Homeomorphisms of products of Boolean separable spaces

by

Věra Trnková (Praha)

Abstract. We construct a Boolean (= compact Hausdorff zero-dimensional) separable space X homeomorphic to X^3 but not to X^2. A more general setting of sum productive representations of ordered commutative semigroups is investigated.

I. Preliminaries and the main theorems. In [4], W. Hanf constructed a Boolean algebra B isomorphic to the product (that is to say, a direct product) $B\times B\times B$ but not to $B\times B$. His result was strengthened by J. Ketonen in [5]: he constructed a countable Boolean algebra B isomorphic to $B\times B\times B$ but not to $B\times B$.

The dual result (i.e. for sums in place of products) depends on the cardinality of the Boolean algebras: there exists no countable Boolean algebra B isomorphic to $B+B+B$ but not to $B+B$, by [9]; however, there exists a large Boolean algebra with this property, by [10]. There is a natural problem: what is the smallest cardinality of a Boolean algebra B isomorphic to $B+B+B$ but not to $B+B$. In the present paper, we construct a Boolean (= compact Hausdorff zero-dimensional) *separable* space X homeomorphic to X^3 but not to X^2. This solves the above problem under the continuum hypothesis: the cardinality of the Boolean algebra B of all clopen (= closed-and-open) subsets of X is $2^{\aleph_0}$ and, by the Stone duality, B is isomorphic to $B+B+B$ but not to $B+B$.

We investigate a more general setting of sum-productive representations of ordered commutative semigroups. Let $\boldsymbol{C}$ be a class of topological spaces, $(S, \oplus, \leqslant)$ an ordered commutative semigroup. A *sum-productive representation of* $(S, \oplus, \leqslant)$ *in* $\boldsymbol{C}$ is any collection $\{X(s)|\ s\in S\}$ of spaces from $\boldsymbol{C}$ such that for every $s, s'\in S$.

(i) $X(s)\times X(s')$ is homeomorphic to $X(s\oplus s')$ and

(ii) $X(s)$ is homeomorphic to a clopen subset of $X(s')$ iff $s\leqslant s'$.

The aim of the present paper is to prove the following theorems.

THEOREM 1. *Every countable ordered commutative semigroup has a sum-productive representation in the class* $\coprod_{\omega}\boldsymbol{BS}$ *of all spaces which are sums (= disjoint unions as clopen subspaces) of countably many Boolean separable spaces.*

THEOREM 2. *Every finite Abelian group with the discrete order has a sum-productive representation in the class* **BS** *of all Boolean separable spaces.*

The special choice of the represented group in Theorem 2 as $\{0, 1\}$ with $1+1 = 0$ gives immediately the result mentioned in the abstract, we obtain a Boolean separable space $X = X(1)$ homeomorphic to $X^3 \simeq X(1+1+1)$ and even no clopen subset of $X^2 \simeq X(0)$ is homeomorphic to X and vice versa.

Let us mention that some strengthenings of the above theorems are presented in the part IV. "Concluding Remarks" of the present paper.

II. The general construction and the proof of Theorem 1.

II.1. Let $(\omega, +)$ be the additive semigroup of all finite cardinals. Then for any set M, ω^M is also a commutative semigroup (with the addition $+$ defined by $(f+g)(m) = f(m)+g(m)$ for all $m \in M$) and the set $\exp\omega^M$, ordered by the inclusion, is an ordered commutative semigroup. Let us denote by $\boldsymbol{U}$ the ordered subsemigroup of $(\exp\omega^\omega, +, \subseteq)$ consisting of all nonempty countable subsets of ω^ω. Then

$\boldsymbol{U}$ is universal for all countable ordered commutative semigroups,

i.e. for every countable ordered commutative semigroup $(S, \oplus, \leqslant)$ there exists a map $\varphi\colon S \to \boldsymbol{U}$ such that for every $s_1, s_2 \in S$,

(a) $\varphi(s_1 \oplus s_2) = \varphi(s_1)+\varphi(s_2)$ and

(b) $\varphi(s_1) \subseteq \varphi(s_2)$ iff $s_1 \leqslant s_2$.

In fact, in [7], for every commutative semigroup $(S, \oplus)$ a disjoint homomorphism $\psi\colon (S, \oplus) \to (\exp\omega^M, +)$ is constructed with $M = \aleph_0 \cdot \operatorname{card} S$ (a *disjoint* homomorphism means a homomorphism such that $\psi(s_1) \cap \psi(s_2) = \emptyset$ whenever $s_1 \neq s_2$). In [1], a given ordered commutative semigroup $(S, \oplus, \leqslant)$ is embedded into an ordered commutative semigroup $(\bar{S}, \oplus, \leqslant)$ such that $\operatorname{card} \bar{S} = \aleph_0 \cdot \operatorname{card} S$ and

if $x, y, z \in \bar{S}$ and $z \leqslant x \oplus y$ then $z = x_1 \oplus y_1$ for some $x_1, y_1 \in \bar{S}$ with $x_1 \leqslant x$ and $y_1 \leqslant y$.

This condition implies that if $\psi\colon (\bar{S}, \oplus) \to (\exp\omega^M, +)$ is a disjoint homomorphism, then the map

$$\varphi\colon (S, \oplus, \leqslant) \to (\exp\omega^M, +, \subseteq)$$

defined by $\varphi(s) = \bigcup_{t \in \bar{S}, t \leqslant s} \psi(t)$ is an order preserving embedding; hence the class of all $(\exp\omega^M, +, \subseteq)$, M ranging over all sets, is shown in [1] to be a universal class for all ordered commutative semigroups. The result of [7] has been improved in [8]: for every commutative semigroup $(S, \oplus)$ there exists a disjoint homomorphism ψ of $(S, \oplus)$ into $(\exp\omega^M, +)$ such that $M = \aleph_0 \cdot \operatorname{card} S$ and

$$\operatorname{card} \psi(s) = \aleph_0 \cdot \operatorname{card} S \quad \text{for every } s \in S.$$

Thus, by [1] and [8], $\boldsymbol{U}$ is really universal for all countable ordered commutative semigroups. To prove Theorem 1, we show that U has a sum-productive representation in the class $\coprod_\omega \boldsymbol{BS}$.

II.2. Now, we describe a general method of constructions of representations. Let $\{X_n, n \in \omega\}$ be a collection of topological spaces. For every $f \in \omega^\omega$ put

$$X(f) = \prod_{n \in \omega} X_n^{f(n)},$$

i.e. $X_n^{f(n)}$ is the product of $f(n)$ copies of the space X_n ($X_n^{f(n)}$ is a one-point space whenever $f(n) = 0$) and $X(f)$ is the product of all of them. Clearly,

$$X(f) \times X(g) \text{ is homeomorphic to } X(f+g).$$

For every nonempty countable $A \subseteq \omega^\omega$ put

$$X(A) = \coprod_{n \in \omega} \Big(\coprod_{f \in A} X(f)\Big)_n,$$

i.e. $X(A)$ is a sum (= disjoint union as clopen subsets) of $\aleph_0$ copies of each $X(f)$ with $f \in A$ (a sum is denoted by the symbol $\coprod$ in the present paper). Then

$$X(A) \times X(B) \text{ is homeomorphic to } X(A+B).$$

In fact, each summand of $X(A+B)$ is of the form $X(f+g)$ so that it is homeomorphic to the summand $X(f) \times X(g)$ of $X(A) \times X(B)$; and vice versa. Since both $X(A+B)$ and $X(A) \times X(B)$ contain each its summand in $\aleph_0$ copies, they are homeomorphic. Hence

$$\{X(A) |\ A \in \boldsymbol{U}\}$$

is a sum-productive representation of $\boldsymbol{U}$ whenever the following implication $(*)$ is fulfilled.

$(*)$ $\quad X(A)$ is homeomorphic to a clopen subset of $X(B) \Rightarrow A \subseteq B$.

The aim of the next parts II.3–II.13 is to construct the starting collection $\{X_n |\ n \in \omega\}$ such that each X_n is a Boolean separable space and $(*)$ is valid for this collection. This will prove Theorem 1.

II.3. Let α be a nontrivial ultrafilter on ω. Denote by

$$P(\alpha) = \omega \cup \{\alpha\}$$

the subspace of the β-compactification $\beta\omega$ of the discrete space ω consisting of all the isolated points of $\beta\omega$ and the point α; all the neighbourhoods of α intersect ω precisely in the ultrafilter α. Let us recall (see e.g. [2]) that

if $f\colon P(\alpha) \to P(\alpha)$ is a continuous map such that $f(\alpha) = \alpha$ and $f(\omega) \subseteq \omega$, then there exists $\mathcal{U} \in \alpha$ such that $f(j) = j$ for all $j \in \mathcal{U}$.

This assertion will be used without any explicit reference to it.

We construct a space Seq by means of α as follows: the underlying set of Seq is the set of all finite strings $q_1 q_2 \ldots q_k$ of elements of ω, the empty string Λ included, i.e. $\mathrm{Seq} = \bigcup_{n\in\omega} \omega_n$. For every string $s = q_1 \ldots q_k \in \mathrm{Seq}$ we have a canonical map

$$\psi_s\colon P(\alpha) \to \mathrm{Seq}$$

defined by

$$\psi_s(\alpha) = s\,,$$
$$\psi_s(j) = sj = q_1 \ldots q_k j \text{ for every } j \in \omega\,.$$

We investigate Seq being endowed with the finest topology, for which each map ψ_s, $s \in S$, is a homeomorphism of $P(\alpha)$ into Seq. This topology is obtained by iterations of the following closure operator cl: for every $A \subseteq \mathrm{Seq}$,

$$\mathrm{cl}A = A \cup \{s \in \mathrm{Seq} \mid \text{there exists } \mathscr{U} \in \alpha \text{ such that } sj \in A \text{ for all } j \in \mathscr{U}\}\,.$$

LEMMA. *Let $\mathcal{O}$ be a subset of* Seq, *s be in its closure $\overline{\mathcal{O}}$ and $s \notin \mathcal{O}$. Then*

$$\{j \in \omega \mid sj \in \overline{\mathcal{O}}\}$$

is in the ultrafilter α.

Proof. Let us suppose the contrary. Then $\mathscr{U} = \{j \in \omega \mid sj \notin \overline{\mathcal{O}}\}$ is in α. If sj is not in $\overline{\mathcal{O}}$, there exists its neighbourhood $\mathscr{V}_j$ in Seq disjoint with $\mathcal{O}$. Then

$$\{s\} \cup \bigcup_{j\in\mathscr{U}} \mathscr{V}_j$$

is a neighbourhood of s disjoint with $\mathcal{O}$, which is a contradiction.

PROPOSITION. Seq *is extremally disconnected.*

Proof. Let $\mathcal{O}$ be an open subset of Seq, s be in the closure $\overline{\mathcal{O}}$ and $s \notin \mathcal{O}$. By the previous lemma, $\overline{\mathcal{O}} \cap \psi_s(P(\alpha))$ is a neighbourhood of s in $\psi_s(P(\alpha))$. Analogously, for every $t \in \overline{\mathcal{O}} \cap \psi_s(P(\alpha))$, the set $\overline{\mathcal{O}} \cap \psi_t(P(\alpha))$ is a neighbourhood of t in $\psi_t(P(\alpha))$. Repeating this procedure, we see that $\overline{\mathcal{O}}$ is a neighbourhood of s in Seq. Thus $\overline{\mathcal{O}}$ is open, Seq is extremally disconnected.

COROLLARY. *Since a β-compactification of an extremally disconnected space is extremally disconnected again* (see e.g. [3]), *β*Seq *is extremally disconnected.*

II.4. Let X be the space which is obtained from $\beta\mathrm{Seq}$ and the Cantor discontinuum C by glueing a point $a \in C$ with the empty string $\Lambda \in \mathrm{Seq} \subseteq \beta\mathrm{Seq}$; denote the obtained point by a again. Let us suppose, for shortness, that $\mathrm{Seq} \subseteq \beta\mathrm{Seq} \subseteq X$ and let us denote by C the copy of the Cantor discontinuum contained in X, so that

$$X = C \cup \beta\mathrm{Seq} \quad \text{and} \quad C \cap \beta\mathrm{Seq} = \{a\}\,.$$

Observation. X is a Boolean separable space.

II.5. Let $\mathscr{A} = \{\alpha_n \mid n \in \omega\}$ be a collection of nontrivial ultrafilters on ω. Let us use the construction described in II.3–4 for each $\alpha = \alpha_n$ (and let us add the index n to each constructed object), so we have constructed an extremally disconnected space Seq_n and a space X_n such that

$$X_n = C_n \cup \beta\mathrm{Seq}_n, \qquad C_n \cap \beta\mathrm{Seq}_n = \{a_n\}$$

(where C_n is the copy of the Cantor discontinuum contained in X_n). Thus, we have defined a collection $\{X_n \mid n \in \omega\}$ of Boolean separable spaces. Let $X(f)$ and $X(A)$ be defined by means of this collection as in II.2. Let us denote by $a(f)$ the point of $X(f)$ with all its coordinates equal to a_n. Denote by (+) the following statement.

(+) If a clopen neighbourhood of $a(f)$ is homeomorphic to a clopen subset of $X(g)$, then $f = g$.

PROPOSITION. *If the statement* (+) *is fulfilled, then the statement* (*) *of* II.2 *is also fulfilled.*

Proof. Let $A, B \subseteq \omega^\omega$ be given, let h be a homeomorphism of $X(A)$ onto a clopen subset of $X(B)$. For any $f \in A$, $X(f)$ is a clopen subset of $X(A)$. Since $X(f)$ is compact, $h(X(f))$ is covered by finitely many spaces $X(g_1), \ldots, X(g_m)$ with $g_i \in B$. Choose that g among the g_i's for which $h(a(f)) \in X(g)$. Hence $a(f)$ has a clopen neighbourhood homeomorphic to a clopen subset of $X(g)$, so that, by (+), $f = g \in B$. We conclude that $A \subseteq B$.

Remark. Let us mention that the heart of the proof of Theorem 1 is to prove that if the collection $\mathscr{A} = \{\alpha_n \mid n \in \omega\}$ of ultrafilters has suitable properties (namely (1) and (2) of II.8) then the statement (+) is fulfilled for the collection $\{X_n \mid n \in \omega\}$ constructed by means of $\mathscr{A}$. This is given in the parts II.6 – II.12 below.

II.6. For any topological space Z, we denote

$$\boldsymbol{S}(Z) = \{x \in Z \mid x \text{ has an uncountable character but any its neighbourhood contains a point with a countable character}\}.$$

For every $x \in \boldsymbol{S}(Z)$ and every $n \in \omega$, let us denote by $\mathscr{H}_n^x(Z)$ the set of all homeomorphisms h of $P(\alpha_n)$ into Z such that $h(\alpha_n) = x$. We define a binary relation $\boldsymbol{R}$ on the set $\mathscr{H}_n^x(Z)$ as follows.

$h\boldsymbol{R}g$ iff there exist

(a) a continuous map $c\colon Z \to \beta\mathrm{Seq}_n$ such that $c(x) = a_n$ and

(b) a set $\mathscr{U} \in \alpha_n$ such that $c \circ h$ and $c \circ g$ coincide on $\mathscr{U} \cup \{\alpha_n\}$ and this restricted map is a homeomorphism of $\mathscr{U} \cup \{\alpha_n\}$ into $\beta\mathrm{Seq}_n$.

Let us say that $\mathscr{H} \subseteq \mathscr{H}_n^x(Z)$ is $\boldsymbol{R}$-independent iff no pair of distinct elements of $\mathscr{H}$ is in the relation $\boldsymbol{R}$; let us denote by $f_x(n)$ the supremum of the cardinalities of all $\boldsymbol{R}$-independent subsets of $\mathscr{H}_n^x(Z)$. Hence, for every $x \in \boldsymbol{S}(Z)$, we have defined a function

$$f_x\colon \omega \to \mathrm{Card}\,.$$

Denote

$$M(Z) = \{x \in S(Z)|\ \text{there exists a neighbourhood } \mathcal{U} \text{ of } x \text{ such that } f_y < f_x \text{ for every } y \neq x,\ y \in \mathcal{U} \cap S(Z)\},$$

where $f_y < f_x$ means that $f_y(n) \leqslant f_x(n)$ for all $n \in \omega$ and $f_y \neq f_x$.

OBSERVATION. *If Z_1 is a clopen subset of a space Z_2, then $S(Z)_1 = Z_1 \cap S(Z_2)$.* Moreover, for every $x \in S(Z_1)$ and every n,

$\mathcal{H}_n^x(Z_1) \subseteq \mathcal{H}_n^x(Z_2)$ and for any $h \in \mathcal{H}_n^x(Z_2)$ there exists $h' \in \mathcal{H}_n^x(Z_1)$ such that hRh',

hence

$$M(Z_1) = Z_1 \cap M(Z_2).$$

II.7. Since we are going to deal with coordinates of points in $X(f)$, we express $X(f)$ in a form more suitable for the handling of coordinates. Let us denote

$$L(f) = \{(j, n)|\ n \in \omega,\ j = 1, \dots, f(n)\}$$

and for every $l = (j, n) \in L(f)$ put $\bar{l} = n$. Then

$$X(f) = \prod_{l \in L(f)} X_{\bar{l}}.$$

Denote by $\pi_l\colon X(f) \to X_{\bar{l}}$ the lth projection.

OBSERVATION. *$S(X(f))$ is precisely the set of all $x \in X(f)$ such that $\pi_l(x) \in C_{\bar{l}}$ for all $l \in L(f)$ and, for at least one $l_0 \in L(f)$, $\pi_{l_0}(x)$ is equal to $a_{\bar{l}_0}$.*

II.8. Now, let us suppose that the collection $\mathcal{A} = \{\alpha_n|\ n \in \omega\}$ of ultrafilters fulfils the following two conditions:

(1) each α_n is a weak P-point of $\omega^* = \beta\omega \setminus \omega$ (i.e. it is an accumulation point of no countable subset of ω^*);

(β) if $n \neq m$, then the types of α_n and α_m are incomparable in the Rudin–Keisler order (i.e. there exists no continuous map $g\colon P(\alpha_n) \to P(\alpha_m)$ such that $g(\alpha_n) = \alpha_m$, $g(\omega) \subseteq \omega$ and vice versa).

A collection $\mathcal{A}$ with these properties really does exist, by [6].

For each $n \in \omega$, let us denote by P_n the subspace $\{a_n\} \cup \{j|\ j \in \omega\}$ of X_n (i.e. P_n consists of the empty string $\Lambda = a_n$ and all the strings of the length 1 in Seq_n). Clearly, P_n is homeomorphic to $P(\alpha_n)$, by the homeomorphism $j \leftrightarrow j$, $a_n \leftrightarrow \alpha_n$.

LEMMA. *Let m, n be in ω. Let $k\colon P(\alpha_m) \to X_n$ be a continuous map such that $k(\alpha_m) = a_n$. Then*

either there exists $\mathcal{U} \in \alpha_m$ such that $k(\mathcal{U}) \subseteq C_n$

or $m = n$ and there exists $\mathcal{U} \in \alpha_m$ such that $k(\mathcal{U}) \subseteq P_n$ and $k(j) = j$ for all $j \in \mathcal{U}$.

Proof. (a) Since $X_n = C_n \cup \beta\mathrm{Seq}_n$ and $C_n \cap \beta\mathrm{Seq}_n = \{a_n\}$, either $\{j \in \omega|\ k(j) \in C_n\}$ or $\{j \in \omega|\ k(j) \in \beta\mathrm{Seq}_n \setminus \{a_n\}\}$ is in α_m. Let us suppose the last case. We have to prove that $m = n$ and there exists $\mathcal{U} \in \alpha_m$ such that $k(\mathcal{U}) \subseteq P_n$ and $k(j) = j$ for all $j \in \mathcal{U}$.

Any countable subset of a compact Hausdorff extremally disconnected space is C^*-embedded in it, see [2]. Hence P_n is C^*-embedded in $\beta\mathrm{Seq}_n$ by II.3, so the closure $\bar{R}$ of $R = P_n \setminus \{a_n\}$ in $\beta\mathrm{Seq}_n$ is homeomorphic to $\beta\omega$; denote by $h\colon \bar{R} \to \beta\omega$ the homeomorphism with $h(j) = j$ for all $j \in \omega$, $h(a_n) = \alpha_n$.

(b) Put

$$\mathcal{V} = k^{-1}(R).$$

We prove that $\mathcal{V}$ is in α_m. Let us suppose the contrary, i.e. the set $Z = k^{-1}(\beta\mathrm{Seq}_n \setminus P_n)$ is in α_m. Then, for every $Y \subseteq Z$ with $Y \in \alpha_m$, α_m is in the closure of Y; hence a_n is in the closure of $k(Y)$. Since α_n is supposed to be a weak P-point of ω^* (see II.8 (1)), a_n is an accumulation point of no countable subset of $\bar{R} \setminus R$, hence there exists $Y \subseteq Z$ in α_m such that

$$k(Y) \cap \bar{R} = \emptyset.$$

(c) Since R is homeomorphic to ω, we can choose, for every $j \in R$, a clopen neighbourhood $\mathcal{O}_j$ in $\beta\mathrm{Seq}_n$ such that

$$\mathcal{O}_j \cap R = \{j\} \text{ and } \mathcal{O}_j \cap \mathcal{O}_{j'} = \emptyset \text{ for } j \neq j'.$$

Then

for each $\mathcal{W} \subseteq Y$ in α_m, a_n is not in the closure of

$$J(\mathcal{W}) = \{j \in R|\ \mathcal{O}_j \cap k(\mathcal{W}) = \emptyset\}.$$

In fact, let us suppose that a_n is in the closure of $J(\mathcal{W})$ for some $\mathcal{W} \subseteq Y$ in α_m. Since each point j of $J(\mathcal{W})$ has a neighbourhood disjoint with $k(\mathcal{W})$, namely $\mathcal{U}_j$,

$$\overline{k(\mathcal{W})} \cap J(\mathcal{W}) = \emptyset.$$

Conversely,

$$k(\mathcal{W}) \cap \overline{J(\mathcal{W})} = \emptyset$$

because $\mathcal{W} \subseteq Y$, $J(\mathcal{W}) \subseteq R$ and $k(Y) \cap \bar{R} = \emptyset$.
Since $\beta\mathrm{Seq}_n$ is extremally disconnected, necessarily

$$\overline{k(\mathcal{W})} \cap \overline{J(\mathcal{W})} = \emptyset.$$

Since $a_n \in \overline{k(\mathcal{W})}$, a_n cannot be an element of $\overline{J(\mathcal{W})}$, which is a contradiction.

(d) Let $h\colon \bar{R} \to \beta\omega$ be as in (a). For each $w \in K(\mathcal{W})$ define

$$t(w) = j \quad \text{whenever } w \in \mathcal{O}_j \text{ (such } j \text{ is unique!)},$$
$$t(w) = a_n \quad \text{if no such } j \text{ exists},$$
$$t(a_n) = a_n.$$

By the conclusion of (b), $\varkappa = h \circ t \circ k$ is a continuous mapping of $P(\alpha_m)$ into $P(\alpha_n)$ such that $\varkappa^{-1}(\omega) \in \alpha_m$. Since α_m and α_n are supposed to be incomparable whenever $m \neq n$ (see II.8. (2)), necessarily $m = n$. Then there exists $\mathcal{W} \in \alpha_m$ such that $\varkappa(j) = j$ for all $j \in \mathcal{W}$. We may suppose $\mathcal{W} \subseteq Y$, i.e. $k(\mathcal{W}) \cap \bar{R} = \emptyset$. For every $j \in \mathcal{W}$ choose open (in $\beta\,\mathrm{Seq}_n$) neighbourhoods M_j of $k(j)$ and N_j of $t(k(j))$ such that $M_j \cap N_j = \emptyset$ and put $M = \bigcup_{j\in\mathscr{W}} (M_j \cap \mathcal{O}_j)$, $N = \bigcup_{k\in\mathscr{W}} (N_j \cap \mathcal{O}_j)$. Then M and N are disjoint open sets, hence both $k(\mathscr{W}) \subseteq M$ and $t(k(\mathscr{W})) \subseteq N$ are open in $k(\mathscr{W}) \cup t(k(\mathscr{W}))$. Since $k(\mathscr{W}) \cup t(k(\mathscr{W}))$ is countable, it is C^*-embedded in $\beta\,\mathrm{Seq}_n$. This is a contradiction again because the function equal to zero on $k(\mathscr{W})$ and equal to 1 on $t(k(\mathscr{W}))$ cannot be continuously extended to a_n. We conclude that

$$\mathscr{V} = k^{-1}(R) \text{ is in } \alpha_m .$$

(e) If $\mathscr{V} = k^{-1}(R)$ is in α_m, then necessarily $m = n$, hence there exists $\mathscr{U} \subseteq \mathscr{V}$ in α_m such that $k(j) = j$ for all $j \in \mathscr{U}$, which has to be proved.

Remark. In the preliminary version of this lemma, the ultrafilters α_n were presumed to be selective. The present proof, using only weak P-points, was done by Petr Simon.

II.9. Let $\mathscr{A} = \{\alpha_n |\ n \in \omega\}$ be as in II.8 and $X(f)$ be as in II.2.

Lemma. *Let there exist a homeomorphism h of $P(\alpha_n)$ into $X(f)$ such that $h(\alpha_n) \in S(X(f))$. Then $f(n) > 0$ and*

$\pi_l(h(\alpha_n)) \in C_{\bar{l}}$ for all $l \in L(f)$ and

$\pi_l(h(\alpha_n)) = a_{\bar{l}}$ for at least one $l \in L(f)$ with $\bar{l} = n$.

Proof. If $\pi_l(\alpha_n) \in X_{\bar{l}} \setminus C_{\bar{l}}$ for some $l \in L(f)$, then $h(\alpha_n) \notin S(X(f))$, which is a contradiction. Hence $\pi_l(h(\alpha_n)) \in C_{\bar{l}}$ for all $l \in L(f)$. Let us suppose that $\pi_l(h(\alpha_n)) \neq a_{\bar{l}}$ for all $l \in L(f)$ with $\bar{l} = n$ (this happens necessarily whenever $f(n) = 0$). By II.8, for every $l \in L(f)$ there exists $\mathscr{U}_l \in \alpha_n$ such that $\pi_l(h(\mathscr{U}_l)) \subseteq C_{\bar{l}}$. Denote by $\tilde{X}_k$ the space on the same set as X_k such that C_k is its clopen subspace and $\tilde{X}_k$ is discrete outside, let $i_k\colon \tilde{X}_k \to X_k$ be the identical map. Then $\pi_l \circ h$ factorizes through $i_{\bar{l}}$ so that h factorizes through the identical map $i\colon \prod_{l\in L(f)} \tilde{X}_{\bar{l}} \to \prod_{l\in L(f)} X_{\bar{l}}$. Since $\prod_{l\in L(f)} \tilde{X}_{\bar{l}}$ is metrizable, this is impossible.

II.10. Now, we prove two auxiliary lemmas.

Lemma A. *Let G be a G_δ-subset of $\omega^* = \beta\omega\setminus\omega$; let α be in G. Let $g\colon \beta\omega \to \beta\omega$ be a continuous map such that $g(z) = z$ for all $z \in G$. Then there exists $\mathscr{U} \subseteq \omega$ in the ultrafilter α such that $g(n) = n$ for all $n \in \mathscr{U}$.*

Proof. Let G_k, $k = 1, 2, \ldots$, be open in $\beta\omega$ such that $G = \omega^* \cap \bigcap_{k=1}^{\infty} G_k$. For each k find $\mathscr{V}_k$ in α such that $\overline{\mathscr{V}}_k \subseteq G_k$. Then $H = \omega^* \cap \bigcap_{k=1}^{\infty} \overline{\mathscr{V}}_k$ (where the bars denote the closures in $\beta\omega$) is a subset of G so that $g(z) = z$ for all $z \in H$. We may suppose $\mathscr{V}_{k+1} \subseteq \mathscr{V}_k$ and $\bigcap_{k=1}^{\infty} \mathscr{V}_k = \emptyset$. Then there exists k_0 such that $g(n) = n$ for all $n \in \mathscr{V}_{k_0}$. In fact, let us suppose the contrary, i.e., for every k there exists $x_k \in \mathscr{V}_k$ such that $g(x_k) \neq x_k$. Then $g(z) \neq z$ for every z in the closure $\overline{\mathscr{W}}$ of an infinite $\mathscr{W} \subseteq \{x_k |\ k = 1, 2, \ldots\}$. This is a contradiction because $\overline{\mathscr{W}} \cap H \neq \emptyset$.

Lemma B. *Let $\vartheta\colon (\beta\omega)^2 \to \beta\omega$ be a continuous map. Then there exists no G_δ-subset G of ω^* and its element α such that*

$$\vartheta(z, \alpha) = z \text{ and } \vartheta(\alpha, z) = z \quad \text{for all} \quad z \in G .$$

Proof. Let us suppose that there exists a G_δ-subset G of ω^* and $\alpha \in G$ with the above property. Put $g(-) = \vartheta(-, \alpha)\colon \beta\omega \to \beta\omega$. By Lemma A, there exists $\mathscr{U} \in \alpha$ such that $g(n) = n$ for all $n \in \mathscr{U}$. Since the map $\vartheta(n, -)\colon \beta\omega \to \beta\omega$ sends α to n, there exists $\mathscr{V}_n \in \alpha$ such that $\vartheta(n, m) = n$ for every $m \in \mathscr{V}_n$. Hence for every $n \in \mathscr{U}$ and every $x \in \overline{\mathscr{V}}_n \setminus \mathscr{V}_n$, $\vartheta(n, x) = n$. Put $H = \bigcap_{n\in\mathscr{U}} (\overline{\mathscr{V}}_n \setminus \mathscr{V}_n)$. Then H is a G_δ-subset of ω^*, $\alpha \in H$, and for every $z \in H$,

$$\vartheta(n, z) = n \quad \text{for all} \quad n \in \mathscr{U},$$

hence $\vartheta(\alpha, z) = \alpha$. Since $\alpha \in H \cap G$ and $H \cap G$ is G_δ in ω^*, $H \cap G \setminus \{\alpha\}$ is nonempty. Thus for $\gamma \in H \cap G \setminus \{\alpha\}$ we have $\vartheta(\alpha, \gamma) = \alpha$, which is a contradiction.

II.11. Let $f \in \omega^\omega$, $\mathscr{H}_n^x(X(f))$, $\boldsymbol{R}$ be as in II.6. Let us denote for every $h \in \mathscr{H}_n^x(X(f))$

$$p(h) = \{l \in L(f) |\ \bar{l} = n \text{ and there exists } \mathscr{U} \in \alpha_n \text{ such that } \pi_l \circ h \text{ maps } \mathscr{U} \text{ into } P_n \setminus \{a_n\}\}.$$

Lemma. *Let h_1, h_2 be in $\mathscr{H}_n^x(X(f))$. Then*

$$h_1 \boldsymbol{R} h_2 \quad \text{iff} \quad p(h_1) \cap p(h_2) \neq \emptyset .$$

Proof. If $p(h_1) \cap p(h_2) \neq \emptyset$, choose l in their intersection and define $c\colon X(f) \xrightarrow{\pi_l} X_{\bar{l}} \xrightarrow{r} \beta\,\mathrm{Seq}_n$, where r is the retraction sending any point of C_n to a_n. This map c and a suitable $\mathscr{U} \in \alpha_n$ fulfil the requirements of the definition of $\boldsymbol{R}$, hence $h_1 \boldsymbol{R} h_2$. To prove the converse, let us suppose that $p(h_1) \cap p(h_2) = \emptyset$ but there exists a continuous map $c\colon X(f) \to \beta\,\mathrm{Seq}_n$ and a set $\mathscr{U} \in \alpha_n$ such that $c(x) = a_n$, $c(h_1(j)) = c(h_2(j))$ for every $j \in \mathscr{U}$ and $c \circ h_1$ is a homeomorphism of $\mathscr{U} \cup \{\alpha_n\}$ into $\beta\,\mathrm{Seq}_n$. By II.8, we may suppose that $c(h_1(j)) = j \in P_n \setminus \{a_n\}$ for all $j \in \mathscr{U}$. Since both $p(h_1)$ and $p(h_2)$ are finite, there exists $\mathscr{V} \subseteq \mathscr{U}$, $\mathscr{V}$ in α_n, such that

$$\text{for } i = 1, 2 \text{ and for all } l \in p(h_i), \quad \pi_l(h_i(j)) = j \in P_{\bar{l}} \quad \text{for all } j \in \mathscr{V} .$$

For every $l \in L(f)$ choose a countable system $\{\mathcal{O}_{l,k} |\ k = 1, 2, \ldots\}$ of clopen subsets of $C_{\bar{l}}$ such that $\bigcap_{k=1}^{\infty} \mathcal{O}_{l,k} = \{a_{\bar{l}}\}$ and find $\mathscr{U}_{l,k} \subseteq \mathscr{V}$, $\mathscr{U}_{l,k}$ in α_n such that

for every $l \in L(f) \setminus p(h_1) \cup p(h_2)$,

$$\pi_l(h_i(\mathscr{U}_{l,k})) \subseteq \mathcal{O}_{l,k} \quad \text{for} \quad i = 1, 2 ,$$

for every $l \in p(h_i)$,

$$\pi_l(h_j(\mathscr{U}_{l,k})) \subseteq \mathcal{O}_{l,k} \quad \text{for} \quad \{i, j\} = \{1, 2\}$$

and put

$$G = \bigcap_{\substack{l \in L(f) \\ k=1,2,\ldots}} (\overline{\mathcal{U}_{l,k}} \setminus \mathcal{U}_{l,k}) .$$

Then G is a G_δ-subset of $\beta\mathcal{V}\setminus\mathcal{V}$ and $\alpha_n \in G$. Let h_i^*: $\beta(\mathcal{V}) \to X(f)$ be the continuous extension of h_i: $\mathcal{V} \cup \{\alpha_n\} \to X(f)$, $i = 1, 2$. Let us define a map h: $(\beta\mathcal{V})^2 \to X(f)$ by

$$\pi_l(h(y_1, y_2)) = \pi_l(h_1^*(y_1)) \quad \text{for} \quad l \in p(h_1) ,$$
$$\pi_l(h(y_1, y_2)) = \pi_l(h_2^*(y_2)) \quad \text{for} \quad l \in p(h_2) ,$$
$$\pi_l(h(y_1, y_2)) = \pi_l(x) = a_{\bar{l}} \quad \text{for} \quad l \in L(f)\setminus p(h_1) \cup p(h_2) .$$

Since each $\pi_l \circ h$ is continuous, h is continuous. One can verify that $h(z, \alpha_n) = h_1^*(z)$ and $h(\alpha_n, z) = h_2^*(z)$ for all $z \in G$. Put

$$\vartheta\colon (\beta\mathcal{V})^2 \xrightarrow{h} X(f) \xrightarrow{c} \beta\,\mathrm{Seq}_n \xrightarrow{r} \beta\omega ,$$

where r is the retraction sending any string $q_1 \ldots q_k$ with $k > 1$ to q_1. Then ϑ is a continuous map which sends (z, α_n) to z and (α_n, z) to z for all $z \in G$. This contradicts II.10B.

II.12. Let $f \in \omega^\omega$, let $\boldsymbol{S}(X(f))$, $\mathcal{H}_n^x(X(f))$ and $f_x(n)$ be as in II.6.

Lemma. *For every $x \in \boldsymbol{S}(X(f))$ and $n \in \omega$, the number $f_x(n)$ is precisely the number of all $l \in L(f)$ such that $\pi_l(x) = a_n$.*

Proof. By II.9, $p(h)$ is nonempty for every $h \in \mathcal{H}_n^x(X(f))$. If $h \in \mathcal{H}_n^x(X(f))$ and $l \in p(h)$, then $\pi_l(x) = a_n$. Hence by II.11, $f_x(n)$ is the maximal cardinality of a pairwise disjoint system of nonempty subsets of the set $T_{x,n} = \{l \in L(f) |\ \bar{l} = n$ and $\pi_l(x) = a_n\}$, which is equal to $\operatorname{card} T_{x,n}$.

The proof of Theorem 1. Let us finish the proof of Theorem 1. Let $a(f)$ be the point of $X(f)$ with all coordinates equal to $a_{\bar{l}}$. If $x \in \boldsymbol{S}(X(f))\setminus\{a(f)\}$, then there exists $l_0 \in L(f)$ such that $\pi_{l_0}(x)$ is in $C_{\bar{l}_0}\setminus\{a_{\bar{l}_0}\}$. Then any neighbourhood of x in $X(f)$ contains a point $y \in \boldsymbol{S}(X(f))\setminus\{x\}$ with $f_y = f_x$. In fact, choose y such that

$$\pi_l(y) = \pi_l(x) \text{ for all } l \in L(f)\setminus\{l_0\} ,$$

$\pi_{l_0}(y)$ is in $C_{\bar{l}_0}\setminus\{a_{\bar{l}_0}\}$, sufficiently near to $\pi_{l_0}(x)$ but distinct from it.

Clearly, $f_x < f_{a(f)}$ for every $x \in \boldsymbol{S}(X(f))\setminus\{a(f)\}$. Hence

$$\boldsymbol{M}(X(f)) = \{a(f)\} \quad \text{and} \quad f_{a(f)} = f .$$

This implies (+), by Observation II.6. This completes the proof of Theorem 1, by II.5 and (*) in II.2.

Remark. Clearly, if $A \subseteq \omega^\omega$, then $A = \{f_x |\ x \in \boldsymbol{M}(X(A))\}$.

III. Compactifications and the proof of Theorem 2.

III.1. In the proof of Theorem 2, we use the construction described in II. If A is a nonvoid countable subset of ω^ω, we denote by $X(A)$ the space constructed in II.2, by means of a system $\mathcal{A} = \{\alpha_n |\ n \in \omega\}$ of ultrafilters on ω, satisfying (1) and (2) of II.8.

First, we show that "the set A can be recognized from the topological structure of a suitable compactification of the space $X(A)$". Let $\boldsymbol{S}(Z)$ and $\boldsymbol{M}(Z)$ be as in II.6. Let $W(A)$ be a compactification of $X(A)$. Since each copy of $X(f)$ in $X(A)$ is clopen in $W(A)$, the set $\boldsymbol{M}(X(A))$ is contained in $\boldsymbol{M}(W(A))$ and each point of $\boldsymbol{M}(X(A))$ is an isolated point of $\boldsymbol{M}(W(A))$. Let us suppose that the compactification $W(A)$ fulfils that

(**) $\boldsymbol{M}(X(A))$ is precisely the set of all isolated points of $\boldsymbol{M}(W(A))$.

Then, by II.6 – II.12, A is precisely the set of all f_x with x being an isolated point of $\boldsymbol{M}(W(A))$. We conclude that for any $A, B \in \boldsymbol{U}$, $X(A)$ and $X(B)$ being as in II.2 and $W(A)$, $W(B)$ being their compactifications which satisfy (**),

$W(A)$ is homeomorphic to a clopen subset of $W(B)$ implies that $A \subseteq B$.

III.2. To prove Theorem 2, it is sufficient to construct compactifications $W(A)$ for each $A \in \boldsymbol{U}$ such that (**) is fulfilled and $W(A) \times W(B)$ is always homeomorphic to $W(A+B)$. Such construction is unknown, in general; this is the reason why Theorem 2 speaks only about a sum-productive representation of finite Abelian groups with the discrete order.

First, we present a construction of the compactifications in the case that the represented group is a cyclic one (with the discrete order), say $c_{t-1} = \{1, \ldots, t-1\}$, $t > 1$ (where the group-operation of c_{t-1} is the addition modulo $t-1$). Let φ: $c_{t-1} \to \boldsymbol{U}$ be an embedding which satisfies (a), (b) of II.1. Put $A = \varphi(1)$ and for every positive natural number denote

$$kA = A + \ldots + A \ (k \text{ times}) .$$

For every k choose a bijection

$$b_k\colon (\omega \times A)^k \to \omega \times kA$$

such that

$$b_k((n_1, f_1), \ldots, (n_k, f_k)) = (n, f) \quad \text{where } n \text{ is an element of } \omega \text{ and } f_1 + \ldots + f_k = f .$$

Since $1 \equiv t \pmod{t-1}$, we have $A = tA$; hence b_t is a bijection of $(\omega \times A)^t$ onto $\omega \times A$.

Let $X(A)$ be as in II.2; we can express it as

$$X(A) = \coprod_{(n,f) \in \omega \times A} (X(f))_n ,$$

where $(X(f))_n$ is a copy of $X(f)$. Then b_k defines a homeomorphism h_k of $(X(A))^k$ onto $X(kA)$ as follows: if $(x_1, \ldots, x_k) \in (X(A))^k$, then for each $s = 1, \ldots, k$ find the unique pair (n_s, f_s) such that $x_s \in (X(f_s))_{n_s}$, i.e. x_s is a tuple $\{(x_s)_l |\ l \in L(f_s)\}$, where $L(f_s)$ is as in II.7; then $h_k(x_1, \ldots, x_k)$ is the tuple $\{(x_s)|_l\ s = 1, \ldots, k,\ l \in L(f_s)\}$ situated in the copy $(X(f))_n$ with $(n, f) = b_k((n_1, f_1), \ldots, (n_k, f_k))$. Since b_t is a bijection of $(\omega \times A)^t$ onto $\omega \times A$, h_t is a homeomorphism of $(X(A))^t$ onto $X(A)$.

Let $W_0 = \{\xi\} \cup X(A)$ be a one-point compactification of $X(A)$, let us denote by

$$h_{0,1}\colon W_1 = W_0^t \to W_0$$

the continuous extension of the homeomorphism h_t; $h_{0,1}$ sends the whole set $W_0^t \setminus (X(A))^t$ to ξ. Consider the following chain (= inverse spectrum) $\mathscr{W}$ over ω:

$$W_0 \xleftarrow{h_{0,1}} W_1 \xleftarrow{h_{1,2}} W_2 \xleftarrow{h_{2,3}} W_3 \longleftarrow \ldots,$$

where $W_{k+1} = (W_k)^t$ and $h_{k,k+1} = (h_{k-1,k})^t$. Let W be its inverse limit, let X be its subspace, which is an inverse limit of the following subchain $\mathscr{X}$ of $\mathscr{W}$:

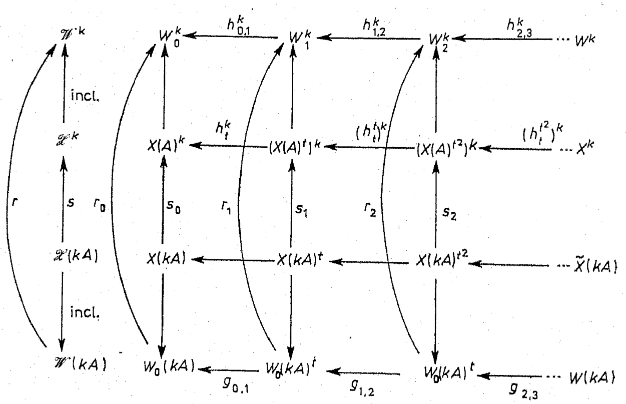
$$X(A) \xleftarrow{h_t} X(A)^t \xleftarrow{(h_t)^t} X(A)^{t^2} \xleftarrow{(h_t)^{t^2}} X(A)^{t^3} \longleftarrow \ldots$$

Since all the $(h_t)^{t^k}$'s are homeomorphisms, X is homeomorphic to $X(A)$. Clearly, W is a compactification of X. The product of t copies of the chain $\mathscr{W}$ is $\mathscr{W}$ again, shifted one step to the right, hence W^t is homeomorphic to W (because products commute with inverse limits of chains).

III.3. We are going to prove that $\{W^k |\ k = 1, \ldots, t-1\}$ is a sum-productive representation of the group $c_{t-1} = \{1, \ldots, t-1\}$ in the class $\boldsymbol{BS}$. Clearly, each space W^k is really in $\boldsymbol{BS}$; by III.2, W^t is homeomorphic to W. Hence it is sufficient to show that

if $k, p \in \{1, \ldots, t-1\}$ and W^k is homeomorphic to a clopen subset of W^p, then $k = p$.

Let $k \in \{1, \ldots, t-1\}$ be given. Let us denote by s_0 the homeomorphism of $X(kA)$ onto $X(A)^k$, inverse to h_k. Let $W_0(kA)$ be a compactification of $X(kA)$ such that there is a homeomorphism r_0 of $W_0(kA)$ onto W_0^k, which extends s_0. Consider the following diagram.



In the above diagram, we define

$$s_n\colon X(kA)^{t^n} \xrightarrow{s_0^{t^n}} (X(A)^k)^{t^n} \xrightarrow{p_n} (X(A)^{t^n})^k,$$

where p_n permutes the coordinates and analogously

$$r_n\colon W_0(kA)^{t^n} \xrightarrow{r_0^{t^n}} (W_0^k)^{t^n} \xrightarrow{q_n} (W_0^{t^n})^k = W_n^k$$

where q_n permutes the coordinates. Finally, we define

$$g_{n,n+1} = r_n^{-1} \circ h^k_{n,n+1} \circ r_{n+1}.$$

Clearly, $g_{n,n+1} = (g_{n-1,n})^t$.

Let us denote by $\tilde{X}(kA)$ an inverse limit of $\mathscr{X}(kA)$ and by $W(kA)$ an inverse limit of $\mathscr{W}(kA)$. Then $\tilde{X}(kA)$ is homeomorphic to $X(kA)$ and $W(kA)$ is its compactification. Hence it is sufficient to prove (**) for $W(kA)$.

The inverse limit $W(kA)$ is the subspace of $\prod_{n=0}^{\infty} (W_0(kA))^{t^n}$ consisting of all sequences $w = \{w_n\}_{n=0}^{\infty}$ such that $g_{n-1,n}(w_n) = w_{n-1}$; for each n, $r_n(w_n)$ is a k-tuple of elements of $W_n = (W_0)^{t^n}$, say $\{w_{n,i}|\ i = 1, \ldots, k\}$; and each $w_{n,i}$ is a t^n-tuple of elements of W_0, say $\{w_{n,i,j}|\ j = 1, \ldots, t^n\}$. If for every $n \in \omega$, $i = 1, \ldots, k$ and $j = 1, \ldots, t^n$, $w_{n,i,j}$ is distinct from ξ, i.e. it belongs to $X(A)$, then $w \in \tilde{X}(kA)$. If there exists n_0 such that $w_{n_0,i,j} = \xi$ for some $i \in \{1, \ldots, k\}$ and $j \in \{1, \ldots, t^{n_0}\}$, then $w_{n_0} \notin (X(kA))^{t^{n_0}}$; hence $w_n \notin (X(kA))^{t^n}$ for all $n \in \omega$. Let us denote

$$\boldsymbol{K}(W(kA)) = \{w \in \boldsymbol{S}(W(kA))|\ \text{for every } n \in \omega,\ i \in \{1, \ldots, k\},\ j \in \{1, \ldots, t^n\},$$
$$\text{either } w_{n,i,j} = \xi \text{ or } w_{n,i,j} \in \boldsymbol{M}(X(A))\},$$

$$\boldsymbol{E}(W(kA)) = \boldsymbol{S}(W(kA)) \setminus (\boldsymbol{K}(W(kA)) \cup \tilde{X}(kA)).$$

We prove that no point of $\boldsymbol{M}(W(kA)) \setminus \tilde{X}(kA)$ is an isolated point of $\boldsymbol{M}(W(kA))$ such that we prove that

if $w \in \boldsymbol{K}(W(kA)) \setminus \tilde{X}(kA)$, then any its neighbourhood in $W(kA)$ contains infinitely many points of $\boldsymbol{M}(W(kA))$ and

if $w \in \boldsymbol{E}(W(kA))$, then w is not in $\boldsymbol{M}(W(kA))$.

The first statement is evident: any neighbourhood of ξ in W_0 contains infinitely many points of $\boldsymbol{M}(X(A))$; if $w \in \boldsymbol{K}(W(kA)) \setminus \tilde{X}(kA)$, then each $w_{n,i,j}$ is either ξ or in $\boldsymbol{M}(X(A))$ and at least one is really equal to ξ (and each point $v \in W(kA)$ with all $v_{n,i,j}$ being in $\boldsymbol{M}(X(A))$ is in $\boldsymbol{M}(\tilde{X}(kA))$, hence in $\boldsymbol{M}(W(kA))$.

The second statement follows easily from the lemma below.

LEMMA. *Let w be in $\boldsymbol{E}(W(kA))$. Then for every neighbourhood $\mathscr{U}$ of w in $W(kA)$ there exists a homeomorphism τ of $W(kA)$ onto itself such that $\tau(w) \in \mathscr{U} \setminus \{w\}$.*

In fact, since $w \in S(W(kA))$, $\tau(w)$ is also in $S(W(kA))$; for every $n \in \omega$, $\mathcal{H}_n^w(W(kA))$ and $\mathcal{H}_n^{\tau(w)}(W(kA))$ are in one-to-one correspondence preserving the relation $\boldsymbol{R}$, hence $f_w = f_{\tau(w)}$; thus, w is not in $\boldsymbol{M}(W(kA))$.

Proof of the lemma. Since w is in $\boldsymbol{E}(W(kA))$, there exist $n_0 \in \omega$, $i_0 \in \{1, \dots, k\}$ and $j_0 \in \{1, \dots, t^{n_0}\}$ such that $\tilde{w} = w_{n_0, i_0, j_0}$ is in $X(A) \setminus \boldsymbol{M}(X(A))$. Find $f \in A$ and $p \in \omega$ such that $\tilde{w}$ is in the pth copy of $X(f)$ in $X(A)$, say $X(f)_p$. Since $w \in S(W(kA))$ and since $\tilde{w} \notin \boldsymbol{M}(X(f)_p)$, there exists its coordinate, say the lth one $\tilde{w}_l$, such that $\tilde{w}_l \in C_{\bar{l}} \setminus \{a_{\bar{l}}\}$ (where $C_{\bar{l}}$ and $a_{\bar{l}}$ are as in II.5). Choose a homeomorphism μ_l of $X_{\bar{l}}$ (where $X_{\bar{l}}$ is as in II.5) onto itself such that $\mu_l(y) = y$ for all $y \in X_{\bar{l}} \setminus C_{\bar{l}}$, $\mu_l(\tilde{w}_l) \neq \tilde{w}_l$, $\mu_l(\tilde{w}_{\bar{l}})$ is in $C_{\bar{l}} \setminus \{a_{\bar{l}}\}$ and sufficiently near to $\tilde{w}_l$. Then we define a homeomorphism μ_p of $X(f)_p$ onto itself such that for each point of $X(f)_p$ we change at most its lth coordinate and we use for this change the homeomorphism μ_l. Since $X(f)_p$ is clopen in W_0, we can extend μ_p to a homeomorphism μ of W_0 onto itself as follows: μ is identical outside $X(f)_p$ and equal to μ_p on $X(f)_p$. The special forms of μ and $h_{0,1}\colon W_0^t \to W_0$ admit to find a homeomorphism $\tilde{\mu}$ of W_0^t onto itself such that $h_{0,1} \circ \tilde{\mu} = \mu \circ h_{0,1}$ and that there exists $j \in \{1, \dots, t\}$ such that $\tilde{\mu}$ changes at most the jth coordinate of some points of W_0^t (in fact, μ is identical outside $X(f)_p$ and $h_{0,1}^{-1}(X(f)_p) = X(f_1)_{p_1} \times \dots \times X(f_t)_{p_t}$ with $((f_1, p_1), \dots, (f_t, p_t)) = b_1^{-1}(f, p)$; $h_{0,1}$ restricted to $X(f_1)_{p_1} \times \dots \times X(f_t)_{p_t}$ only collects coordinates in $L(f_1), \dots, L(f_t)$; hence the lth coordinate in $X(f)_p$ corresponds to precisely one coordinate in precisely one of the spaces $X(f_1)_{p_1}, \dots, X(f_t)_{p_t}$, say in $X(f_j)_{p_j}$, $j \in \{1, \dots, t\}$).

Now, we define a homeomorphism ϱ_{n_0} of $W_0^{t^{n_0}}$ onto itself such that we change at most the j_0th coordinate of points of $W_0^{t^{n_0}}$, by the homeomorphism μ. The special forms of μ and $h_{n,n+1} = (h_{0,1})^{t^n}$ admit to find a homeomorphism ϱ_n of $W_0^{t^n}$ onto itself for each $n \geqslant n_0$ such that ϱ_n changes at most one coordinate of points in $W_0^{t^n}$ and $h_{n,n+1} \circ \varrho_{n+1} = \varrho_n \circ h_{n,n+1}$. If we put $\tau_n = (\varrho_n)^k \circ r_n$, then for all $n \geqslant n_0$, $g_{n,n+1} \circ \tau_{n+1} = \tau_n \circ g_{n,n+1}$ and the homeomorphism $\tau = \lim_{n \in \omega} \tau_n$ of $W(kA)$ onto itself has the required properties.

III.4. Now, we present a generalization of III.2 to obtain a sum-productive representation of a given finite Abelian group G in the class $\boldsymbol{BS}$.

First, let us express G as $\prod_{i=0}^{m-1} c_{t_i-1}$, where c_{t_i-1} are cyclic groups of order $t_i - 1$, i.e. any element g of G is an m-tuple $g = (g_0, \dots, g_{m-1})$ with $g_i \in \{1, \dots, t_i - 1\}$. We choose a collection $\mathscr{A} = \{\alpha_{i,n} \mid i \in m, n \in \omega\}$ of ultrafilters on ω such that the whole collection fulfils (1) and (2) of II.8 and construct the sum-productive representation of the cyclic group c_{t_i-1} by means of $\mathscr{A}_i = \{\alpha_{i,n} \mid n \in \omega\}$ as in II. – III.2, i.e. we choose $\varphi_i\colon c_{t_i-1} \to \boldsymbol{U}$, denote $A_i = \varphi_i(1)$, construct $X(A_i)$ by means of $\mathscr{A}_i$ as in II.2–5 and for $k \in \{1, \dots, t_i - 1\}$, W_i^k is a compactification of $X(kA_i)$ constructed by means of the chain $\mathscr{W}_i^k$ as in III.2. For each $g = (g_0, \dots, g_{m-1}) \in G$ we put

$$r(g) = \prod_{i \in m} W_i^{g_i}.$$

Then $\{r(g) \mid g \in G\}$ is a sum-productive representation of G in $\boldsymbol{BS}$. In fact, the $r(g)$'s are in $\boldsymbol{BS}$, evidently; $r(g+g')$ is homeomorphic to $r(g) \times r(g')$ because for each $i \in m$, $W_i^{g_i} \times W_i^{g_i'}$ is homeomorphic to any W_i^z with $z \equiv (g_i + g_i') \bmod t_i - 1$. Moreover, all the sets $g_i A_i$, $i \in m$, can be recognized from the topology of $r(g)$. We have only to investigate the sets $\mathcal{H}_{i,n}^x(Z)$ of all homeomorphisms h of $P(\alpha_{i,n})$ into Z (with $h(\alpha_{i,n}) = x \in S(Z)$), the relation $\boldsymbol{R}_{i,n}$ on $\mathcal{H}_{i,n}^x(Z)$ and functions $f_{i,x}$ as in II.6, but depending on a given $i \in m$. Finally, we define

$$\boldsymbol{M}(Z) = \{x \in S(Z) \mid \text{there exists a neighbourhood } \mathscr{U} \text{ of } x \text{ in } Z \text{ such that for every } y \in (\mathscr{U} \cap S(Z)) \setminus \{x\},\ f_{i,y} \leqslant f_{i,x} \text{ for all } i \in m \text{ and } f_{i,y} < f_{i,x} \text{ for at least one } i \in m\}.$$

Similarly as in II.–III.2, we can see that

$$g_i A_i = \{f_{i,x} \mid x \text{ is an isolated point of } \boldsymbol{M}(r(g))\}.$$

This shows that if $g \neq g'$, then $r(g)$ is not homeomorphic to a clopen subset of $r(g')$.

IV. Concluding Remarks. Let us present some strengthenings of the Theorem 1 and Theorem 2.

IV.1. First, let us recall that, in the proof of Theorem 1, we constructed a sum-productive representation of the ordered semigroup $\boldsymbol{U}$ in $\prod_{\omega} \boldsymbol{BS}$. The semigroup $\boldsymbol{U}$ contains all countable ordered commutative semigroups, but also some natural uncountable ones.

Examples. (a) *The additive group of all real numbers with its natural order* Let $\varphi\colon (Q, +, \leqslant) \to \boldsymbol{U}$ be an embedding of the additive group of all rational numbers (with their natural order) into $\boldsymbol{U}$. For every real number r put

$$\bar{\varphi}(r) = \bigcup_{\substack{q \in Q \\ q \leqslant M}} \varphi(q).$$

Then $\bar{\varphi}$ is an embedding of the additive group $(R, +, \leqslant)$ of all real numbers with their natural order into $\boldsymbol{U}$; hence $(R, +, \leqslant)$ has a sum-productive representation in $\prod_{\omega} \boldsymbol{BS}$.

(b) *Many nonhomeomorphic square roots.* Let $(\exp\omega, \oplus, \subseteq)$ be the semigroup of all subsets of ω with an operation $\oplus$ defined by

$$s_1 \oplus s_2 = \emptyset \quad \text{for all} \quad s_1, s_2 \subseteq \omega$$

and ordered by the inclusion. Let $(S, \oplus, \subseteq)$ be its ordered subsemigroup consisting of all finite subsets of ω. Since S is countable, there exists an embedding $\varphi\colon (S, \oplus, \subseteq) \to \boldsymbol{U}$. For any $t \subseteq \omega$ put

$$\bar{\varphi}(t) = \bigcup_{\substack{t \in S \\ s \subseteq t}} \varphi(s).$$

Then $\bar{\varphi}$ is an embedding of $(\exp\omega, \oplus; \subseteq)$ into $\boldsymbol{U}$; thus, $(\exp\omega, \oplus, \subseteq)$ has a sum-productive representation $\{X(s)|\ s \in \exp\omega\}$ in $\coprod_{\omega} \boldsymbol{BS}$. Then the space $X = X(\emptyset)$ has $2^{\aleph_0}$ nonhomeomorphic square roots in $\coprod_{\omega} \boldsymbol{BS}$: each $X(s)$ with $s \subseteq \omega$ fulfils

$$X(s) \times X(s) \simeq X.$$

IV.2. Both Theorem 1 and Theorem 2 can be strengthened as follows: given a space P in $\boldsymbol{C}$, then $(S, \oplus, \leqslant)$ has a sum-productive representation $\{X(s)|\ s \in S\}$ in $\boldsymbol{C}$ such that the space P is a retract of each representing space $X(s)$, where either

$$\boldsymbol{C} = \coprod_{\omega} \boldsymbol{BS} \quad \text{and} \quad (S, \oplus, \leqslant) = \boldsymbol{U}$$

or $\boldsymbol{C} = \boldsymbol{BS}$ and $(S, \oplus, \leqslant)$ is an arbitrary finite Abelian group.

In fact, let N be a space in $\boldsymbol{C}$ such that any its point has an uncountable character. If $A \in \boldsymbol{U}$, replace the space $X(A)$ in II.2 by the space

$$Y(A) = \coprod_{n \in \omega} (P \times N)^n \times X(A).$$

Clearly, P is a retract of $Y(A)$: we choose $y_0 \in N$, $x_0 \in X(A)$ and embed P onto $P \times \{y_0\} \times \{x_0\}$; the summand $(P \times N)^1 \times X(A)$ can be retracted onto $P \times \{y_0\} \times \{x_0\}$ by the first projection, the other summands are mapped into it e.g. by a constant map. Since each $X(A)$ is homeomorphic to a sum of ω copies of itself $Y(A) \times Y(B)$ is homeomorphic to $Y(A+B)$, for every $A, B \in \boldsymbol{U}$. If $n = 0$, $(P \times N)^0$ is a single point, so we can identify $(P \times N)^0 \times X(A)$ with $X(A)$. Since any point of $(P \times N)^n \times X(A)$ with $n \geqslant 1$ has an uncountable character, we have

$$\boldsymbol{S}(Y(A)) = \boldsymbol{S}(X(A)) \quad \text{and} \quad \boldsymbol{M}(Y(A)) = \boldsymbol{M}(X(A)).$$

Thus, $\{Y(A)|\ A \in \boldsymbol{U}\}$ is a sum-productive representation of $\boldsymbol{U}$ in $\coprod_{\omega} \boldsymbol{BS}$ by spaces with the given retract P. (We can combine this result with IV.1; we obtain e.g. that

every space $P \in \coprod_{\omega} \boldsymbol{BS}$ can be embedded as a retract into a space $X \in \coprod_{\omega} \boldsymbol{BS}$ which has $2^{\aleph_0}$ nonhomeomorphic square roots in $\coprod_{\omega} \boldsymbol{BS}$.)

If P is in $\boldsymbol{BS}$ and $A = tA$, we can construct a compactification of $Y(A)$ similarly as in III: we choose a homeomorphism h of $Y(A)^t$ onto $Y(A)$ such that it maps $X(A)^t$ onto $X(A)$ as in III.1 and define a one-point compactification W_0 of $Y(A)$ and $h_{0,1}$: $W_0^t \to W_0$ extending h. Then we construct the chain $\mathscr{W}$ as in III. The limit W of $\mathscr{W}$ has P as its retract (in fact, choose $f \in A$; then $P \times N \times X(f)$ is clopen in W and P is its retract). The proof that $\{W, W^2, \ldots, W^{t-1}\}$ is a sum-productive representation of the cyclic group c_{t-1} is quite analogous to III. as well as the step from the cyclic groups to the finite Abelian groups.

IV.3. Let $\{X(s)|\ s \in S\}$ and $\{X'(s)|\ s \in S\}$ both be sum-productive representations of an ordered commutative semigroup $(S, \oplus, \leqslant)$. We say that these representations are nonhomeomorphic if none of the spaces $X(s)$ is homeomorphic to any of the spaces $X'(s)$. All the results of this paper can be strengthened in the way that there are many nonhomeomorphic representations. Let $\mathscr{A} = \{\alpha_{i,n}|\ i \in I, n \in \omega\}$ be a system of ultrafilters on ω such that (1) and (2) of II.8 are satisfied. Let us construct a sum-productive representation $\{X_i(s)|\ s \in S\}$ as in II. and III. by means of the system $\mathscr{A}_i = \{\alpha_{i,n}|\ n \in \omega\}$. Similarly as in III.3, one can prove that if $i \neq i'$, then the representations $\{X_i(s)|\ s \in S\}$ and $\{X_{i'}(s)|\ s \in S\}$ are nonhomeomorphic. Since there exists $\mathscr{A} = \{\alpha_{i,n}|\ i \in I, n \in \omega\}$ with (1) and (2) of II.8 such that $\operatorname{card} I \geq 2^{\aleph_0}$, by [6], we obtain that there is at least $2^{\aleph_0}$ nonhomeomorphic representations.

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MATH. INSTITUTE OF THE CHARLES UNIVERSITY
18600 Praha 8, Sokolowská 83, Czechoslovakia

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