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On some dynamical properties of S-unimodal maps on an interval

by

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Abstract. A globally expanding mapping is introduced. It has a uniform hyperbolic structure on the set of periodic points and on the set of preimages of the critical point. For S-unimodal mappings the existence of one such structure is equivalent to the existence of another. When the iterates of the critical point are away from the critical point itself, then the mapping is globally expanding.

0. Introduction. The aim of this paper is to present some results on the dynamics of S-unimodal mappings of an interval. The results are related to the results of Collet and Eckmann [1], Guckenheimer [3] and Misiurewicz [4].

In Section 1 we introduce two notions:

I. *global expanding* — which means that the length of every interval with two consecutive critical points of f^n as endpoints expands exponentially under f^n.

II. *uniform hyperbolic structure* on the set $\mathrm{Per}(f)$ which means that there are two constants $K>0$; $\lambda>1$ such that if $f^s(x) = x$ then $|f^{s\prime}(x)| > K\lambda^s$ and on the set of preimages of the critical point $C_{-\infty}$, i.e., if $f^n(x) = c$ then $|f^{n\prime}(x)| > K\lambda^n$. This notion in another form appears in [1].

In Section 2 we prove that if f is globally expanding, then f has a uniform hyperbolic structure on $\mathrm{Per}(f)$.

In Section 3 we show that a mapping has a uniform hyperbolic structure on $\mathrm{Per}(f)$ if and only if it has a uniform hyperbolic structure on $C_{-\infty}$.

In Section 4 we demonstrate that if f has an uniform hyperbolic structure on $\mathrm{Per}(f)$ then for n large enough f^n has no restrictive central point. Hence f has sensitivity on initial conditions (see [3]).

In Section 5 we prove that if f has a uniform hyperbolic structure on $C_{-\infty}$ and for some $K>0$; $\lambda>1$ and every n we have $|f^{n\prime}(f(c))| > K\lambda^n$ then the length of the interval of monotonicity of f^n diminishes exponentially with n. Hence, if f has no sinks and the images of the critical point are separated from the critical point itself, then f is globally expanding (see [1] and [4]).

In Section 6 we estimate how fast an iterate of the critical point comes back to its neighbourhood (see Lemma 2.7 [1]).

I would like to acknowledge many very helpful and stimulating discussions with W. Szlenk.

1. Assumptions, notation and definitions. Throughout this paper we shall deal with a C^3 mapping $f\colon \langle 0;1\rangle \to \langle 0;1\rangle$ with $f(0)=f(1)=0$.
We assume that

(1.1) $$f \text{ is } S \text{ unimodal},$$

i.e., there exists a unique $c\in(0;1)$ such that f is strictly increasing on $(0,c)$ and strictly decreasing on $(c,1)$ and f has a nonpositive Schwarzian derivative:

$$Sf(x)=\frac{f(x)}{f'(x)}-\frac{3}{2}\left(\frac{f''(x)}{f'(x)}\right)^2\leqslant 0 \quad \text{for} \quad x\neq c.$$

For technical reasons we assume moreover that

(1.2) $$|f'(x)|>\vartheta|x-c|$$

for some $\vartheta>0$ and every $x\in(0,1)$.

We define η and L by

$$\eta=\sup_{x(0,1)}|f''(x)| \qquad L=\sup_{x(0,1)}|f'(x)|.$$

We shall use the following notation: for $n=0,1,2,\ldots$

$$f^0=\mathrm{id}; \quad f^{n+1}=f^n\circ f; \quad x_n=f^n(x).$$

We define:

$$C_{-n}=\{x\colon (f^n)'(x)=0\}, \qquad C_{-\infty}=\bigcup_{n=1}^{\infty}C_{-n},$$

$$C_n=\{c_i\}_{i=1,\ldots,n}, \qquad C_\infty=\bigcup_{n=1}^{\infty}C_n,$$

$$\mathrm{Per}_n(f)=\{x\colon f^n(x)=x\}, \qquad \mathrm{Per}(f)=\bigcup_{n=1}^{\infty}\mathrm{Per}_n(f).$$

We define Δ_n as the family of connected components of $(0,1)\setminus C_{-n}$, $\operatorname{card}C_{-n}=r_n$,

$$\Delta_n=\{\Delta_n^i\}_{i=0,\ldots,r_n}.$$

Observe that:

(1.3) for $m\leqslant n$ we have either $\Delta_m^j\supset\Delta_n^i$ or $\Delta_m^j\cap\Delta_n^i=\varnothing$.

We state following properties (see II.4 in [2]):

(1.4) If $Sf\leqslant 0$ and $Sg\leqslant 0$ then $S(f\circ g)\leqslant 0$ and $Sf^n\leqslant 0$.

(1.5) If $Sf\leqslant 0$ then $|f'|$ has no positive strict local minima.

(1.6) If $z\in C_{-n}$ then for some $k<n$, $z_k=c$.

(1.7) The endpoints of $f^n(\Delta_n^i)$ belong to $C_n\cup\{0\}$.

We denote by $|A|$ the Lebesgue measure of a set A.
We mean by (a,b) an interval with endpoints a and b, not necessarily $a<b$.
For a given point a different from c we define a' by

$$f(a')=f(a) \quad \text{and} \quad a'\neq a;$$

by definition $c'=c$.

Throughout the paper the letter c is reserved for the critical point, z for preimages of c, p and q for periodic points. The natural numbers will be denoted by N, n, k, s, j, i, l. λ_E, λ_H, λ_M, λ_C are fixed estimates (larger than 1) of f'. K_E, K_H, K_M, K_C ... are fixed positive numbers, defined later on.

DEFINITION 1. We say that f is *globally expanding* if there exist two constants $K_E>0$ and $\lambda_E>1$ such that for every $n=1,2,\ldots$ and $i=0,\ldots,r_n$

$$|f^n(\Delta_n^i)|>K_E\lambda_E^n|\Delta_n^i|.$$

DEFINITION 2. We say that f has a *uniform hyperbolic structure* on the set of periodic points if there exist two constants $K_H>0$ and $\lambda_H>1$ such that, for every $n=1,2,\ldots$ and every $p\in\mathrm{Per}_n(f)$ (i.e. $p_n=p$)

$$|(f^n)'(p)|>K_H\lambda_H^n.$$

2. Global expanding.

Proposition 2.1. *Assume that f is globally expanding and fulfils* (1.1) *and* (1.2). *Then f has a uniform hyperbolic structure on the set of periodic points.*

Before the proof we note that under the assumptions of Proposition 2.1 we have

(2.1) $$\operatorname{diam}\Delta_n=\max_{0\leqslant i\leqslant r_n}|\Delta_n^i|<\frac{1}{K_E}\lambda_E^{-n},$$

and hence we have

LEMMA 2.2. *Under the assumptions of Proposition* 2.1, *f has no periodic attracting orbits.*

Proof of Proposition 2.1. Suppose that for a periodic point p of a sufficiently large period we have

(2.2) $$p_s=p$$

and

(2.3) $$|(f^s)'(p)|\leqslant\lambda^s \quad \text{for some } \lambda<\lambda_E.$$

We assume that

(2.4) $$p \notin C_s, \quad \text{and hence} \quad \{p_j\}_{j=0,\dots,s-1} \cap C_{-s} = \emptyset .$$

The other case will be discussed at the end of the proof. For simplification we set $g(x) = f^s(x)$. By (1.1) and (1.4) we have $Sg \leqslant 0$. For any n there is an $i = i(p) \leqslant r_{ns}$ such that $p \in \Delta^i_{ns}$. We denote the endpoints of Δ^i_{ns} by α^n and β^n, $\alpha^n < p < \beta^n$. Since $Sg \leqslant 0$ we have by (1.4) and (1.5) $|(g^n)'(p)| \geqslant |(g^n)'(x)|$ either for all $x \in (\alpha^n, p)$ or for all $x \in (p, \beta^n)$.

There are two possibilities:

1° The above inequality holds for any n large enough for the same side of p.

2° There are infinitely many changes of sides.

Suppose 1° holds for α^n, the other case being handled similarly. By the Mean Value Theorem and by (2.3) we have for some $y \in (\alpha^n, p)$

(2.5) $$|g^n(p) - g^n(\alpha^n)| = (g^n)'(y)|\,|p - \alpha^n| \leqslant |(g^n)'(p)|\,|\Delta^i_{ns}| \leqslant \lambda^{ns}\frac{1}{K_E}\lambda_E^{-ns} = \frac{1}{K_E}\left(\frac{\lambda}{\lambda_E}\right)^{ns}.$$

By (2.5) and (2.3) we have $g^n(\alpha^n) \to p$ as $n \to \infty$. This together with (2.4) implies that for large n $g^n(\alpha^n, p) \cap C_{-s} = \emptyset$, and hence $\alpha^n = \alpha^{n+1} = \dots$, which contradicts $\operatorname{diam}\Delta_n \to 0$. Suppose now that 2° holds.

We choose n such that

$$|(g^n)'(p)| > |(g^n)'(x)| \quad \text{for} \quad x \in (\alpha^n, p)$$

and

$$|(g^{(n+1)})'(p)| > |(g^{(n+1)})'(x)| \quad \text{for} \quad x \in (p, \beta^{n+1}) .$$

We claim that $\alpha^n = \alpha^{n+1}$ and $\beta^n = \beta^{n+1}$, for n large enough. Inequality (2.5) holds for α^n and β^{n+n}. Suppose $\alpha^n \neq \alpha^{n+1}$. Then

$$\alpha^{n+1} \in C_{-(n+1)s} \setminus C_{-ns} \quad \text{and} \quad g^{n+1}(\alpha^{n+1}) \in C_s .$$

By the Mean Value Theorem we have for some $y \in (g^n(\alpha^{n+1}), g^n(p))$ as $\alpha^{n+1} \in (\alpha^n, p)$

$$\operatorname{dist}(C_s, p) \leqslant |g^{n+1}(\alpha^{n+1}) - p| = |g'(y)|\,|g^n(\alpha^{n+1}) - g^n(p)| \leqslant \sup_{0<x<1}|g'(x)|\,|g^n(\alpha^n) - g^n(p)| \leqslant L^s|g^n(\alpha^n) - g^n(p)| ;$$

recall that

$$L = \sup_{x\in(0,1)} |f'(x)| .$$

As C_s is a finite set, by (2.4) $\operatorname{dist}(C_s, p) > 0$. Thus the above inequality contradicts (2.5) and proves $\alpha^n = \alpha^{n+1}$. The proof of $\beta^n = \beta^{n+1}$ is similar.

In view of (1.7) we have $g^n(\alpha^n), g^n(\beta^n) \in C_\infty$, and so we can set $g^n(\alpha^n) = c_{k_n}$ and $g^n(\beta^n) = c_{l_n}$. We have

$$c_{k_{n+1}} = g^{n+1}(\alpha^{n+1}) = g^{n+1}(\alpha^n) = g(c_{k_n}) = c_{k_n+s} .$$

By Definition 1 of global expanding and (2.3) we have

(2.6) $$|\alpha^n - \beta^n| \leqslant \frac{1}{K_E}\lambda_E^{-(n+1)s}|c_{k_{n+1}} - c_{l_{n+1}}| \leqslant \frac{1}{K_E}\lambda_E^{-(n+1)s}(|c_{k_n+s} - p| + |c_{l_n+s} - p|) ,$$

(2.7) $$|c_{k_n+s} - p| = |g(c_{k_n}) - g(p)| \leqslant L^s|c_{k_n} - p| = L^s|g^n(\alpha^n) - g^n(p)| \leqslant L^s|(g^n)'(p)|\,|\alpha^n - p| \leqslant L^s\lambda^{ns}|\alpha^n - p| ,$$

(2.8) $$|p - c_{l_n+s}| = |g^{n+1}(p) - g^{n+1}(\beta^n)| \leqslant |(g^{n+1})'(p)|\,|p - \beta^n| \leqslant L^s|(g^n)'(p)|\,|p - \beta^n| \leqslant L^s\lambda^{ns}|p - \beta^n| .$$

We put (2.7) and (2.8) in (2.6) and obtain

(2.9) $$|\alpha^n - \beta^n| \leqslant \frac{1}{K_E}\lambda_E^{-(n+1)s}(L^s\lambda^{ns}|\alpha^n - p| + L^s\lambda^{ns}|p - \beta^n|) = \frac{1}{K_E}\lambda_E^{-(n+1)s}L^s\lambda^{ns}|\alpha^n - \beta^n| = \frac{1}{K_E}\left(\frac{L}{\lambda_E}\right)^s\left(\frac{\lambda}{\lambda_E}\right)^{ns}|\alpha^n - \beta^n| .$$

For n large enough (2.9) contradicts (2.3), which completes the proof for any $p \notin C_s$. Suppose now that $p \in C_s$. Hence for some m, $c_m = p$. Thus there is a unique periodic orbit with this property. By Lemma 2.2 there are no periodic attracting orbits; so $|(f^s)'(p)| > 1$ and we can take $\lambda_H = \min(\lambda_E, |(f^s)'(p)|^{1/s}) > 1$.

3. Uniform hyperbolic structure.

DEFINITION 3. We say that f has a *uniform hyperbolic structure* on the set $C_{-\infty}$ if there are two constants $K_C > 0$ and $\lambda_C > 1$ such that for every natural n and every $z \in (0, 1)$ if $z_n = c$ then $|(f^n)'(z)| > K_C\lambda_C^n$.

This definition is similar to Definition 2. Proposition 3.9 states that uniform hyperbolic structures on sets $\operatorname{Per}(f)$ and $C_{-\infty}$ are equivalent.

First we quote some lemmas, which will be useful in the sequel.

LEMMA 3.1 (= Theorem 1.3 in [4]). *Assume that f is S-unimodal and has no sinks. Then for every open interval U containing a critical point c there exists a number $m = m(U)$ such that for every $x \in (0, 1)$ if $x_i \notin U$ for $i = 0, \dots, m-1$ then $|f^{m'}(x) > 1$.*

Taking U smaller, we may assume $|(f^m)'(x)| > \lambda > 1$. Set $\lambda_M = \lambda^{1/m}$. Since U is open, there exists a $K_M > 0$ such that for every $x \in (0, 1)$ and natural n:

$$\text{if } x_i \notin U \quad \text{for} \quad i = 0, \dots, n-1 \quad \text{then} \quad |(f^n)'(x)| > K_M\lambda_M^n .$$

LEMMA 3.2 (= Lemma 2.6 in [1]). *Let f be S-unimodal, $x, y \in \Delta^i_n$. Then*

$$|x_n - y_n| \geqslant ((f^n)'(x)(f^n)'(y))^{1/2}|x - y| .$$

As in Section 1, x' is defined by $f(x') = f(x)$ and $x' \neq x$.

LEMMA 3.3 (= Lemma II.5.6 in [2]). *Let f be S-unimodal with no stable periodic orbit. Define*

$$\mathcal{K}^n = \{x\colon x_i \notin (x, x'),\ i = 1, \ldots, n-1;\ x_n \in (x, x')\}.$$

Then every connected component of $\mathcal{K}^n$ is of the form (p, q') with $p_n = p$ and $q_n = q$. Moreover $\mathcal{K}^n \cap C_{-n} = \emptyset$.

LEMMA 3.4. *Assume that f is S-unimodal and has a uniform hyperbolic structure on* Per(f). *Then there exists a $K > 0$ such that if $p_n = p$ then $|(f^n)'(p')| > K\lambda_H^n$.*

Proof. Since $f(p') = f(p)$ we have

$$(3.1)\quad |(f^n)'(p')| = |(f^{n-1})'(f(p'))\cdot f'(p')| = \left|(f^n)'(p)\frac{f'(p')}{f'(p)}\right| > K_H\lambda_H^n\left|\frac{f'(p')}{f'(p)}\right|.$$

By (1.2) we have

$$(3.2)\qquad |f(p)-c_1| = \left|\int_c^p |f'(x)|\,dx\right| \geqslant \frac{\vartheta}{2}|p-c|^2,$$

and similarly, since $\eta = \sup_{x\in(0,1)} |f''(x)|$, we have

$$(3.3)\qquad |f(p')-c_1| \leqslant \frac{\eta}{2}|p'-c|^2.$$

Hence

$$(3.4)\qquad \left|\frac{p'-c}{p-c}\right| \geqslant \left(\frac{\vartheta}{\eta}\right)^{1/2}.$$

We use again (1.2) and the definition of η

$$(3.5)\qquad \left|\frac{f'(p')}{f'(p)}\right| \geqslant \frac{\vartheta}{\eta}\left|\frac{p'-c}{p-c}\right| \geqslant \left(\frac{\vartheta}{\eta}\right)^{3/2}.$$

In order to complete the proof we put (3.5) in (3.1).

In definition 2 we can replace K_H by $K_H\left(\frac{\vartheta}{\eta}\right)^{3/2}$. Hence we may assume that

$$|(f^n)'(p')| > K_H\lambda_H^n.$$

COROLLARY 3.5. *Under the assumptions of Lemma* 3.4

$$|(f^n)'_{|\mathcal{K}^n}| > K_H\lambda_H^n.$$

For every component of $\mathcal{K}^n$ this is directly implied by Lemma 3.3 and (1.5).

LEMMA 3.6. *Under the assumptions of Lemma* 3.4 *there exist two constants $K > 0$, $\lambda > 1$ such that for every $x \in (0, 1)$*

$$\textit{if } x_i \notin (x_n, x_n') \textit{ for } i = 0, \ldots, n-1 \textit{ then } |(f^n)'(x)| > K\lambda^n.$$

Proof. Suppose $x_i \notin (x_n, x_n')$, $i < n$. There exists a sequence

$$0 = j_m < \ldots < j_1 < j_0 = n$$

such that

$$x_{j_1} \in \mathcal{K}^{n-j_1};\ x_{j_2} \in \mathcal{K}^{j_1-j_2};\ \ldots;\ x = x_{j_m} \in \mathcal{K}^{j_{m-1}}.$$

It is obvious that $x_s \notin (x_{j_i}, x_{j_i}')$ for $s < j_i$. By Corollary 3.5 for fixed j large enough we can find a $\lambda_1 > 1$ such that for $s > j$ $|f^{s\prime}_{|\mathcal{K}^s}| > \lambda_1^s$. We take an open interval $U \ni c$ containing no periodic points of period less than $j+1$. Let k be minimal with $x_k \in U$. We see that for some $r \leqslant m$ we have

$$k = j_r \quad\text{and}\quad j_i - j_{i+1} > j \quad\text{for}\quad i < r.$$

We now have

$$|(f^n)'(x)| = |(f^{n-k})'(x_k)|\,|f^{k\prime}(x)| = \prod_{i=0}^{r-1} |(f^{j_i-j_{i+1}})'(x_{j_{i+1}})|\,|(f^k)'(x)|.$$

By Lemma 3.1 $|(f^k)'(x)| > K_M\lambda_M^k$, and the assertion follows for

$$\lambda = \min(\lambda_1, \lambda_M) \quad\text{and}\quad K = K_M.$$

For the simplicity of further estimations we can modify

$$\lambda_H \text{ to } \lambda \quad\text{and}\quad K_H \text{ to } \min(K_H, K).$$

COROLLARY 3.7. *Under the assumptions of Lemma* 3.4, *if (a, b) is such that*

$$(a_i, b_i) \cap [(a_n, a_n') \cup (b_n, b_n')] = \emptyset \quad\text{for}\quad 0 \leqslant i < n$$

then

$$|(a_n, b_n)| > |(a, b)|\,K_H\lambda_H^n.$$

Proof. This is an immediate consequence from the Mean Value Theorem and Lemma (3.6).

LEMMA 3.8 (= Corollary II.5.8 in [2]). *If f is S-unimodal and has no stable periodic orbit then the set $C_{-\infty}$ is dense in (c_2, c_1).*

PROPOSITION 3.9. *Assume that f is S-unimodal with no stable periodic orbit. Then f has a uniform hyperbolic structure on* Per(f) *if and only if f has a uniform hyperbolic structure on $C_{-\infty}$.*

Proof. The only if part of the proof is an easy consequence of Lemma 3.6.

The if part: We assume that, for every $z \in C_{-\infty}$, if $z_n = c$ then

$$|(f^n)'(z)| > K_C\lambda_C^n, \quad K_C > 0, \quad \lambda_C > 1.$$

Suppose there is a periodic point p of period s and $a\lambda < \lambda_C$ such that $(f^s)'(p) < \lambda^s$. We may assume $(f^s)'(p) > 0$, otherwise we take $2s$ instead of s. Let z_1, z_2 be two points of $C_{-\infty}$ nearest to p with $p \in (z_1, z_2)$. By assumption f has no stable periodic orbit; hence we have $f^s(z_1) < z_1 < p$ and $f^s(z_2) > z_2 > p$. In consequence $(z_1, z_2) \subset f^s(z_1, z_2)$.

We define two sequences α^n and β^n by:

$$\alpha^1 = z_1, \quad \beta^1 = z_2,$$

$$\alpha^{n+1} = \left(f^s|(z_1, z_2)\right)^{-1}(\alpha^n), \quad \beta^{n+1} = \left(f^s|(z_1, z_2)\right)^{-1}(\beta^n) \quad \text{(see Fig. 1)}.$$

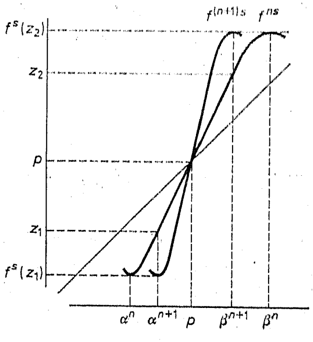

Fig. 1

By an analogous argument we have for every n

$$\alpha^n < \alpha^{n+1} < p < \beta^{n+1} < \beta^n.$$

Since $Sf \leqslant 0$, we have for every n: either $(f^{ns})'(\alpha^{n+1}) \leqslant (f^{ns})'(p)$ or $(f^{ns})'(\beta^{n+1}) \leqslant (f^{ns})'(p)$. Suppose that the first possibility holds. We have for some $j < s$, $f^j(z_1) = c$ and, by the definition of α^n, $f^{ns+j}(\alpha^{n+1}) = c$. Hence

$$\lambda^{ns} > (f^{ns})'(p) \geqslant (f^{ns})'(\alpha^{n+1})$$

$$= \frac{(f^{ns+j})'(\alpha^{n+1})}{(f^j)'(f^{ns}(\alpha^{n+1}))} > \frac{K_C \lambda_C^{ns+j}}{L^j}$$

$$= K_C \left(\frac{\lambda_C}{L}\right)^j \lambda_C^{ns} > K_C \left(\frac{\lambda_C}{L}\right)^s \lambda^{ns}$$

For n large enough we obtain a contradiction, which completes the proof.

4. Central points and sensitivity.

DEFINITION 4. For $n > 1$, we say that $x \in (0, 1)$ is the *central fixed point of* f^n if $x_n = x$ and $(f^n)'|_{(x,c)} > 0$.

Remember that in (x, c) we do not necessarily have $x < c$.

DEFINITION 5. We say that the central fixed point x of f^n is *restrictive* if $f^n(x, x') \subset (x, x')$.

LEMMA 4.1 (= Lemma II.7.8 in [2]). *Assume that f is S-unimodal and has no stable periodic orbit. Let p be the restrictive central point for f^n. Then $f^i(p, p') \cap (p, p') = \emptyset$ for $i = 1, \ldots, n-1$.*

PROPOSITION 4.2. *Assume that f is S-unimodal and has a uniform hyperbolic structure on the set* Per(f). *Then there is an N such that, for all $n > N$, f^n has no restrictive central points.*

Before the proof we quote a theorem proved by Guckenheimer in [3], which will show some consequences of Proposition 4.2.

Let us first introduce the notion of sensitive dependence. We say that f has sensitive dependence on initial conditions if there is a set $Y \subset \langle 0, 1 \rangle$ of positive Lebesgue measure and an $\varepsilon > 0$ such that for every $x \in Y$ and every neighbourhood U of x we have a $y \in U$ and an $n \geqslant 0$ such that $|x_n - y_n| > \varepsilon$.

THEOREM G (see [3]). *Let f be S-unimodal with no stable periodic orbit. Then f has sensitive dependence on initial conditions if and only if there is an N such that, for all $n > N$, f^n has no restrictive central point.*

Proof of Proposition 4.2. (See Fig. 2.) Suppose that p is a restrictive centra point for f^n. There exists a unique $q \in (c, p')$ such that $q_n = q$ and a unique $u \in (c, q)$ with $|(f^n)'(u)| = 1$. We define $y \in \langle c, q \rangle$ by

$$|(f^n)'(y)| = \sup_{x \in (c,q)} |(f^n)'(x)|.$$

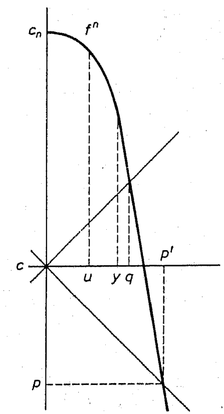

Fig. 2

We shall estimate $|(f^n)'(y) - (f^n)'(u)|$:

$$(4.1) \qquad |(f^n)'(y) - (f^n)'(u)| = \left| \int_u^y (f^n)''(x)\,dx \right| \leqslant \left| \int_u^y |(f^n)''(x)|\,dx \right|.$$

For $x \notin C_{-n}$ we have

(4.2) $$(f^n)''(x) = \Big(\prod_{s=0}^{n-1} f'(x_s)\Big)' = \sum_{s=0}^{n-1} f''(x_s)\cdot(f^s)'(x) \prod_{\substack{i=0\\ i\neq s}}^{n-1} f'(x_i)$$

$$= (f^n)'(x) \sum_{s=0}^{n-1} \frac{f''(x_s)\cdot(f^s)'(x)}{f'(x_s)}.$$

In view of (1.2) we have

(4.3) $$|(f^n)''(x)| \leqslant \frac{\eta}{\vartheta}\,\frac{|(f^n)'(x)|}{|x-c|} + \frac{\eta}{\vartheta}|(f^n)'(x)|\cdot \sum_{s=1}^{n-1} \frac{|(f^s)'(x)|}{|x_s-c|}.$$

By Lemma 4.1 and by (3.4) we have, for $s<n$ and $x \in (u, y)$,

(4.4) $$|x_s-c| \geqslant \Big(\frac{\vartheta}{\eta}\Big)^{1/2} |x-c| \geqslant \Big(\frac{\vartheta}{\eta}\Big)^{1/2} |u-c|.$$

By this and the definition of y we get from (4.3)

(4.5) $$|(f^n)''(x)| \leqslant \Big(\frac{\eta}{\vartheta}\Big)^{3/2} \frac{1}{|u-c|}\Big(|(f^n)'(x)| + |(f^n)'(y)| \sum_{s=1}^{n-1} |(f^s)'(x)|\Big).$$

We put the above inequality in (4.1) and obtain by integration

(4.6) $$\big||(f^n)'(y)|-1\big| \leqslant \Big(\frac{\eta}{\vartheta}\Big)^{3/2} \frac{1}{|u-c|}\Big(|y_n-u_n| + |(f^n)'(y)| \sum_{s=1}^{n-1} |y_s-u_s|\Big).$$

By Lemma 4.1 we infer that Corollary 3.7 holds for (u_s, y_s) as (a, b) and f^{n-s} as f^n. Hence

(4.7) $$|y_n-u_n| \geqslant K_H \lambda_H^{n-s} |y_s-u_s|$$

and

(4.8) $$\sum_{s=1}^{n-1} |u_s-y_s| \leqslant \frac{|y_n-u_n|}{K_H} \sum_{s=1}^{n-1} \lambda_H^{s-n} \leqslant \frac{|y_n-u_n|}{K_H}\cdot\frac{1}{\lambda_H-1};$$

in consequence we have

$$\big||(f^n)'(y)|-1\big| \leqslant \Big(\frac{\eta}{\vartheta}\Big)^{3/2}\Big(1+\frac{|(f^n)'(y)|}{K_H(\lambda_H-1)}\Big)\cdot\Big|\frac{y_n-u_n}{u-c}\Big|.$$

Thus, in view of the fact that $|(f^n)'(y)| \to \infty$ as $n \to \infty$ for some $K>0$ independent of n, we have

(4.9) $$\Big|\frac{u_n-q}{u-c}\Big| = \Big|\frac{u_n-q_n}{u-c}\Big| \geqslant \Big|\frac{u_n-y_n}{u-c}\Big| > K;$$

in order to complete the proof it is now enough to show that the quotient on the left side of the last inequality tends to 0 as $n \to \infty$.

We set $|(c, p')| = d$; so by (3.4) we have

(4.10) $$|(p, p')| \leqslant d\Big(1+\Big(\frac{\eta}{\vartheta}\Big)^{1/2}\Big).$$

In view of Lemma 3.2 and Lemma 3.4 we have by $|(f^n)'(u)| = 1$

(4.11) $$|u_n-p| \geqslant |u-p'|(K_H\lambda_H^n)^{1/2}.$$

Since p is restrictive, $u_n \in (p, p')$, and we have

(4.12) $$|u-p'| \leqslant \frac{d\Big(1+\Big(\frac{\eta}{\vartheta}\Big)^{1/2}\Big)}{(K_H\lambda_H^n)^{1/2}} = K_1 d(\lambda_H^n)^{-1/2} \quad \text{for some } K_1>0$$

and

(4.13) $$|u-c| = |p'-c|-|u-p'| \geqslant d(1-K_1\lambda_H^{-n/2}).$$

Using again Lemma 3.2 and Lemma 3.4, we obtain

(4.14) $$|p-q| = |f^n(p')-f^n(q)| > K_H\lambda_H^n|p'-q|,$$

but as

$$|p-q|+|p'-q| = |p-p'| \leqslant d\Big(1+\Big(\frac{\eta}{\vartheta}\Big)^{1/2}\Big)$$

we can write

(4.15) $$d\Big(1+\Big(\frac{\eta}{\vartheta}\Big)^{1/2}\Big) > (K_H\lambda_H^n+1)|p'-q|$$

and

(4.16) $$|u_n-q| < |p'-q| < \frac{d\Big(1+\Big(\frac{\eta}{\vartheta}\Big)^{1/2}\Big)}{K_H\lambda_H^n+1}.$$

By (4.13) and (4.16) for some $K_2>0$ we have

(4.17) $$\Big|\frac{u_n-q}{u-c}\Big| < \frac{d\Big(1+\Big(\frac{\eta}{\vartheta}\Big)^{1/2}\Big)}{(K_H\lambda_H^n+1)(1-K_1\lambda_H^{-n/2})d} < K_2\lambda_H^{-n}.$$

The last inequality contradicts inequality (4.9) for large n, and thus the proof is complete.

We can obtain a stronger result if we assume that

$$|f^{n''}(c)| = |(f^{n-1})'(c_1)|\cdot|f''(c)|$$

is not too small.

PROPOSITION 4.3. *Under the assumptions of Proposition 4.2, if we have additionally for every n*

$$|(f^n)'(c_1)| > K\lambda_C^{-n/8},$$

where $K>0$ is independent on n, and λ_C is fixed by Definition 3, then for $\Delta_{n+1}^i = (c, a)$ with $a_n = c$ we have

$$|f^n(\Delta_{n+1}^i)| > K_1\lambda_C^{n/8}|\Delta_{n+1}^i|$$

for some n-independent $K_1>0$ (see Fig. 3).

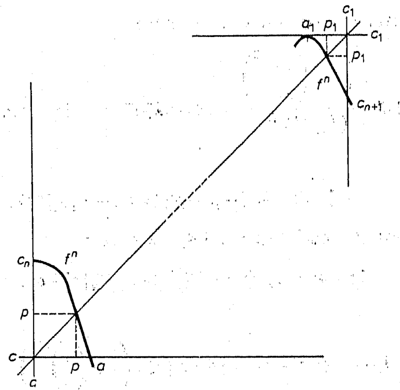

Fig. 3

Proof. Let $\Delta_n^j \supset \Delta_{n+1}^i$; then Δ_n^j contains a central point p (or p') of f^n. Let $p\in\Delta_{n+1}^i$; the case with $p'\in\Delta_{n+1}^i$ can be handled similarly by Lemma 3.4. By Lemma 3.2 and Proposition 3.1 we have for some $K_2>0$

$$(4.18)\qquad |f^n(p, a)| > K_2\lambda_C^n|(p, a)|.$$

Again by Lemma 3.2, (1.2) and the assumption we have for some $K_3>0$

$$\frac{\eta}{2}|c_n-c|^2 \geqslant |c_{n+1}-c_1| \geqslant |c_{n+1}-p_1| > K_3(\lambda_C^n\cdot\lambda_C^{-n/2})^{1/2}|c_1-p_1|.$$

Hence for some $K_4>0$

$$|c_n-c|^2 > K_4\lambda_C^{n/4}|c-p|^2.$$

Combining this inequality with inequality (4.18), we obtain the assertion.

5. The length of the interval of monotonicity. In this section we shall deal with a function for which there exist two constants $K_C>0$ and $\lambda_C>1$ such that for every n

$$(5.0)\qquad |(f^n)'(c_1)| > K_C\lambda_C^n,$$

and if $z_n = c$

$$|(f^n)'(z)| > K_C\lambda_C^n.$$

Such functions have been studied in [1], where it was also proved that, if f has no sinks and $\mathrm{dist}(C_\infty, c)>0$ (see [4]), then f satisfies (5.0) (Appendix in [1]).

First we prove a consequence of Lemma 3.2:

LEMMA 5.1. *Let f be S-unimodal. Let $\Delta_n^i = (a, b)$ with $a_k = c$ for some $k<n$. Let $y\in(a, b)$ such that $(f^n)''(y) = 0$. Then for every $x\in(a, y)$ we have*

$$(5.1)\qquad |x_n-a_n| > \frac{\vartheta}{2\sqrt{\eta}}|(f^{n-k})'(x_k)\cdot(f^{n-k-1})'(c_1)|^{1/2}|x_k-c|^{3/2}$$

and

$$(5.2)\qquad |(f^n)'(x)| > \frac{\vartheta^2}{4\eta}|(f^{n-k-1})'(c_1)\cdot(f^k)'(a)|\,|x_k-c|.$$

Proof. By Lemma 3.2 and (1.2) we have

$$(5.3)\qquad \begin{aligned} |x_n-a_n| &= |f^{n-k-1}(x_{k+1})-f^{n-k-1}(a_{k+1})|\\ &\geqslant \left(\frac{(f^{n-k})'(x_k)}{f'(x_k)}\cdot(f^{n-k-1})'(c_1)\right)^{1/2}|x_{k+1}-c_1|\\ &\geqslant |(f^{n-k})'(x_k)\cdot(f^{n-k-1})'(c_1)|^{1/2}\frac{\vartheta}{2\sqrt{\eta}}|x_k-c|^{3/2},\end{aligned}$$

which proves (5.1). Again by Lemma 3.2 we have

$$(5.4)\qquad |x_k-c|^{3/2} \geqslant |x_k-c|^{1/2}((f^k)'(x)\cdot(f^k)'(a))^{1/2}|x-a|.$$

In view of (1.1) we may estimate

$$(5.5)\qquad |(f^n)'(x)|\,|x-a| > |x_n-a_n|.$$

Now we put (5.4) and (5.5) in (5.3) and simplify

$$|(f^n)'(x)|^2 > \frac{\vartheta^2}{4\eta}|(f^{n-k-1})'(c_1)|\,|(f^{n-k})'(x_k)(f^k)'(x)|\,|(f^k)'(a)|\cdot|x_k-c|.$$

As $(f^{n-k})'(x_k)\cdot(f^k)'(x) = (f^n)'(x)$, the proof is complete.

PROPOSITION 5.2. *Let f be S-unimodal and satisfy (5.0). Then for some $K>0$ and $\lambda>1$ and every n*

$$\mathrm{diam}\Delta_n < K\lambda^{-n}.$$

Proof. Let $\Delta_n^i \in \Delta_n$, $\Delta_n^i = (a,b)$, $a_k = c = b_r$, $r<k<n$. Suppose first that $c \in f^n(\Delta_n^i)$. There is a $z\in(a,b)$ such that $z_n = c$. We shall estimate separately $|(a,z)|$, $|(z,b)|$ being handled similarly. Let $u\in(a,z)$ with $|(f^n)'(u)|=1$. By Lemma 3.2 and (5.0) we have

$$1 > |u_n - z_n| \geq K_C^{1/2}\lambda_C^{n/2}|u-z|.$$

It is now enough to estimate $|u-a|$. By Lemma 5.1 and Lemma 3.2 we have for $x=u$

$$(5.6)\qquad 1 > \frac{\vartheta^2}{4\eta}|(f^{n-k-1})'(c_1)\cdot(f^k)'(a)|\cdot|(f^k)'(a)\cdot(f^k)'(u)|^{1/2}|u-a|.$$

If $|(f^k)'(u)|\geq\lambda_C^{-n}$ then by (5.6) $|u-a|\leq K_2\lambda_C^{-n/2}$. If $|(f^k)'(u)|<\lambda_C^{-n}$ then $|(f^{n-k})'(u_k)|>\lambda_C^n$ and by Lemma 5.1:

$$1>|u_n-a_n|>K_3(\lambda_C^n\lambda_C^{n-k-1})^{1/2}|u_k-c|^{3/2}$$

and hence

$$|u_k-c|<K_4\lambda_C^{-n/3}.$$

Let $\Delta_k^j\supset\Delta_n^i$ and $\Delta_k^j=(w,v)$; $z,u\in(w,a)$, $w_s=c$, $s<k$ (see Fig. 4).

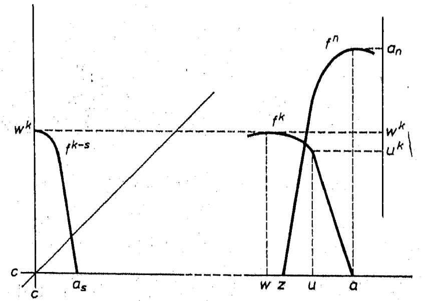

Fig. 4

By Corollary 3.7 and (1.3) $|f^s(\Delta_k^j)|>K_5|\Delta_k^j|$ and by Proposition 4.3 $|f^{k-s}(f^s(w,a))|>K_6|f^s(w,a)|$. Hence

$$|f^k(w,a)|>K_7|(w,a)|>K_7|(z,a)|.$$

But

$$|f^k(w,a)|=|w_k-u_k|+|u_k-c|<|(f^k)'(u)|+K_4\lambda_C^{-n/3}.$$

This proves the assertion in the case $c\in f^n(\Delta_n^i)$.

Suppose $c\notin f^n(\Delta_n^i)$. For some j: $\Delta_n^i\subset\Delta_k^j$ and as $a_k=c$ we have by the first part of the proof $|\Delta_n^i|<|\Delta_k^j|<K_8\lambda^{-k}$. It is now enough to show that $n<2k$ (see Fig. 5).

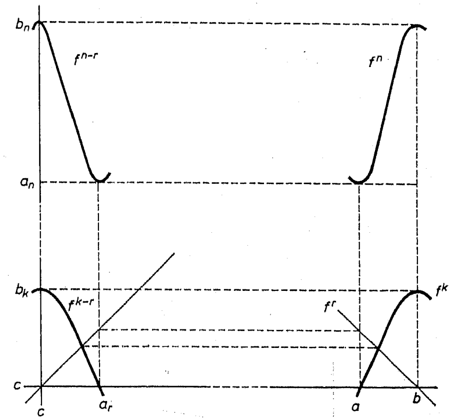

Fig. 5

Let us consider f^{n-r} and f^{k-r} on the interval

$$f^r(\Delta_n^i)=(a_r,b_r)=(a_r,c).$$

Since $f^{k-r}(a_r)=c$ we have a fixed point of f^{k-r} in the interval (a_r,a_r') and hence $f^{2(k-r)}(a_r,a_r')\ni c$. Thus $2(k-r)>(n-r)$, which completes the proof.

COROLLARY 5.3. *Suppose that f is S-unimodal with no sinks. If* $\operatorname{dist}(C_\infty,c)>0$, *then f is globally expanding.*

This has also been proved by K. Ziemian by another method in her Ph. D. thesis (Warsaw University).

The proof of the case where $c\in f^n(\Delta_n^i)$ follows directly from Proposition 5.2. Otherwise the direct use of Lemma 3.1 is necessary. We omit the details.

6. The time of comeback.

Proposition 6.1. *Let f be S-unimodal without a stable periodic orbit. Then there exists a $K>0$, and an $\varepsilon_0>0$ such that, for every $\varepsilon\in(\varepsilon_0,0)$, if $|c_n-c|<\varepsilon$ then $n>K|\log\varepsilon|$.*

Proof (see Fig. 6). Let $\Delta_{n+1}^i=(z,c)$ and $\Delta_{n+1}^{i+1}=(c,z')$; $z_k=c$.

As at the end of the proof of Proposition 5.2 we have $n<2k$ and there is a fixed point q of f^k in (z,z'). Let n be the smallest positive integer with $|c_n-c|<\varepsilon$. We have by (4.2), as $L=\sup_{x\in(0,1)}|f'(x)|$

$$(6.1)\qquad 1<|(f^k)'(q)-(f^k)'(c)|<\sup_{x\in(0,1)}|(f^k)''(x)|\,|q-c|<\eta\frac{|q-c|}{L-1}L^k(L^k-1).$$

If $n = k$ then

$$\varepsilon > |c_n - c| > |q_n - c| = |q-c| > \frac{1}{K_1} L^{-2n}.$$

If $n > k$ then there is no fixed point of f^n on (z, z'). This follows directly from the fact that if f^n has a fixed point in (z, z') then it has a central point, and hence $n = k$. We define y by $|y-c| = \eta^{-1}L^{-2n+1}$. By (6.1) for n we have $y \in (z, z')$. We have

$$|(f^n)'(y)| = |(f^{n-1})'(y_1)|\,|f'(y)| < L^{n-1}\eta\,|y-c| \leqslant L^{-n}.$$

We may assume that, for $x \in (z, z')$, $|x_n - c| > |c_n - c|$; otherwise either $|z_n - c_n| = |c_{n-k} - c| < |c_n - c|$; which contradicts the minimality of n, or there is a $v \in (z, z')$

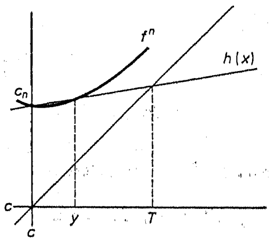

Fig. 6

with $v_n = c$, which contradicts the definition of z. Assume $c_n > c$, the case $c_n < c$ being handled similarly. We construct the line $h(x) = L^{-n}(x-c)+c_n$ and define T by $T = h(T)$. Since there is no fixed point of f^n on (z, z') and by (1.5) $|(f^n)'|_{(c,y)}| \leqslant L^{-n}$, we have $|T-c| \geqslant |y-c|$. By definition $|T-c| = L^{-n}|T-c| + |c_n - c|$; hence for some $K_2 > 0$ independent of n we have

$$\varepsilon > |c_n - c| = |1-L^{-n}|\,|T-c| > |1-L^{-n}|\,|y-c| = K_2 L^{-2n}$$

and

$$2n > \frac{\log K_2/\varepsilon}{\log L},$$

which implies the assertion.

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