$. Put

$$y_n = \pi(\sigma^+(x_n), x_n), \qquad y = \pi(\sigma^+(x), x) .$$

Let $t \in [0, \sigma^-(y)]$ be arbitrary. We have $y_n \in b^+ \setminus d$ and

$$\lim_{n\to\infty} \pi(\min(t, \sigma^-(y_n)), y_n) = \pi(t, y) ,$$

which implies that $\pi(t, y) \in \partial B_1$. This shows that

$$\pi([\sigma^+(x), \sigma^-(x)], x) = \pi([0, \sigma^-(y)], y) \subset \partial B_1 .$$

We will need the following theorem concerning isolated invariant sets.

THEOREM 1.5. *Let π be a dynamical system in X and $S \subset X$ be an isolated invariant set. For any U, an open neighbourhood of S, there exists a block $B \subset U$ such that $S = S(B)$.*

Proof. We extend the system π to $\hat{X}$, the one-point compactification of X. Since X is locally compact and fulfils the second axiom of countability, XI. 8.6 in [4] implies that $\hat{X}$ is compact and metrizable. In this case the theorem was proved in [2] (Th. 3.4).

2. Asymptotically stable sets. A nonempty compact invariant set S is called positively asymptotically stable (PAS) if:

(i) for each open neighbourhood U of S there is an open neighbourhood $V \subset U$ of S such that $\pi^+(V) \subset U$,

(ii) there is an open neighbourhood W of S such that $\Lambda^+(x) \subset S$ for any $x \in W$.

The maximal set W in (ii) is open and invariant. It is called the region of attraction of S. If we change the sign $+$ to $-$ in (i) and (ii), we obtain the definition of a negatively asymptotically stable (NAS) set. Note that asymptotically stable sets are isolated and invariant.

The following criterion for asymptotic stability has been proved in [2], p. 337.

THEOREM 2.1. *Let B be a block. $S(B)$ is PAS (NAS) if and only if $a^- = \emptyset$ (resp. $a^+ = \emptyset$).*

PROPOSITION 2.2. *If S is PAS then for any block B, $S = S(B)$ there exists a block B_1 such that $S = S(B_1)$, $\partial B_1 = b_1^+ = a_1^+$ and ∂B_1, $\emptyset$ are sections associated with B_1. An analogous result for NAS-sets is also valid.*

Proof. Let B be a block, $S = S(B)$. By the previous theorem $a^- = \emptyset$, and so

$$B_1 = B \setminus \bigcup_{x \in b^-} \pi([\sigma^+(x), 0], x) = A^+$$

is also a block (see Prop. 1.4). Thus $b_1^+ = a_1^+$, $b_1^- = \emptyset$ and, since $a_1^+ \subset \operatorname{int} b_1^+(\operatorname{rel} \Sigma^+)$ for any section Σ^+ associated with B_1 (Prop. 1.1), a_1^+ is open in Σ^+ and so a_1^+ is a section associated with B_1. The fact that $a_1^+ = \partial B_1$ is a trivial consequence of condition (c) for B_1.

PROPOSITION 2.3. *If a set S is PAS and U is its region of attraction, then any block B which fulfils the claim of Prop. 2.2 is a strong deformation retract of U.*

Proof. Let $\hat{U}$ be a one-point compactification of U. It is easy to see that $\{\infty\}$ is an isolated invariant set for $\hat{\pi}$ and $B_1 = \hat{U} \setminus \operatorname{int} B$ is a block, $b_1^- = \partial B_1$. Thus $\{\infty\}$ is NAS and the mapping

$$\tau\colon B_1 \setminus \{\infty\} \ni x \to \sigma_1^-(x) \in [0, \infty)$$

is continuous. The homotopy

$$H\colon X \times I \to X$$

($I = [0, 1]$) given by

$$H(t, x) = \begin{cases} x & \text{for } x \in B, \\ \pi(t\tau(x), x) & \text{otherwise} \end{cases}$$

defines a strong deformation retraction.

The main results of this section is the following:

THEOREM 2.4. *If K is a compact subset of X such that, for any $x \in X$, $\pi^+(x) \cap \cap K \neq \emptyset$, then there exists a PAS-set S such that X is its region of attraction. (By definition, S must be compact.)*

Proof. First observe that $\{\infty\}$ is an isolated invariant set for the system $\hat{\pi}$ in $\hat{X}$ since there are no invariant sets in $X \setminus K$. Let $B \subset \hat{X} \setminus K$ be a block, $\{\infty\} = S(B)$. The existence of such a block follows from Theorem 1.5. It is easy to see that the set a^+ is empty, and so $\{\infty\}$ is NAS. By Prop. 2.2, there exists a block B_1, $\{\infty\} = S(B_1)$, such that ∂B_1 is a section associated with B_1 and $a_1^- = \partial B_1$. We put

$$B_2 = X \setminus \operatorname{int} B_1 .$$

One can verify that B_2 is also a block for the system π and the set $S(B_2)$ satisfies the theorem.

3. Euclidean neighbourhood retracts. A topological space X is called a Euclidean neighbourhood retract (ENR) if there exists a positive integer n and $Y \subset \boldsymbol{R}^n$, Y being homeomorphic with X, such that there is an open set U, $Y \subset U \subset \boldsymbol{R}^n$ and Y is a retract of U. The class of ENR's coincides with the class of ANR's (absolute neighbourhood retracts) which are locally compact, metrizable, finite-dimensional and fulfil the second axiom of countability (see [6], Ch. V).

The following lemma is an immediate consequence of 6.1 (iv), p. 90 in [1]:

LEMMA 3.1. *If X is an* ENR *and Y is a closed subset of X such that ∂Y is an* ENR *then Y is an* ENR.

We prove two results concerning isolated invariant sets.

PROPOSITION 3.2. *Assume that X is an* ENR, *B is a block and $\Sigma^{\pm}$ are sections associated with B. If Σ^+, Σ^- and ∂b^+* (relΣ^+) *(or ∂b^-* (relΣ^-)) *are* ENR*'s then B, b^+ and b^- are* ENR*'s.*

Proof. By Lemma 3.1 it suffices to prove that ∂B is an ENR. Since $\partial b^+(\mathrm{rel}\Sigma^+)$ and Σ^+ are ENR's, b^+ is an ENR. The set

$$C = \{\pi(t, x)\colon x \in \partial b^+ \ (\mathrm{rel}\Sigma^+),\ t \in [0, \sigma^-(x)]\}$$

in an ENR since it is homeomorphic with $\partial b^+\ (\mathrm{rel}\Sigma^+)\times I$. The mapping

$$\partial b^+\ (\mathrm{rel}\Sigma^+) \ni x \to \pi(\sigma^-(x), x) \in b^-$$

is a homeomorphism onto ∂b^- (relΣ^-) since Proposition 1.3 is valid. Thus b^- is an ENR by Lemma 3.1. We have

$$\partial B = b^+ \cup b^- \cup C.$$

Since $b^+ \cap C$ and $b^- \cap C$ are ENR's, 2.9, p. 102 in [1] implies that ∂B is an ENR. This completes the proof.

THEOREM 3.3. *If X is a 2-dimensional manifold and S is an isolated invariant set, then for any open neighbourhood U of S there exists a block B, $B \subset U$ and $S = (SB)$ such that B is an* ENR *and b^+ and b^- are 1-dimensional manifolds with boundary.*

Proof. Let B be a block, $B \subset U$ and $S = S(B)$. The existence of such a bolck follows from Theorem 1.5. Assume that $\Sigma^{\pm}$ are sections associated with B and $\delta > 0$ is a number satisfying conditions (a), (b) and (c) in the definition of the block B. We assert that there are 1-dimensional manifolds $\Sigma_1^{\pm} \subset \Sigma^{\pm}$ such that $\Sigma_1^{\pm}$ are sections associated with B. We argue as follows. $\pi^* = \pi|_{(-\delta,\delta)\times\Sigma^+}$ is a homeomorphism with an open range and $b^+ \subset \pi((-\delta, \delta), \Sigma^+)$. Let $W_1, \ldots, W_r$ be connected components of $\pi((-\delta, \delta), \Sigma^+)$ such that $b^+ \subset \bigcup_{i=1}^{r} W_i$. W_i is open; thus there exists a compact and connected set K such that $b_i^+ \subset \operatorname{int} K \subset K \subset W_i$, where $b_i^+ = b^+ \cap W_i$, $i = 1, \ldots, r$. Put

$$\tilde{Z}_i = p(\pi^{*-1}(K)),$$

where $p\colon (-\delta, \delta)\times\Sigma^+ \to \Sigma^+$ is the projection. It is easy to see that $\tilde{Z}_i$ is compact and connected, i.e., it is a continuum and has more than two points. By Th. 1.6, p. 164 in [5], $\tilde{Z}_i$ is homeomorphic with S^1 or I. We put $Z_i = \tilde{Z}_i$ if $\tilde{Z}_i$ is homeomorphic to S^1 and $Z_i = h((0, 1))$ if $h\colon I \to \tilde{Z}_i$ is a homeomorphism. The set $\Sigma_1^+ = \bigcup_{i=1}^{r} Z_i$

is a section which is associated with B. Similarly we construct Σ_1^-. By Prop. 1.1 a^+ is compact and $a^+ \subset \operatorname{int} b^+$ (relΣ_1^+). We can find a 1-manifold with boundary b_0^+ such that b_0^+ is a compact subset of b^+ and $a^+ \subset \operatorname{int} b_0^+$ (relΣ_1^+). We put

$$B_0 = B \setminus \bigcup_{x \in b^+ \setminus b_0^+} \pi([0, \sigma^-(x)], x).$$

By Prop. 1.4 B_0 is a block. The claim of our theorem follows from Prop. 3.2.

4. The Index. Denote by G the class of all pairs (f, U), where $f\colon Z \to X$ is a continuous function, X is an ENR, $U \subset Z \subset X$ and U is open, such that the set

$$\operatorname{Fix}(f, U) = \{x \in U\colon f(x) = x\}$$

is compact.

Let (f, U) belong to G. By $\operatorname{ind}(f, U)$ we will mean the index of $f|_U$ (see [3], Ch. VII). It satisfies the following properties:

(i) If W is open $\operatorname{Fix}(f, U) \subset W \subset U$, then $\operatorname{ind}(f, U) = \operatorname{ind}(f, W)$.

(ii) If f is constant, then $\operatorname{ind}(f, U) = 1$ if $f(U) \in U$ and $\operatorname{ind}(f, U) = 0$ if $f(U) \notin U$.

(iii) If $U = \bigcup_{i=1}^{r} U_i$, U_i is open and $U_i \cap U_j \cap \operatorname{Fix}(f, U) = \emptyset$ for $i \neq j$, then $\operatorname{ind}(f, U) = \sum_{i=1}^{r} \operatorname{ind}(f, U_i)$.

(iv) If $f_t\colon Z \to X$ is a homotopy, $t \in I$, such that $\bigcup_{t \in I} \operatorname{Fix}(f_t, U)$ is compact, then $\operatorname{ind}(f_0, U) = \operatorname{ind}(f_1, U)$.

(v) If $U \subset X$, $U' \subset X'$, $f\colon U \to X'$, $f'\colon U' \to X$ are continuous, then

$$\operatorname{ind}(f \circ f', f'^{-1}(U)) = \operatorname{ind}(f' \circ f, f^{-1}(U'))$$

provided the sets $\operatorname{Fix}(f \circ f', f'^{-1}(U))$ and $\operatorname{Fix}(f' \circ f, f^{-1}(U))$ are compact.

Condition (v) implies the following result:

PROPOSITION 4.1. *Assume that $U \subset W \subset X$, X is an* ENR *and U, W are open. By the Theorem of Hanner, W is also an* ENR. *Let $f\colon U \to W$ be continuous and $i\colon W \to X$ be the inclusion. Then* $\operatorname{ind}(f, U) = \operatorname{ind}(i \circ f, U)$.

A consequence of (iv) is the following

PROPOSITION 4.2. *If $f_t\colon X \to X$ is a homotopy, X is an* ENR, *$U \subset X$ is open with compact closure and, for any $(x, t) \in \partial U \times I$, $f_t(x) \neq x$, then* $\operatorname{ind}(f_0, U) = \operatorname{ind}(f_1, U)$.

Let X be an ENR and π be a dynamical system in X. Assume that U is an open subset of X. If there are no rest points of π in ∂U and U is relatively compact, then $\operatorname{ind}(\pi_t, U)$ has a constant value for $0 < t < \varepsilon$, provided ε is small enough. In fact, the compactness of $\bar{U}$ implies that there is a constant $T > 0$ such that $\pi_t(x) \neq x$ for $x \in \partial U$, $t \in (0, T)$, which by Prop. 4.2 proves the remark.

The index $I(\pi, U)$ of the system π in the set U is defined as the limit

$$I(\pi, U) = \lim_{t\to 0^+} \operatorname{ind}(\pi_t, U)$$

provided that the limit exists.

From the observation made above it follows that the index is well defined for open sets having compact closure.

The index has the following properties:

Proposition 4.3. (1) *If U and W are open, $U \subset W$, $\overline{W}$ is compact and there are no rest points in $\overline{W}\setminus U$, then $I(\pi, W) = I(\pi, U)$.*

(2) *If X is compact, $U_1, \ldots, U_r$ are open, $U_i \cap U_j = \emptyset$ for $i \neq j$ and there are no rest points in $X \setminus \bigcup U_i$, then*

$$I(\pi, X) = \sum_{i=1}^{r} I(\pi, U_i) = \chi(X),$$

where χ denotes the Euler characteristic.

(3) *If there are no rest points in the set $\overline{U}$, then $I(\pi, U) = 0$.*

(4) *If π is generated by the equation $\dot{x} = Ax$, where A is a real nonsingular matrix and k denotes the number of its eigenvalues having the positive real parts, then for any open set U, $0 \in U$: $I(\pi, U) = (-1)^k$.*

Proof. Assertions (1) and (3) are obvious, (2) is a consequence of the Lefschetz Fixed Point Theorem ([3], VII. 6.22) and (4) follows from exercise VII. 5.17.3 in [3].

Now we compute the index of a dynamical system in the interior of a block. A similar result is obtained in [8].

Theorem 4.4. *Let B be a block such that B and b^- are* ENR's. *Then*

$$I(\pi, \operatorname{int} B) = \chi(B) - \chi(b^-).$$

Proof. The continuity of π and the compactness of $S(B)$ imply that there exists a number $T > 0$ and U, an open neighbourhood of $S(B)$, such that $\pi([0, T], U) \subset \operatorname{int} B$. Obviously, in $B \setminus U$ there are no stationary points, and thus:

$$I(\pi, \operatorname{int} B) = I(\pi, U) = \operatorname{ind}(\pi_t, U)$$

for $t \in (0, T)$. Now we describe the construction of a space $\tilde{B}$ and a function $\tilde{\pi}$. The details of this construction are presented in [12].

Let S^1 denote the set $\{\alpha \in \boldsymbol{C}\colon |\alpha| = 1\}$. In the topological direct sum of B and $b^- \times S^1$ we identify any point $x \in b^- \subset B$ with the point $(x, 1) \in b^- \times S^1$. We get the space $\tilde{B}$ with the quotient topology. The space is a compact ENR and, since $\chi(b^- \times S^1) = 0$, we have

$$\chi(\tilde{B}) = \chi(B) - \chi(b^-).$$

Let $\tilde{\pi}\colon [0, \infty) \times \tilde{B} \to \tilde{B}$ be the mapping

$$\tilde{\pi}(t, \tilde{x}) = \begin{cases} \pi(t, x) & \text{if } \tilde{x} = x,\ x \in B \text{ and } t \leqslant \sigma^-(x) \\ (\pi(\sigma^-(x), x),\ \exp 2\pi i(t - \sigma^-(x))) & \text{if } \tilde{x} = x,\ x \in B \text{ and } t \geqslant \sigma^-(x) \\ x, \alpha \exp 2\pi i t & \text{if } \tilde{x} = (x, \alpha) \in b^- \times S^1,\ t \geqslant 0. \end{cases}$$

Prop. 1.1 implies the continuity of $\tilde{\pi}$. For any $t \geqslant 0$ the mapping $\tilde{\pi}_t\colon \tilde{B} \to \tilde{B}$ is homotopic to the identity, and thus by the Lefschetz Fixed Point Theorem

$$\operatorname{ind}(\tilde{\pi}_t, \tilde{B}) = \chi(B) - \chi(b^-).$$

It is easy to check that if $t \in (0, 1)$ then $\operatorname{Fix}(\tilde{\pi}_t, \tilde{B}) \subset S(B)$. Choose $\varepsilon < \min(T, 1)$. By (I) and Prop. 4.1, we have for $t \in (0, \varepsilon)$

$$\operatorname{ind}(\pi_t, U) = \operatorname{ind}(\tilde{\pi}_t, U) = \operatorname{ind}(\tilde{\pi}_t, \tilde{B}).$$

The theorem is proved.

A space Y is said to be of *finite type* if Y has the homotopy type of a compact polytope. Any compact ENR is of finite type (see [14]).

Corollary 4.5. *If S is a* PAS-*set and U is its region of attraction, then U is of finite type and*

$$I(\pi, U) = \chi(U).$$

Proof. U is an ENR since it is an open subset X. By Prop. 2.2 and 2.3 there exists a block B, $S = S(B)$ and $b^- = \emptyset$, such that B is a strong deformation retract of U. Thus B is a compact ENR and has the homotopy type of U. Since $\chi(b^-) = 0$, the claim follows from Theorem 4.4.

5. The case of a smooth dynamical system. Let $f\colon \boldsymbol{R}^n \to \boldsymbol{R}^n$ be continuous and U an open and bounded subset of $\boldsymbol{R}^n$. For $p \notin f(\partial U)$ we define $\deg(p, f, U)$, the degree of f with respect to p and U, as follows (compare [10], Ch. III):

Let f be of class C^1. Denote by Z the set of $x \in U$ such that $j_x f$ (the Jacobian of f at x) vanishes. If the set $f^{-1}(p) \cap Z$ is empty, we put

$$\deg(p, f, U) = \sum_{f(x)=p} \operatorname{sgn} j_x f;$$

if it is not empty, then

$$\deg(p, f, U) = \lim_{q \to p,\, f^{-1}(q)\cap Z = \emptyset} \deg(q, f, U)$$

For f continuous we define

$$\deg(p, f, U) = \lim_{n\to\infty} \deg(p, f_n, U)$$

where $\{f_n\}$ is an arbitrary sequence of C^1-mappings converging uniformly to f.

Observe that if $g\colon \boldsymbol{R}^n \to \boldsymbol{R}^n$ is sufficiently close to f then $\deg(p, g, U) = \deg(p, f, U)$. A similar result is valid for the index of a continuous mapping in $\boldsymbol{R}^n$. More exactly, assume that $\operatorname{Fix}(f, \boldsymbol{R}^n) \cap \partial U = \emptyset$. If for any $x \in \partial U$

$$|g(x) - f(x)| < |f(x) - x|,$$

the homotopy f_t,

$$f_t(x) = tf(x) + (1-t)g(x), \quad t \in I,$$

has no fixed points in ∂U, and thus by Prop. 4.2

$$\operatorname{ind}(f, U) = \operatorname{ind}(g, U).$$

Assume that π is a dynamical system in R^n and a C^1-mapping $v\colon R^n \to R^n$ is the velocity vector field of π. Let $U \subset R^n$ be an open and bounded set, $v(x) \neq 0$ for $x \in \partial U$. The following result has also been obtained by M. Mrozek (unpublished).

THEOREM 5.1 *Under the assumptions stand above,*

$$I(\pi, U) = (-1)^n \deg(0, v, U)$$

Proof. Since v is of class C^1, $(\pi(t, x) - x - tv(x))/t \to 0$ if $t \to 0$, $t \neq 0$, uniformly in some neighbourhood of $\bar{U}$ (see [11], III. 5.3). Put

$$Q = \min\{|v(x)|\colon x \in \partial U\}.$$

There exists a $T > 0$ such that, for any t, $0 < t < T$ and $x \in \partial U$,

$$|\pi(t, x) - x - tv(x)| < tQ.$$

The observation made above implies that if $0 < t < T$ then

$$I(\pi, U) = \operatorname{ind}(\pi_t, U) = \operatorname{ind}(\mathrm{id} + tv, U).$$

Since $\mathrm{id} + v$ and $\mathrm{id} + tv$ are homotopic,

$$I(\pi, U) = \operatorname{ind}(\mathrm{id} + v, U).$$

By Sard's Theorem there exists a C^1 vector field w close to v such that 0 is its regular value and

$$\deg(0, w, U) = \deg(0, v, U), \quad \operatorname{ind}(\mathrm{id} + w, U) = \operatorname{ind}(\mathrm{id} + v, U).$$

By the definition of $\deg(0, w, U)$ and property (iii) of the index, in order to prove the theorem it suffices to assume that $0 \in U$, $w^{-1}(0) = \{0\}$ and $j_0 w \neq 0$.

We can find a real linear operator L having all eigenvalues with nonzero real parts, L close to the differential of w in 0, and an open neighbourhood V of 0, $V \subset U$, such that

$$\deg(0, L, V) = \deg(0, w, U), \quad \operatorname{ind}(\mathrm{id} + L, V) = \operatorname{ind}(\mathrm{id} + w, U).$$

Using a similar argument to that used at the beginning of this proof, we can assert that

$$\operatorname{ind}(\mathrm{id} + L, V) = I(\varrho, V)$$

where ϱ is the dynamical system generated by the equation $\dot{x} = Lx$. By Proposition 4.3 (4), $I(\varrho, V)$ is equal of $(-1)^k$, where k denotes the number of the eigenvalues of L with positive real part. Since $\deg(0, L, V) = (-1)^{n-k} = (-1)^n(-1)^k$, we obtain the theorem.

6. Applications of the index. As a consequence of Theorem 2.4 and Corollary 4.5 we obtain the following result:

THEOREM 6.1. *Let X be an ENR and π be a dynamical system in X. If there is a compact subset K of X such that, for every $x \in X$, $\pi^+(x) \cap K \neq \emptyset$, then:*

(1) *X is of finite type;*

(2) *K has a rest point provided that $\chi(X) \neq 0$;*

(3) *If there are no rest points in ∂K, then*

$$I(\pi, \operatorname{int} K) = \chi(X).$$

Before stating further results, we introduce the following definition.

Let $\check{H}^* = \{\check{H}^q\}$ be the Čech cohomology functor having Q as coefficients. Assume that, for a topological space Y, $\check{H}^q(Y)$ is finitely generated and is trivial for almost all q. We set

$$\check{\chi}(Y) = \sum_{q=0}^{\infty} (-1)^q \dim \check{H}^q(Y).$$

In the sequel we assume that M is a 2-dimensional topological manifold which fulfils the second axiom of countability and π is a dynamical system in M.

LEMMA 6.2. *If S is an isolated invariant set for π and $\check{\chi}(S)$ is defined, then, for any open $U \supset S$ such that S is the maximal invariant subset of U,*

$$I(\pi, U) \leqslant \check{\chi}(S).$$

Proof. By Theorem 3.3 there exists a block B, $S = S(B)$, such that B is a compact ENR and b^- is a compact 1-manifold with boundary. Following Churchill ([2], p. 340), there is an exact sequence

$$\ldots \to \check{H}^q(B, b^-) \to \check{H}^q(S) \to \check{H}^q(a^-) \to \check{H}^{q+1}(B, b^-) \to \ldots$$

The exactness implies that

$$\chi(B, b^-) + \check{\chi}(a^-) = \check{\chi}(S),$$

since the Čech and singular cohomologies of B and b^- are isomorphic. The set a^- is a compact subset of the 1-manifold b^-, and thus it is a disjoint union of several copies of S^1 and compact subsets of R. This implies that $\check{\chi}(a^-) \geqslant 0$. Since

$$\chi(B, b^-) = \chi(B) - \chi(b^-) = I(\pi, U)$$

(see Theorem 4.4), the lemma is proved.

Now we present the main result.

THEOREM 6.3. *Assume that all the Betti numbers of M are finite. Let K be a compact subset of M having no rest points in its boundary. If, for every $x \in M$, $\pi^+(x) \cap K \neq \emptyset$ then*

$$I(\pi, \operatorname{int} K) \geqslant \chi(M).$$

Proof. Observe that if we assume M to be compact the theorem is a trivial consequence of Prop. 4.3 (2).

If M is noncompact with the Betti numbers finite, the classifying theorem for open surfaces (see [9], p. 268) implies the existence of a compact 2-manifold N such that $M \subset N$, $S = N \setminus M$ is finite, $S = \{p_1, \ldots, p_m\}$. We extend the system π to the system $\hat{\pi}$ in N setting $\hat{\pi}(p_i) = p_i$ for $i = 1, \ldots, m$. The assumption that every trajectory of π intersects K implies that S is an isolated invariant set for $\hat{\pi}$ and $N \setminus K$ is its isolating neighbourhood. Any rest point of $\hat{\pi}$ must belong to S or to K. By Prop. 4.3(2)

$$I(\hat{\pi}, \operatorname{int} K) = \chi(N) - I(\hat{\pi}, N \setminus K).$$

By Lemma 6.2

$$I(\hat{\pi}, \operatorname{int} K) \geqslant \chi(N) - \chi(S).$$

Since $\chi(N) - \chi(S) = \chi(M)$ by VI. 7.21 and VIII. 8.6 in [3] and $I(\pi, \operatorname{int} K) = I(\hat{\pi}, \operatorname{int} K)$ by Prop. 4.1, the desired inequality follows.

The following corollary is an immediate application of the previous theorem. I am indebted to the referee for pointing out that this result is also a trivial consequence of Theorem 33 in [7].

COROLLARY 6.4. *If $M = \boldsymbol{R}^2$ and there exists a compact set K such that $\pi(x) \cap K \neq \emptyset$ for every $x \in \boldsymbol{R}^2$, then K has a rest point.*

Note that the contrapositions of Theorems 6.1 and 6.3 supply sufficient conditions for the existence of a trajectory or a positive semitrajectory in the set $X \setminus K$.

As a consequence of Theorems 5.1 and 6.3 we obtain the following result:

COROLLARY 6.5. *Let $v\colon \boldsymbol{R}^2 \to \boldsymbol{R}^2$ be the velocity field of a dynamical system π. Assume that v is of class C^1, $v(0) = 0$, and the differential of v has two real eigenvalues, λ and μ, $\lambda < 0 < \mu$. If U is an open bounded set such that $v^{-1}(0) \cap U = \{0\}$, then there exists an x such that $\pi(x) \subset \boldsymbol{R}^2 \setminus U$.*

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Received 12 January 1984;
in revised form 5 July 1984