Some remarks on embeddings of Boolean algebras and topological spaces, II

by

Ryszard Frankiewicz (Warszawa)

Abstract. We prove the existence of a model of ZFC+MA+$\neg$CH in which the algebra $P(\omega_1)$ is not embeddable in $P(\omega)/[\omega]^{<\omega}$ and the existence of model ZFC+LM/Δ is not embeddable into $P(\omega)/[\omega]^{<\omega}$.

The following theorem is well known in the foundations of set theory and general topology:

Assume CH. Then each Boolean algebra of cardinality c can be embedded in $P(\omega)/[\omega]^{<\omega}$. For the proof see Comfort and Negrepontis [1]. In Kunen's dissertation and in [3] it is shown that the above theorem is not true in some models of ZFC set theory, for example in $L[\{c_\alpha|\ \alpha<\omega_2\}]$, where c_α's are Cohen reals. The question arises whether the theorem follows from Martin's Axiom MA. Analyzing the classical proof, we see that gaps in $P(\omega)/[\omega]^{<\omega}$ are a unique obstacle as regards the existence of embedding of $P(\omega_1)$ in $P(\omega)/[\omega]^{<\omega}$ (under MA). The solution is negative. We shall prove the following:

THEOREM. *Assume the consistency of* ZFC. *There is a model of* ZFC+MA+ +$\neg$CH+$\diamondsuit_c$ *in which the algebra* $P(\omega_1)$ *cannot be embedded in the algebra* $P(\omega)/[\omega]^{<\omega}$.

The above theorem was obtained in 1980 (see [4]) during the TOPSET conference in Toronto. It was proved independently by J. Baumgartner. Two years later J. van Mill in [6] proved a similar result. Namely, in Kunen's model (unpublished) in which there are only (ω_1, ω_1)-gaps and (ω_1, c)-gaps and MA+$\neg$CH holds, the universal (for embeddings) algebra of cardinality c cannot be embedded into $P(\omega)/[\omega]^{<\omega}$. It is not known whether the theorem holds in Kunen's model.

It seems that the existence of (ω_1, ω_1)-gaps will not contradict the existence of embeddings of algebras of power c, which are hereditarily c.c.c. But, unfortunately, that is not true. The following theorem will be proved in Sec. 3.

THEOREM. *If* ZFC *is consistent, then it remains consistent with* LM/Δ *not being embeddable into* $P(\omega)/[\omega]^{<\omega}$.

1. Let $\langle A, \leqslant\rangle$ be a partially ordered set. For arbitrary ordinals $\varkappa$, λ a subset $\mathscr{L} = \{a_\alpha, b_\beta\colon \alpha<\varkappa, \beta<\lambda\}$ of A is a $(\varkappa, \lambda)$-chain if for all $a<\alpha'<\varkappa$ and $\beta<\beta'<\lambda$ we have $a_\alpha<a_{\alpha'}<b_{\beta'}<b_\beta$. $(\varkappa, \lambda)$-chain $\mathscr{L}$ forms a $(\varkappa, \lambda)$-gap, if we do not have $a_\alpha<c<b_\beta$, all $\alpha<\varkappa$ and $\beta<\lambda$ for any c in A.

A classical theorem of Hausdorff says that there are (ω_1, ω_1)-gaps in the algebra $P(\omega)/[\omega]^{<\omega}$.

For a given chain $\mathscr{L} = \{a_\alpha, b_\beta\colon \alpha<\varkappa, \beta<\lambda\}$ in $P(\omega)/[\omega]^{<\delta}$ we denote by $E(\mathscr{L})$ the set of all pairs $\langle s, t\rangle$ where s, t are functions with $\mathrm{dom}(s)\in[\varkappa]^{<\omega}$ $\mathrm{dom}(t)\in[\lambda]^{<\omega}$ and ranges $\subseteq\omega$ satisfying the following condition:

$\bigcup\{a_\alpha\setminus s(\alpha)\colon \alpha\in\mathrm{dom}(s)\}\subseteq\bigcap\{b_\beta\cup t(\beta)\colon \beta\in\mathrm{dom}(t)\}$. We order $E(\mathscr{L})$ by inverse inclusion. Thus, $E(\mathscr{L})$ is a natural partial ordering to kill a gap.

LEMMA 1. *If $\mathscr{L}$ is an (ω_1, ω_1)-gap in $P(\omega)/[\omega]^{<\omega}$, then there is a c.c.c.-set $\boldsymbol{P}$ such that for each generic $G\subseteq \boldsymbol{P}$ we have $V[G]\Vdash$ "$E(\mathscr{L})$ is not a c.c.c.-set".*

Proof. We choose representatives from equivalence classes and work in $P(\omega)$ instead of $P(\omega)/[\omega]^{<\omega}$. Let $\mathscr{L} = \{a_\alpha, b_\beta\colon \alpha, \beta<\omega_1\}$. Thus, for each $\alpha<\omega_1$ there is an $n_\alpha\in\omega$ such that $a_\alpha\setminus n_\alpha\subseteq b_\alpha$, and hence there are an uncountable $A\subseteq\omega_1$, and n such that $a_\alpha\setminus n\subseteq b_\alpha$ for all $\alpha\in A$. Let $\boldsymbol{P}^*$ consist of all finite $s\subseteq A$ with the property:

$$a_\alpha\setminus n\not\subseteq b_\beta, \quad \text{for} \quad \alpha<\beta \text{ in } s.$$

$\boldsymbol{P}^*$ is ordered by inverse inclusion.

We prove first thet $\boldsymbol{P}^*$ is a c.c.c.-set. Suppose, an the contrary that $C = \{s_\alpha\colon \alpha<\omega_1\}$ is an uncountable antichain in $\boldsymbol{P}^*$. Applying the Δ-Lemma, we may assume that s_α's are pairwise disjoint. Moreover, replacing C by a subfamily, we obtain $\max s_\alpha<\min s_\beta$ for $\alpha<\beta$. Now, if $A_s = \bigcap_{\alpha\in s}(a_\alpha\setminus n)$, $B_t = \bigcup_{\eta\in t} b_\eta$, then the chain $\{A_s, B_t\colon s, t\in C\}$ is not an (ω_1, ω_1)-gap and consequently $\mathscr{L}$ is not a gap either. Thus, $\boldsymbol{P}^*$ is a c.c.c.-set.

Let $\boldsymbol{P}\subseteq\boldsymbol{P}^*$ consist of all s such that for uncountably many α we have $s\cup\{\alpha\}\in\boldsymbol{P}^*$. We easily check that $\boldsymbol{P}^*\setminus\boldsymbol{P}$ is at most countable (again using the assumption that $\mathscr{L}$ is an (ω_1, ω_1)-gap), and hence each generic $G\subseteq\boldsymbol{P}$ is uncountable. In $V[G]$ we have an uncountable antichain $\{\langle p_\alpha, q_\alpha\rangle\colon \alpha\in\bigcup G\}$ of $E(\mathscr{L})$, where $p_\alpha = \{\langle\alpha, n\rangle\}$ and $q_\alpha = \{\langle\alpha, 0\rangle\}$, which finishes the proof.

Using the Δ-Lemma and Ramsey's Theorem, one can prove the following (unpublished) result of Kunen:

LEMMA 2. *If $\mathscr{L}$ is not an (ω_1, ω_1)-gap in $P(\omega)/[\omega]^{<\omega}$, then $E(\mathscr{L})$ is a c.c.c.-set.*

2. In this section we shall prove our theorem, formulated in the introduction. We start from a model L of constructible sets and fix a regular cardinal $\varkappa\geqslant\omega_2$. Enlarge 2^{ω_1} to $\varkappa$, using a standard set of conditions. Then, in the model V thus obtained, the principle $\diamondsuit_\varkappa$ still holds true, (see [2]). Let $\langle S_\alpha\colon \alpha<\varkappa\rangle$ be a $\diamondsuit_\varkappa$-sequence in V and $F\colon \varkappa\to H(\varkappa)$ — a fixed bijection of $\varkappa$ onto the family $H(\varkappa)$ of all sets of hereditary power $<\varkappa$.

We now define a sequence P_α, for $\alpha<\varkappa$, of partial orderings $\boldsymbol{P}_\alpha\in H(\varkappa)$. Assume that $\boldsymbol{P}_\alpha$ has been defined. Suppose that $F(S_\alpha)$ is a term in V^{P_α} and $\Vdash_{P_\alpha}$ "$F(S_\alpha)$ is an embedding of $\mathrm{BA}([\omega_1]^{<\omega_1})$ in $P(\omega)/[\omega]^{<\omega}$". (Here and later $\mathrm{BA}(X)$ denotes the subalgebra generated by a subset X of a given Boolean algebra.) Since $\mathrm{BA}([\omega_1]^{<\omega_1})$ contains $2^{\omega_1} = \varkappa$ (ω_1, ω_1)-gaps and $P(\omega)/[\omega]^{<\omega}$ has cardinality $\mathfrak{c} = 2^\omega<\varkappa$, there exists an (ω_1, ω_1)-gap $\mathscr{L}$ in $\mathrm{BA}([\omega_1]^{<\omega_1})$, (in V^{P_α}), such that: $\Vdash_{P_\alpha}$ "$F(S_\alpha)(\mathscr{L})$ is a gap in $P(\omega)/[\omega]^{<\omega}$".

Let $\boldsymbol{P}$ be a c.c.c.-set corresponding to the (ω_1, ω_1)-gap $F(S_\alpha)(\mathscr{L})$ as in Lemma 1 and put $\boldsymbol{P}_{\alpha+1} = \boldsymbol{P}_\alpha * \boldsymbol{P}$.

Suppose now that

$$\Vdash_{P_\alpha}\text{“}F(S_\alpha) \text{ is a partial ordering with c.c.c.”}$$

In this case we define $\boldsymbol{P}_{\alpha+1} = \boldsymbol{P}_\alpha * F(S_\alpha)$. Finally, if $F(S_\alpha)$ is not as above, we put $\boldsymbol{P}_{\alpha+1} = \boldsymbol{P}_\alpha$. For limit ordinals $\lambda\leqslant\varkappa$, $\boldsymbol{P}_\lambda$ is defined as the direct limit of the preceding $\boldsymbol{P}_\alpha$'s.

It is clear that in $V^{P_\varkappa}$ both MA and $\diamondsuit_\varkappa$ hold true. It remains to prove that in $V^{P_\varkappa}$ the algebra $P(\omega_1)$ is not embeddable in $P(\omega)/[\omega]^{<\omega}$. Suppose, on the contrary, that

$$\Vdash_{P_\varkappa}\text{“}f \text{ is an embedding of } P(\omega_1) \text{ in } P(\omega)/[\omega]^{<\omega}\text{”}.$$

Then, f embeds the subalgebra $\mathrm{BA}([\omega_1]^{<\omega_1})$ in $P(\omega)/[\omega]^{<\omega}$. Since $\boldsymbol{P}_\varkappa$ satisfies c.c.c., we may assume that $f\subseteq H(\varkappa)$. There exists a closed unbounded set $C\subseteq\varkappa$ such that, for limit $\alpha\in C$, $f_\alpha = f\cap V^{P_\alpha}$ is an embedding of $\mathrm{BA}([\omega_1]^{<\omega_1})$ in $P(\omega)/[\omega]^{<\omega}$ in V^{P_α}. The set $T = \{\alpha<\varkappa\colon f\cap F[\alpha] = f_\alpha\}$ is again closed and unbounded and for a limit $\alpha\in C\cap T$ we have $F(S_\alpha) = f\cap F[\alpha] = f_\alpha$. Hence, by the construction of $\boldsymbol{P}_{\alpha+1}$ and Lemma 1, the set $E(F(S_\alpha)(\mathscr{L}))$ does not, in V^{P_α} and a fortiori in $V^{P_\varkappa}$, satisfy the c.c.c.-condition. Since there are no (ω_1, ω_1)-gaps in $P(\omega_1)$, it follows that $f(\mathscr{L}) = F(S_\alpha)(\mathscr{L})$ cannot be an (ω_1, ω_1)-gap in $P(\omega)/[\omega]^{<\omega}$ and consequently, by Lemma 2, $E(f(\mathscr{L}))$ is a c.c.c.-set, which gives the desired contradiction.

3. The continuum hypothesis (CH) implies that the algebra LM/Δ (the Lebesgue measurable sets of the unit interval modulo the sets of measure zero) is embeddable in $P(\omega)/[\omega]^{<\omega}$. Moreover, there is an embedding φ which satisfies

$$\text{measure of } a = \text{density of } \varphi(a)$$

(the density of a subset $A\subseteq\omega$ is defined as the limit $\lim_n |A\cap n|/n$, if it exists). We shall now prove that the assumption CH is essential here.

THEOREM. *If* ZFC *is consistent, then it remains consistent with the following additional statements:*

(a) LM/Δ *is not embeddable in* $P(\omega)/[\omega]^{<\omega}$.

(b) $\mathfrak{c} = 2^{\omega_0} = \omega_2$.

The proof is obtained as a modification of Shelah's proof from [7]. More precisely, properties (a) and (b) of the Theorem are both true in modification of Shelah's model, in which the Lebesgue measure has no Borel lifting.

Since our proof is identical with that of Shelah from a certain point on, we shall indicate below only the principal differences. In view of [8], IV, See 3, it is sufficient to prove the following lemma:

LEMMA. *Let $\overline{M}$ be an ω_1-oracle and h an embedding of* LM/Δ *in* $P(\omega)/[\omega]^{<\omega}$. *Then there exist a forcing* $\boldsymbol{P}$ *satisfying the $\overline{M}$-chain condition, and a $\boldsymbol{P}$-name X of a Borel set such that, for every generic $G\subseteq \boldsymbol{P}\times Q$ over V (Q is a Cohen forcing), there is no $A\subseteq\omega$ in $V[G]$ satisfying the following conditions.*

(i) for every $B\in \mathrm{Borel}^V$, if $B^{V[G]}\subseteq X[G] \bmod \Delta$, then $h(B)\subseteq A \bmod [\omega]^{<\omega}$

(ii) for every $B\in \mathrm{Borel}^V$, if $B^{V[G]}\cap X[G] = \emptyset \bmod \Delta$, then $h(B)\cap A = \emptyset \bmod [\omega]^{<\omega}$.

Outline of proof. $\boldsymbol{P}$ is defined as in Shelah [7]. Let Se denote the set of monotonic sequences $\bar{a} = \langle a_i\colon i\leqslant\omega\rangle$, where a_i, for $i<\omega$, are rationals, $a_i\neq a_{i+1}$, $\lim_{i<\omega} a_i = a_\omega$, and a_ω is an irrational. Let $P^\beta = P(\langle\bar{a}^\alpha\colon \alpha<\beta\rangle)$, where $\beta\leqslant\omega_1$, $\bar{a}^\alpha\in Se$ and a^α_ω is are pairwise distinct, consist of the conditions $p = \langle u_p, f_p\rangle$ where

a. $u_p\subseteq(0,1)$ is open, $\mathrm{cl}(u_p)$ has measure $\leqslant\frac{1}{2}$, f_p is a function from u_p to $\{0,1\}$;

b. there are n, b_l, J_l such that $0 = b_0<b_1<\ldots<b_{n-1}<b_n = 1$, $u_p = \bigcup_{l=0}^{n-1} J_l$ and J_l are open with $\mathrm{cl}(J_l)\subseteq(b_l, b_{l+1})$;

c. J_l is either a rational interval and $f_p|J_l$ is constant or $J_t = \bigcup_{n(l)\leqslant m<\omega}(a^\alpha_{2m}, a^\alpha_{2m+1})$, for some $\alpha<\beta$ and $n(l)<\omega$, and f_p on $(a^\alpha_{4m+2k}, a^\alpha_{4m+2k+1})$ is constantly k if $n(l)\leqslant 2m+k$, $k\in\{0,1\}$.

The ordering $\leqslant$ on P is as follows:

$$p\geqslant q \quad\text{iff}\quad u_p\subseteq u_q \quad\text{and}\quad f_p\subseteq f_q \quad\text{and}\quad \mathrm{cl}(u_p)\cap u_q = u_p .$$

Let X_p be the union of all rational intervals (a,b) such that for some $p\in G_p$ we have $(a,b)\subseteq u_p$ and f_p on (a,b) is constantly zero.

Define inductively a sequence $\langle P_\delta\colon \delta<\omega_1\rangle$ as follows: assume that P_δ has been defined, let $M_\delta\in\overline{M}$, $(p^*, q^*)\in P_\delta\times Q$ and let $\underset{\sim}{A}$ be a $P_\delta\times Q$ — name of a set A. Then we can find an $a^\delta\in Se$ and an infinite subset $\varphi(a^\delta)$ of ω satisfying $\varphi(a^\delta)\subseteq h(b_1, b_2)$, where b_1, b_2 are rationals, $b_1<a^\delta_\omega<b_2$, and $h(b_1, a^\delta_\omega)$ does not split $\varphi(a^\delta_\omega)$ into two infinite sets, so that all the following holds: if $P_{\delta+1} = P(\langle\bar{a}^\alpha; \alpha<\delta\rangle)$, then

(A) every predense subset of P_δ belonging to M_δ is predense in $P_{\delta+1}$;

(B) there is a $\langle p', r'\rangle\in P_{\delta+1}\times Q$ such that $\langle p', r'\rangle\leqslant\langle p^*, q^*\rangle$ and either, for some $n\in\omega$,

$$\langle p', r'\rangle \Vdash_{P_{\delta+1}\times Q} {}''\varphi(a^\delta_\omega)\subseteq A \bmod [\omega]^{<\omega}$$

and

$$\bigcup_{n<m<\omega}(a^\delta_{4m+2}, a^\delta_{4m+3})\cap X\neq\emptyset$$

and

$$\varphi(a^\delta_\omega)\subseteq^* h\Big(\bigcup_{n<m}(a^\delta_{4m+2}, a^\delta_{4m+3n})\Big)''$$

or

$$\langle p', r'\rangle \Vdash_{P_{\delta+1}\times Q} {}''\varphi(a^\delta)\cap A = \emptyset \bmod [\omega]^{<\omega} \quad\text{and}\quad \bigcup_{n<m<\omega}(a^\delta_{4m}, a^\delta_{4m+1})\subseteq X$$

and

$$\varphi(a_\omega)\subseteq^* h\Big(\bigcup_{n<m<\omega}(a^\delta_{4m}, a^\delta_{4m+1})\Big)'' .$$

Now, we show how to find a^δ and $\varphi(a^\delta_\omega)$ with the properties described above. Let λ be a cardinal and take a countable $N\prec\langle H(\lambda), \in\rangle$ containing P_σ, $\langle\bar{a}^\alpha\colon \alpha<\delta\rangle$, A, M_δ and h. Let $\bar{a}^\delta_\omega$ be a random (over N) real, $a^\delta_\omega\in(0,1)\setminus\mathrm{cl}(u_{p^*})$. We work in $N[a^\delta_\omega]$: let $\varphi(a^\delta_\omega)$ be an infinite set almost contained in each $h(b, a^\delta_\omega)$, where $b<a^\delta_\omega$ is a rational, and let $\langle b_n\colon n<\omega\rangle$ be a strictly increasing sequence of rationals converging to a^δ_ω. Define, in $N[a^\delta_\omega]$, a forcing R consisting of pairs $\langle f, g\rangle$, where f is a finite sequence of natural numbers satisfying $f(i)>i$ for all $i<n = \mathrm{dom}(f)$ and g is an arbitrary function from ω to ω, and order R as follows:

$$\langle f, g\rangle\geqslant\langle f', f'\rangle \quad\text{iff}\quad f\subseteq f' \text{ and } \forall_i[g(i)\leqslant g'(i)]$$

and

$$\forall_i[i\in\mathrm{dom}(f')\setminus\mathrm{dom}(f)\to f'(i)\geqslant g(i)] .$$

Let f^* be R-generic over $N[a^\delta_\omega]$ and define in $N[a^\delta_\omega][f^*]$ a sequence $\langle n(b)\colon b<\omega\rangle$ of natural numbers as follows:

$$n(0) = 0 \quad\text{and}\quad n(l+1) = f^*(n(l)) .$$

Now, if for $m<4$ and $k<\omega$,

$$A^k_m = \bigcup_{k\leqslant l<\omega}(b_{n(4l+m)}, b_{n(4l+m+1)}) ,$$

then A^0_m $(m = 0, 1, 2, 3)$ is a partition of (b_0, a^δ_ω) and hence, for some unique m^*, we have

$$\varphi(a^\delta_\omega)\subseteq h(A^0_{m^*}) \bmod [\omega]^{<\omega} .$$

Since we can change f^* at finitely many places, we can assume that $a^\delta_\omega\in A^0_{m^*}$. Put

$$\bar{a}^\delta = \langle b_{n(l)}\colon l<\omega\rangle^\frown\langle a^\delta_\omega\rangle .$$

From now on the proof goes exactly as in Shelah [7].

References

[1] W. Comfort and S. Negrepontis, *The theory of ultrafilters*, Springer Verlag, Berlin 1974.

[2] D. Devlin, *Aspects of Constructibility*, Lecture Notes in Math. 354 (1973).

[3] R. Frankiewicz and A. Gutek, *Some remarks on embeddings of Boolean Algebras and Topological Spaces* I, Bull. Acad. Pol. Sci. 29 (1981).

[4] — *Six Theorems*, Abstracts of Amer. Math. Soc., Nov. 1980.

[5] T. Jech, *Set Theory*, Academic Press, New York 1978.

[6] J. van Mill, in *Handbook of Set Theoretical Topology*, North Holland 1984.
[7] S. Shelah, *Lifting Problem of the Measure Algebra*, Israel J. Math. 45. 1.
[8] — *Proper Forcing*, Lecture Notes in Math. 918, Springer Verlag.

INSTYTUT MATEMATYCZNY PAN
Śniadeckich 8,
00-950 Warszawa

Received 22 December 1983;
in revised form 19 April 1984

On rest points of dynamical systems

by

Roman Srzednicki (Kraków)

Abstract. For a dynamical system π in an ENR-space X we define the index of rest points $I(\pi, U)$ in a relatively compact open set U as $\lim_{t\to 0^+} \operatorname{ind}(\pi_t, U)$. We prove, that if K is a compact set such that any positive semitrajectory intersects K then $I(\pi, \operatorname{int} K)$ is equal to the Euler characteristics $\chi(X)$. If X is a separable 2-manifold having the Betti numbers finite and any trajectory intersects K then $I(\pi, \operatorname{int} K) \geq \chi(X)$.

In the present paper we apply the theory of isolated invariant sets and the fixed point theory to prove some results concerning rest points of dynamical systems. The theory of isolated invariant sets was introduced and developed by C. C. Conley, R. W. Easton, R. C. Churchill and others, but the main idea comes from paper [13] of T. Ważewski. We base ourselves on paper [2], in which there are references and historical remarks concerning this theory. We use the fixed point index constructed by A. Dold and presented in [3].

In Section 1 we present the basic facts in the theory of dynamical systems and isolated invariant sets. Section 2 is devoted to asymptotically stable sets. The main result of this section is Theorem 2.4. It presents a sufficient condition for the existence of an asymptotically stable set in a space X, such that X is its region of attraction. In Section 3 we present the notion of an ENR and we prove that for dynamical systems in 2-manifolds there are blocks which are ENR's (Theorem 3.3). The index of a rest point is defined in Section 4. We compute the index in the interior of a block (Theorem 4.4). In Section 5 we assume that a dynamical system is generated by a C^1 vector field v and we compare the index with the degree of v. In Section 6 we apply the properties of the index to prove some results concerning the existence of trajectories and rest points in a given set.

The author expresses his gratitude to Professors A. Pelczar and S. Sędziwy for their valuable remarks during the preparation of this paper.

1. Preliminaries. In this paper X denotes a topological locally compact space satisfying the second axiom of countability. We say that

$$\pi\colon \boldsymbol{R}\times X \to X$$