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Tiling with smooth and rotund tiles

by

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Abstract. Without assuming local finiteness, this paper studies tilings of topological vector spaces by convex sets that are bounded or finitely bounded. The paper was motivated by a wish to ascertain, in the infinite as well as the finite-dimensional case, to what extent the tiles can be smooth or rotund. Various limitations are established. For example, no space of dimension $\geqslant 2$ admits a countable rotund tiling, and tilings that are uniformly smooth or uniformly rotund are excluded under certain hypotheses. On the other hand, some nonseparable locally convex spaces admit tilings in which each tile is both smooth and rotund. Several unsolved problems remain.

Introduction. A collection $\mathscr{C}$ of subsets of a topological space S is a *covering* if $S = \bigcup \mathscr{C}$. It is a *packing* if $|\mathscr{C}| > 1$, each member of $\mathscr{C}$ is the closure of its nonempty interior, and the interiors are disjoint. A *tiling* is a collection that is both a covering and a packing, and the members of a tiling are *tiles*.

A subset of a topological vector space is here called a *bc-set* (resp. *fc-set*) if it is closed, convex and bounded (resp. *finitely bounded* (has bounded intersection with each finite-dimensional subspace)). Along with certain other adjectives (e.g. closed, convex, smooth, rotund), the prefixes *bc* and *fc* are applied to a collection $\mathscr{C}$ if they apply to each member of $\mathscr{C}$. However, some adjectives refer to $\mathscr{C}$ as a collection or to the interactions among members of $\mathscr{C}$, and we rely on context for the necessary distinctions. For example, $\mathscr{C}$ is *countable* if $|\mathscr{C}| \leqslant \aleph_0$, *disjoint* if no two members intersect, and *locally finite* if each point of the space has a neighborhood that intersects only finitely many members of $\mathscr{C}$.

In a locally finite bc-tiling of $\boldsymbol{R}^d$, each tile is a d-polytope [3] [17], and at least for $d \leqslant 3$ an arbitrary d-polytope P may serve as a prototile in the sense that $\boldsymbol{R}^d$ admits a locally finite tiling in which each tile is combinatorially equivalent to P [4] [13]. However, without the assumption of local finiteness, little is known even in the plane, and that assumption is inappropriate for the study of bc-tilings of in-

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finite-dimensional spaces. It is true that each separable normed linear space admits a locally finite fc-tiling (by sets that are in a sense "parallelotopes") [6]. However, a theorem of Corson [1] asserts that for any bc-covering $\mathscr{C}$ of a normed linear space that has an infinite-dimensional reflexive subspace, there is a finite-dimensional (hence compact) parallelotope that intersects infinitely many members of $\mathscr{C}$; thus $\mathscr{C}$ is not locally finite.

As the terms are used here, a subset of a topological vector space is *smooth* (resp. *rotund*) if it is closed, convex, has nonempty interior and has a unique supporting hyperplane at each boundary point (resp. no segment in its boundary). When local finiteness is not required, some of the tiles in a bc-tiling of $\boldsymbol{R}^d$ may be smooth and rotund, for each bc-packing $\mathscr{P}$ of $\boldsymbol{R}^d$ whose union $\bigcup \mathscr{P}$ is closed can be extended to a bc-tiling by adding d-cubes of various sizes. Nevertheless, it does seem "intuitively obvious" that in a bc-tiling of a normed linear space of dimension $\geqslant 2$, at least some of the tiles must have boundaries with "sharp" places (hence fail to be smooth) or "flat" places (hence fail to be rotund). However, we are able to prove this only under various supplementary hypotheses, and in view of our examples it would not surprise us if certain nonseparable normed linear spaces admit bc- (or at least fc-) tilings in which all tiles are smooth and rotund.

Throughout the paper, X denotes a normed linear space. A *body* in X is a bc-set with nonempty interior, and a *rooted body* is a pair (K, r) consisting of a body K and an interior point r (the *root*) of K. For each $\varrho > 0$, B_ϱ is the ball $\{x \in X: \|x\| \leqslant \varrho\}$. For an arbitrary $c \in X$, the *ball* with *center* c and *radius* ϱ is the set $c+B_\varrho$.

Use is made of the space $\mathscr{G}^d$ of all bodies $G \subset \boldsymbol{R}^d$ such that $0 \in \operatorname{int} G$; this space is metrized by the Hausdorff metric based on the Euclidean metric for $\boldsymbol{R}^d$. An important role is played by the fact that for each $\varrho \in \,]1, \infty[$ the set

$$\mathscr{G}^d_\varrho = \{G \in \mathscr{G}^d:\ B_1^d \subset G \subset B_\varrho^d\} \subset \mathscr{G}^d$$

is compact, where B_ϱ^d is the ball in $\boldsymbol{R}^d$ with center 0 and radius ϱ. This is used in conjunction with the theorem of Mahler [10] and Macbeath [9], asserting that for each d there exists $\varrho_d \in \,]1, \infty[$ such that each member of $\mathscr{G}^d$ that is symmetric about the origin can be mapped, by a suitable linear transformation of $\boldsymbol{R}^d$, onto a member of $\mathscr{G}^d_{\varrho d}$. Other main tools are as follows:

Sierpiński's theorem [8] [14] that a continuum does not admit a countable disjoint covering by closed proper subsets;

a method of transfinite construction used in [5] [6] to show that for each infinite cardinal m such that $m^{\aleph_0} = m$, the space $l^2(m)$ admits a bc-tiling and $l^1(m)$ admits a disjoint tiling by balls of unit radius;

some machinery developed in [7] (based on the Mahler–Macbeath theorem) for dealing with uniform properties of collections of rooted bodies;

the theorem of Corson [1] mentioned earlier.

Our section headings are as follows: 1. Countable convex tilings; 2. Some examples of uncountable fc-tilings; 3. Finding points that are covered by many nonoverlapping translates; 4. Some limitations on tile-shape.

1. Countable convex tilings. This section establishes some general properties of countable convex tilings, with special attention to consequences for the smoothness or rotundity of tiles.

1.1. Lemma. *If $\mathscr{C}$ is a countable closed covering of a Hausdorff space S and*

$$\mathscr{K} = \{C_1 \cap C_2:\ C_1, C_2 \in \mathscr{C},\ C_1 \neq C_2\}$$

then no subcontinuum of $S \sim \bigcup \mathscr{K}$ intersects more than one member of $\mathscr{C}$.

Proof. For an arbitrary subcontinuum Y of $S \sim \bigcup \mathscr{K}$, the collection $\{C \cap Y:\ C \in \mathscr{C}\}$ is a countable disjoint closed covering of Y. By Sierpiński's theorem [14], only one member of this collection is nonempty. ∎

The following result is sharpened in Corollary 1.7 below.

1.2. Corollary. *A topological vector space of dimension $\geqslant 2$ does not admit a countable rotund tiling.*

Proof. Suppose there is such a tiling $\mathscr{C}$, and let $\mathscr{K}$ be as in Lemma 1.1. Since the tiles are rotund and their interiors are disjoint, each nonempty member of $\mathscr{K}$ is a singleton. Thus the set $\bigcup \mathscr{K}$ is countable, and since the space is of dimension $\geqslant 2$, there is a segment that misses $\bigcup \mathscr{K}$ and has its ends in the interiors of different tiles. That contradicts 1.1 and completes the proof. ∎

1.3. Lemma. *Suppose that p and r are points of a topological vector space X that is not the union of countably many closed proper subflats, and that the segment $[p, r]$ crosses a closed hyperplane H and lies in an open set V. Suppose that $\mathscr{Q}$ is a countable collection of closed flats of codimension $\geqslant 2$ in X. Then there exists $q \in H$ such that*

$$]p, q] \cup [q, r[\subset V \sim \bigcup \mathscr{Q}.$$

Proof. We may assume without loss of generality that $0 \in [p, r] \cap H$, whence the space X is both algebraically and topologically the direct sum of the hyperplane H and the line $L = \boldsymbol{R}p$. For each $J \in \mathscr{Q}$ let

$$J_p = H \cap \operatorname{aff}(J \cup \{p\}) \quad \text{and} \quad J_r = H \cap \operatorname{aff}(J \cup \{r\}),$$

closed proper subflats of H, and let

$$\mathscr{G} = \{J_p:\ J \in \mathscr{Q}\} \cup \{J_r:\ J \in \mathscr{Q}\}.$$

Each neighborhood of 0 in H includes a point $q \notin \bigcup \mathscr{G}$, for if a neighborhood of 0 in H were covered by $\mathscr{G}$ then X would be covered by the countable collection

$$\{\varrho G + L:\ \varrho \text{ rational},\ G \in \mathscr{G}\}$$

of closed proper subflats of X. If $q \in H \sim \bigcup \mathscr{G}$ then the 2-gon $]p, q] \cup [q, r[$ misses $\bigcup \mathscr{Q}$, and by choosing q sufficiently close to 0 we may insure that this 2-gon lies in V. ∎

1.4. LEMMA. *Suppose that X is a topological vector space which has a closed hyperplane and is not the union of countably many closed proper subflats. Suppose that $\mathscr{C}$ is a countable closed covering of X,*

$$\mathscr{K} = \{C_1 \cap C_2\colon\ C_1, C_2 \in \mathscr{C},\ C_1 \neq C_2\},$$

$$\mathscr{M} = \{K \in \mathscr{K}\colon\ \operatorname{claff} K \text{ is of codimension } \leqslant 1 \text{ in } X\},$$

and V is a component of $X \sim \operatorname{cl}\bigcup\mathscr{M}$. Then any two points of V are joined in V by a polygonal path which, when its end are omitted, intersects only one member of $\mathscr{C}$.

Proof. Let G be a closed hyperplane in X, and use the term *G-path* to denote a polygonal path in which no edge is parallel to G. Since the space X is locally connected, the component V of the open set $X \sim \operatorname{cl}\bigcup\mathscr{M}$ is itself an open subset of X. Consider an arbitrary pair v, v' of points of V, and let V' denote the set of all points of V that are joined to v by a G-path in V. Then $v \in V'$, and a routine argument shows that V' is both open and closed relative to V, so it follows from V's connectedness that $V' = V$ and hence there is a G-path P from v to v' in V. Since P is compact and V is open, there is a neighborhood U_0 of the origin in X such that $P + U_0 \subset V$, and since X is a topological vector space there is a neighborhood U_1 of the origin such that

$$U_1 = [0, 1]U_1 \quad \text{and} \quad U_1 + U_1 + U_1 + U_1 \subset U_0.$$

Let the polygonal path

$$Q = [v_0, v_1] \cup [v_1, v_2] \cup \ldots \cup [v_{2l-1}, v_{2l}]$$

be obtained by subdividing the edges of P (in particular, $v_0 = v$ and $v_{2l} = v'$) in such a way that whenever $0 \leqslant i < j \leqslant 2l$ and $j \leqslant i+2$ then the segment $[v_i, v_j]$ is not parallel to G and $v_{i+1} - v_i \in U_1$. Let the neighborhood U of the origin be such that $U \subset U_1$ and such that whenever $0 \leqslant i < j \leqslant 2l$ and $j \leqslant i+2$, $w_i \in v_i + U$, and $w_j \in v_j + U$, then the segment $[w_i, w_j]$ is not parallel to G. Note that $[w_i, w_j] \subset V$ when $i < j \leqslant i+2$, for then

$$w_j \in v_i + U_1 + U_1 + U$$

and for each $\lambda \in [0, 1]$ it is true that

$$\lambda w_i + (1-\lambda) w_j \in \lambda v_i + [0, 1]U + (1-\lambda) v_i + [0, 1]U_1 + [0, 1]U_1 + [0, 1]U$$
$$\subset v_i + U_0 \subset P + U_0 \subset V.$$

Now let $\mathscr{Q} = \{\operatorname{cl}\operatorname{aff} K\colon\ K \in \mathscr{K} \sim \mathscr{M}\}$ and apply Lemma 1.3 l times (with $p = v_{2i-2}, r = v_{2i}$, $1 \leqslant i \leqslant l$) to obtain a G-path

$$Q' = [v_0, q_1] \cup [q_1, v_2] \cup \ldots \cup [q_{2l-1}, v_{2l}]$$

from v to v' in V such that

$$Q' \cap (\bigcup \mathscr{Q}) \subset \{v_0, v_2, \ldots, v_{2l}\}$$

and $q_i \in v_i + U$ for each odd i. Then apply Lemma 1.3 $l-1$ more times (with $p = q_i$, $r = q_{i+2}$, $1 \leqslant i \leqslant l-1$) to obtain a G-path

$$Q'' = [v_0, q_1] \cup [q_1, q_2] \cup \ldots \cup [q_{2l-1}, v_{2l}]$$

from v to v' in V such that

$$Q'' \cap (\bigcup \mathscr{Q}) \subset \{v, v'\}.$$

Each closed subarc of $Q'' \sim \{v, v'\}$ is a subcontinuum of $X \sim \bigcup\mathscr{K}$ and hence by Lemma 1.1 intersects only one member of $\mathscr{C}$. But then the same is obviously true of the set $Q'' \sim \{v, v'\}$ itself. ■

The following result is stated for simplicity in a Banach space, but the proof clearly applies to more general topological vector spaces.

1.5. THEOREM. *Suppose that $\mathscr{T}$ is a countable convex tiling of a Banach space X, and $\mathscr{F}$ is the collection of all sets F such that F is the intersection of two tiles and the affine hull of F is dense in a closed hyperplane of X. Then the interiors of the various tiles are precisely the components of $X \sim \operatorname{cl}\bigcup\mathscr{F}$.* *

Proof. If F is the intersection of a tile T with another tile, then F is a convex subset of T's boundary, and the affine hull of F, being the union of all lines that contain a segment in F, misses the interior of T. This implies that the present collection $\mathscr{F}$ is (when $\mathscr{T} = \mathscr{C}$) the same as the collection $\mathscr{M}$ of Lemma 1.4. If a component V of $X \sim \operatorname{cl}\bigcup\mathscr{F}$ intersects two tiles, then it intersects their interiors and the conclusion of 1.4 is contradicted. Hence V is contained in a unique tile T. Since V is open and $\operatorname{int} T$ is a connected subset of $X \sim \operatorname{cl}\bigcup\mathscr{F}$, it must be the case that $V = \operatorname{int} T$. ■

1.6. COROLLARY. *If X, $\mathscr{T}$ and $\mathscr{F}$ are as in 1.5, and $\mathscr{F}$ is finite, then $\mathscr{T}$ is finite.*

Proof. Since $\bigcup\mathscr{F}$ is closed, it follows from Theorem 1.5 that the boundary of any tile is the union of a finite subcollection of $\mathscr{F}$. There are only finitely many such subcollections, and no set is the boundary of more than two tiles. ■

1.7. COROLLARY. *If $\mathscr{T}$ is a countably infinite tiling of a Banach space of dimension $\geqslant 2$, there are infinitely many sets that are the intersections of two nonrotund tiles.*

For no $d \geqslant 2$ we know whether $\boldsymbol{R}^d$ admits a smooth bc-tiling. The remaining results of this section are relevant but they do not settle the question. *

1.8. THEOREM. *If $\boldsymbol{R}^d$ does not admit a smooth bc-tiling then neither does $\boldsymbol{R}^e$ for any $e \geqslant d$.*

Proof. The proof is by induction on e. Suppose that $e \geqslant d$, $\mathscr{T}$ is a smooth bc-tiling of $\boldsymbol{R}^e$, and H is a hyperplane in $\boldsymbol{R}^e$. Each member of $\mathscr{T}$ is supported by two translates of H, and since there are only countably many tiles there is a translate J of H that does not support any tile. The collection

$$\{T \cap J\colon\ T \in \mathscr{T},\ T \cap J \neq \emptyset\}$$

is easily seen to provide a smooth bc-tiling of J and hence of $\boldsymbol{R}^{e-1}$. ■

* See note added at end of paper.

1.9. THEOREM. *Suppose that*

$\mathcal{T}$ *is a smooth bc-tiling of* $\boldsymbol{R}^2$;

B is the union of the boundaries of the tiles;

$\mathcal{K}'$ *(resp.* $\mathcal{K}''$*) is the collection of all singletons* ⟨*resp. segments*⟩ *that are the intersections of two tiles;* $\mathcal{K} = \mathcal{K}' \cup \mathcal{K}''$;

$\mathcal{S}$ *is the collection of all segments* $[p, q]$ *such that* $[p, q]$ *is a maximal segment in the boundary of some tile and* $]p, q[$ *intersects no other tile.*

Then the following statements are all true:

(i) *No point belongs to more than two tiles.*

(ii) *The members of* $\mathcal{K}$ *are pairwise disjoint and the set* $\boldsymbol{R}^2 \sim \bigcup \mathcal{K}$ *is connected but not continuumwise connected; indeed, no continuum in* $\boldsymbol{R}^2 \sim \bigcup \mathcal{K}$ *intersects more than one tile.*

(iii) *If* $S_1, S_2, \ldots$ *is a sequence of pairwise disjoint segments in B, converging to a limit* S_0 *that is neither empty nor a singleton (boundedness of* S_0 *not assumed), then* S_0 *lies in a member S of* $\mathcal{S}$ *and* $S_0 = \lim_{i\to\infty} (S_i \cap J)$, *where J is the closed halfplane that contains S and misses the interior of the tile that contains S.*

(iv) *If* $k_1, k_2, \ldots$ *is a sequence of points converging to a point* k_0, *where* $k_i \in K_i$ *and the* K_i*s are distinct members of* $\mathcal{K}$, *then* $K_0 = \{k_0\} \in \mathcal{K}'$ *or*

$$\lim_{i\to\infty} (\text{length of } K_i) = 0$$

and k_0 *is an endpoint of the segment* $K_0 \in \mathcal{K}''$.

Proof. (i) Consider an arbitrary point $p \in B$. For each tile $T \ni p$ there is, by smoothness, a unique line $L(T)$ that supports T at p. For each pair of tiles T_1 and T_2 such that $p \in T_1 \cap T_2$ there is a line $L(T_1, T_2)$ that separates T_1 from T_2, and plainly $L(T_1) = L(T_1, T_2) = L(T_2)$. If p should belong to a third tile T_3 then $L(T_1, T_2) = L(T_2, T_3) = L(T_3, T_1)$, which is plainly impossible.

(ii) It is immediate from Sierpiński's theorem that no continuum in $\boldsymbol{R}^2 \sim \bigcup \mathcal{K}$ intersects more than one tile. Plainly $\mathcal{K}$ is countable and no member of $\mathcal{K}$ separates $\boldsymbol{R}^2$, so if the segments that belong to $\mathcal{K}$ are pairwise disjoint it follows from a theorem of Mullikin [12] (see also Miller [11]) that $\boldsymbol{R}^2 \sim \bigcup \mathcal{K}$ is connected. That the members of $\mathcal{K}$ are pairwise disjoint is a consequence of (i).

(iii) Since boundedness of S_0 is not assumed, convergence of (S_i) to S_0 is assumed only in the sense [16] that

$$\liminf(S_i) = S_0 = \limsup(S_i).$$

Plainly S_0 is a segment, ray or line and hence lies in a line L. For an appropriate closed halfplane J bounded by L, there is an infinite set I of positive integers such that the midpoint of S_i belongs to J for each $i \in I$, and such that either $S_i \cap L$ is empty for each $i \in I$, or $S_i \cap L$ is a singleton for each $i \in I$ and these singletons progress monotonically on L as $i \in I$ increases. In conjunction with the fact that

$S_0 = \lim_{i\in I} S_i$, this implies that the length of $S_i \sim J$ converges to 0 as $i \in I$ increases, whence $S_0 = \lim_{i\in I} (S_i \cap J)$. From this it follows that $(\operatorname{int} T) \cap J = \emptyset$ for each tile T that intersects the interior of S_0, and hence the interior of S_0 is covered by the boundaries of tiles that miss $\operatorname{int} J$. By smoothness, no point of L lies in two such tiles, and hence by Sierpiński's theorem there is a unique tile that intersects (and hence contains) the interior of S_0. This establishes the existence of S as described in (iii), and it is now also clear that $S_0 = \lim_{i\to\infty} (S_i \cap J)$.

(iv) Suppose that the set $K_0 \in \mathcal{K}$ is not a singleton, whence it is a segment that is the intersection of two tiles T_1 and T_2. The point k_0, being the limit of the sequence (k_i), is not interior to $T_1 \cup T_2$ and hence is an endpoint of K_0. If the length of K_i does not converge to 0 as $i \to \infty$ then the sequence (K_i) admits a subsequence (S_i) which converges, as in (iii), to a segment S_0 that lies in a member of $\mathcal{S}$. But then $S_0 \cap K_0 = \{k_0\}$ and k_0 belongs to three tiles, contradicting (i). ∎

We do not know of any bc-tiling of $\boldsymbol{R}^2$ that satisfies conditions (i)–(iv) of theorem 1.9. However, A. H. Stone (private communication) has constructed a tiling of $\boldsymbol{R}^d$, by rectangular boxes with their edges parallel to the coordinate axes, such that no point belongs to more than two tiles. When $d = 2$ his tiling satisfies (ii) as well as (i).

We include the following result because of our belief that it may be useful in deciding whether $\boldsymbol{R}^2$ admits a smooth tiling.

1.10. THEOREM. *Suppose that* $\mathcal{C}$ *is a countable compact covering of* $\boldsymbol{R}^2$,

$$\mathcal{G} = \{C_1 \cap C_2 \colon C_1, C_2 \in \mathcal{C},\ C_1 \neq C_2,\ |C_1 \cap C_2| > \aleph_0\},$$

H is a closed subset of $\boldsymbol{R}^2$ *that contains* $\bigcup \mathcal{G}$, *and* $\mathcal{L}$ ⟨*resp.* $\mathcal{M}$⟩ *is the collection of all components of H that do (resp. do not) separate* $\boldsymbol{R}^2$. *If each member of* $\mathcal{M}$ *is bounded then the set* $\bigcup \mathcal{L}$ *is unbounded.*

Proof. Assuming that each member of $\mathcal{M}$ is bounded and the set $\bigcup \mathcal{L}$ is bounded, we shall derive a contradiction. Let S^2 denote the one-point compactification of $\boldsymbol{R}^2$, with p the point at infinity. Let $\mathcal{N}$ denote the collection of all singletons in $S^2 \sim H$ and let $\mathcal{D} = \mathcal{L} \cup \mathcal{M} \cup \mathcal{N}$. From results of Whyburn [15] it follows that:

$\mathcal{L} \cup \mathcal{M}$ is countable;

$\mathcal{D}$ is an upper semicontinuous decomposition of S^2;

there is a continuous mapping f of S^2 onto a continuum Z such that the f-inverses of the points of Z are precisely the members of $\mathcal{D}$;

the cutpoints of Z are precisely the f-images of the members of $\mathcal{L}$;

each true cyclic element of Z is topologically a 2-sphere.

Since the set $\bigcup \mathcal{L}$ is bounded in $\boldsymbol{R}^2$, the point $f(p)$ admits a neighborhood U in Z such that U is homeomorphic with $\boldsymbol{R}^2$ and

$$U \subset f[S^2 \sim H] \cup f[\bigcup \mathcal{M}].$$

Let

$$Y = \{y \in \mathbf{R}^2\colon \{y\} = C_1 \cap C_2 \text{ for some } C_1, C_2 \in \mathscr{C}\} \cup \{p\}.$$

From the relationship of $\mathscr{M}$ to $\mathscr{C}$, the compactness of $\mathscr{M} \cup \mathscr{C}$, and the countability of $\mathscr{M}$ and of Y, it follows that there exist

$$v, w \in f^{-1}[U] \sim ((\bigcup M) \cup Y)$$

such that v and w belong to different members of $\mathscr{C}$ and each to a unique member of $\mathscr{C}$. Then $f(v)$ and $f(w)$ are distinct points of the arcwise-connected set

$$U \sim f[(\bigcup \mathscr{M}) \cup Y]$$

and hence are joined by an arc A in this set. The set $f^{-1}A$ is compact and the restriction of f to this set is biunique and continuous, so $f^{-1}A$ is an arc in $\mathbf{R}^2$ that is simply covered by its intersections with the members of $\mathscr{C}$. That contradicts Sierpiński's theorem and completes the proof. ∎

2. Some examples of uncountable fc-tilings. Many basic questions concerning tilability are still open. For example, it is apparently unknown for each $d \geqslant 2$ whether $\mathbf{R}^d$ admits a smooth bc-tiling, * and unknown whether the separable Hilbert space $l^2(\aleph_0)$ admits any bc-tiling at all. However, on the basic of evidence from [5] and [6] it seems reasonable to expect that uncountable tilings may behave quite differently from countable tilings. That expectation is strengthened by the present section, whose main goal is to construct, in a certain locally convex space, an uncountable disjoint fc-tiling whose tiles are all smooth and rotund — in fact, they are "canonical ellipsoids". The construction is based on ideas from [5] and [6] and is first described in an abstract setting, for it may be useful in other contexts as well.

The general setup is a bit complicated, but seems to be necessary for our purposes. The following assumptions are made:

(i) n is an infinite initial (hence a limit) ordinal and $(S_\alpha)_{\alpha \leqslant n}$ is a system of sets indexed by the ordinal numbers $\alpha \leqslant n$;

(ii) $\alpha < \beta \leqslant n \Rightarrow S_\alpha \subset S_\beta$;

(iii) $S_n = S_n^<$, where for each ordinal $\beta \leqslant n$, $S_\beta^< := \bigcup_{\alpha<\beta} S_\alpha$;

(iv) for each $\alpha \leqslant n$, $\mathbf{K}_\alpha$ is a family of collections of subsets of S_α, with $|K| > 1$ whenever $K \in \mathscr{K} \in \mathbf{K}_\alpha$; the empty collection belongs to $\mathbf{K}_\alpha$;

(v) for each $\alpha \leqslant n$, I_α is the function whose domain is the collection

$$\{K\colon K \subset S_n \text{ and } K \cap S_\alpha \neq \emptyset\},$$

with $I_\alpha(K) = K \cap S_\alpha$ for each set K in this domain;

(vi) if $\alpha < n$, $\mathscr{K} \in \mathbf{K}_\alpha$, and $\mathscr{K}' = \mathscr{K}$ or $\mathscr{K}' = \mathscr{K} \cup \{\{p\}\}$ for a point $p \in S_\alpha \sim \bigcup \mathscr{K}$, then there exists $\mathscr{L} \in \mathbf{K}_{\alpha+1}$ such that the restriction of I_α to the collection $\mathscr{L}$ is a one-to-one mapping of $\mathscr{L}$ onto the collection $\mathscr{K}'$;

* See note added at end of paper.

(vii) for each limit ordinal $\lambda \leqslant n$, J_λ is the function whose domain is the collection

$$\{K\colon K \subset S_n \text{ and } |K \cap S_\lambda^<| > 1\},$$

with $J_\lambda(K) = K \cap S_\lambda^<$ for each set K in this domain;

(viii) if λ is a limit ordinal $\leqslant n$ and $\mathscr{K}$ is a collection of subsets of $S_\lambda^<$ each of cardinality > 1, such that for each $\alpha < \lambda$,

$$\{K \cap S_\alpha\colon K \in \mathscr{K} \text{ and } |K \cap S_\alpha| > 1\} \in \mathbf{K}_\alpha,$$

then there exists $\mathscr{L} \in \mathbf{K}_\lambda$ such that the restriction of J_λ to the collection $\mathscr{L}$ is a one-to-one mapping of $\mathscr{L}$ onto $\mathscr{K}$.

Since n is an initial ordinal number, it is also a cardinal number, and the set

$$O_\beta := \{\text{ordinal } \alpha\colon \alpha < \beta\}$$

is of cardinality $< n$ for each ordinal $\beta < n$. The cardinal n is said to be *regular* if each cofinal subset of O_n is of cardinality n.

2.1. THEOREM. *If the above conditions* (i)–(viii) *are satisfied, the cardinal number n is regular, and the set S_n is of cardinality n, then there is a member of $\mathbf{K}_n$ that covers S_n.*

Proof. Since $|S_n| = n$, we may assume that the set S_n is well-ordered by an antireflexive relation $\prec$ so as to be order-isomorphic with O_n. Let a denote the first member of S_n and define the functions

$$\xi\colon S_n \to O_n \quad \text{and} \quad \zeta\colon S_n \to O_n$$

as follows:

$$\xi(t) = \min\{\alpha \in O_n\colon t \in S_\alpha\} \qquad (t \in S_n),$$
$$\zeta(a) = \xi(a)+1,$$
$$\zeta(t) = \max\{\xi(t), \sup_{s \prec t} \zeta(s)\}+1 \qquad (t \in S_n \sim \{a\}).$$

The last definition involves a routine transfinite induction, and it follows from n's regularity that $\zeta(t) < n$.

Now by a second transfinite induction we are going to define a function

$$\mathscr{K}(\cdot)\colon S_n \to 2^{(2^{S_n})}$$

so as to satisfy the following conditions:

$$\mathscr{K}(t) \in \mathbf{K}_{\zeta(t)}, \qquad t \in \bigcup \mathscr{K}(t) \qquad (t \in S_n),$$
$$I_{\zeta(s)} K(t) = K(s) \ (\text{or } K(s) \cup \{p\}) \qquad (s \prec t \in S_n).$$

To start the construction, note that by (iv), the empty collection belongs to $\mathbf{K}_{\zeta(a)}$, and hence the application of (vi) with $\mathscr{K}' = \{\{a\}\}$ yields a collection $\mathscr{K}(a)$ $(= \mathscr{L}) \in \mathbf{K}_{\zeta(a)}$ with $a \in \bigcup \mathscr{K}(a)$.

Let b denote the second member of S_n with respect to the well-ordering $\prec$, and proceed as follows:

if $\xi(b) \preceq \zeta(a)$ then $b \in S_{\zeta(a)}$ and $\zeta(b) = \zeta(a)+1$; construct $\mathscr{K}(b) \in \boldsymbol{K}_{\zeta(b)}$ by applying (vi) with $\mathscr{K}' = \mathscr{K}(a)$ or $\mathscr{K}' = \mathscr{K}(a) \cup \{\{b\}\}$ according as $b \in \bigcup \mathscr{K}(a)$ or $b \notin \bigcup \mathscr{K}(a)$;

if $\zeta(a) \prec \xi(b)$ then $b \notin S_{\zeta(a)}$ and $\zeta(b) = \xi(b)+1$; apply (vi) and (viii) appropriately to "extend" the collection $\mathscr{K}(a) \in \boldsymbol{K}_{\zeta(a)}$ to a collection in $\boldsymbol{K}_{\zeta(a)+1}, \ldots$, working through the intermediate stages and arriving eventually at an extension $\mathscr{K} \in \boldsymbol{K}_{\xi(b)}$. Then proceed as in the first case, applying (vi) to $\mathscr{K}$ or $\mathscr{K} \cup \{\{b\}\}$ according as $b \in$ or $b \notin \bigcup K$, to obtain $\mathscr{L} = \mathscr{K}(b)$.

For an arbitrary member t of $S_n \sim \{a\}$, the construction of $\mathscr{K}(t)$ from the collections $\mathscr{K}(s)$ for $s \prec t$ is entirely analogous to the construction of $\mathscr{K}(b)$ from $\mathscr{K}(a)$. By induction, then, $\mathscr{K}(t)$ is defined for all $t \in S_n$ so as to satisfy the stated conditions. The desired member of $\boldsymbol{K}_n$, covering S_n, is then obtained by a final application of (viii) with $\lambda = n$. ∎

For a cardinal number n and for $1 \leqslant p < \infty$, let $l^p(n)$ denote the space of all real-valued functions x on O_n such that $\sum_{\alpha \in O_n} |x(\alpha)|^p < \infty$, and let $l_f(n)$ denote the subspace consisting of all members x of $l^p(n)$ for which the support

$$\operatorname{supp} x := \{\alpha \in O_n \colon x(\alpha) \neq 0\} \text{ is finite.}$$

2.2. Corollary. *Suppose that m and n are infinite cardinal numbers, $m^{\aleph_0} = m$, and either $n = m^+$ (the successor of m) or m is regular and $n = m$. Let S_n denote $l^p(n)$ or $l_f(n)$, and for each $\alpha \in O_n$ let*

$$S_\alpha = \{x \in S_n \colon \operatorname{supp} x \subset O_\alpha\}.$$

If families $\boldsymbol{K}_\alpha \in 2^{2^{S_\alpha}}$ are given, for each $\alpha \leqslant n$, such that conditions (iv)–(viii) are satisfied, there is a member $\mathscr{K}$ of $\boldsymbol{K}_n$ that covers S_n. For each choice-function $c\colon \mathscr{K} \to S_n$ (i.e., $c(K) \in K$ for each $K \in \mathscr{K}$) there exists $U \subset O_n$ for which $|U| = m$ and the subspace

$$S_U = \{x \in S_n \colon \operatorname{supp} x \subset U\}$$

is covered by the collection

$$\{K \cap S_U \colon K \in K \text{ and } c(K) \in S_U\}.$$

Proof. Plainly the system $(S_\alpha)_{\alpha \leqslant n}$ satisfies conditions (i) and (ii). The hypotheses imply that $n^{\aleph_0} = n$, whence $|S_n| = n$ (see 1.2 of [5] for the argument). They imply also that n is regular, and from this it follows, since $n > \aleph_0$ and each member of $l^p(n)$ has countable support, that condition (iii) is satisfied. Since conditions (iv)–(viii) are part of our hypotheses, Theorem 2.1 is applicable and there exists $\mathscr{K} \in \boldsymbol{K}_n$ with $\bigcup \mathscr{K} = S_n$.

To deal with the choice-function c, we adapt part of the reasoning (suggested by W. Henson) of 3.2 of [6]. For each point $s \in S_n$, choose $K(s) \in \mathscr{K}$ with $s \in K(s)$

and then set $k(s) = c(K(s)) \in K(s)$. Set $U_0 = O_m$ and then construct the transfinite sequences

$$C_1 \subset C_2 \subset \ldots \subset C_\beta \subset \ldots \quad \text{and} \quad U_1 \subset U_2 \subset \ldots \subset U_\beta \subset \ldots \quad (\beta < \omega_1)$$

inductively as follows:

when β is a limit ordinal, $C_\beta = \bigcup_{\alpha<\beta} C_\alpha$ and $U_\beta = \bigcup_{\alpha<\beta} U_\alpha$;

when $\beta = \alpha+1$, $C_\beta = C_\alpha \cup k(S_{U_\alpha})$ and $U_\beta = U_\alpha \cup \bigcup_{s \in C_\beta} \operatorname{supp} s$.

In the latter case,

$$|U_\beta \sim U_\alpha| \leqslant |\bigcup_{s \in C_\beta \sim C_\alpha} \operatorname{supp} s| \leqslant \aleph_0 |k(S_{U_\alpha})| \leqslant \aleph_0 |S_{U_\alpha}| \leqslant \aleph_0 |U_\alpha|^{\aleph_0}$$

and from $|U_\alpha| = m = m^{\aleph_0}$ it follows that $|U_\beta| = m$. In the former case,

$$|U_\beta| \leqslant \aleph_0 \sup\{|U_\alpha| \colon \alpha < \beta\},$$

and when all relevant $|U_\alpha|$s are equal to m, so is $|U_\beta|$. Hence by induction, $|U_\beta| = m$ for all $\beta < \omega_1$, and then with

$$C = \bigcup_{\beta<\omega_1} C_\beta \quad \text{and} \quad U = \bigcup_{\beta<\omega_1} U_\beta,$$

we have $|U| \leqslant 2^{\aleph_0} m = m$ and hence $|U| = m$. Plainly the subspace S_U has the stated property. ∎

The following result was proved in [5], [6].

2.3. Corollary. *If $1 \leqslant p < \infty$ and m is an infinite cardinal for which $m^{\aleph_0} = m$, then the space $l^p(m)$ is covered by a collection of balls of unit radius whose centers form a $2^{1/p}$-dispersed set. For $p = 1$ these balls provide a disjoint tiling of the space.*

Proof. Apply 2.2 with $n = m^+$. For each $\alpha \leqslant n$, let $\boldsymbol{K}_\alpha$ consist of all collections of balls of unit radius in S_α such that the centers form a $2^{1/p}$-dispersed set (pairwise distances $> 2^{1/p}$). It follows from reasoning in [6] that the families $\boldsymbol{K}_\alpha$ satisfy conditions (iv)–(viii). For each ball K in the collection $\mathscr{K}$ of 2.2, let $c(K)$ denote the center of K. With $|U| = m$, the subspace S_U of 2.2 is in effect a replica of $l^p(m)$, and the desired conclusion follows. ∎

When X is $l^2(n)$ or $l_f(n)$, $c \in X$, and $a\colon O_n \to [0, 1]$, we define the set

$$E(c;\, a) = \Big\{x \in X \colon a(i) = 0 \Rightarrow x(i) = c(i);\ \sum_{i \in O_n, a(i)>0} \Big(\frac{x(i)-c(i)}{a(i)}\Big)^2 \leqslant 1\Big\}.$$

Being closed for the l^2-norm topology, it is also closed for any finer topology, and being of l^2-diameter $\leqslant 2$, it is certainly finitely bounded. The set of all points of $E(c;\, a)$ for which $=$ ⟨resp. $<$⟩ holds in $\leqslant$ is the *radial boundary* $\operatorname{rbd} E(c;\, a)$ (resp. *radial interior* $\operatorname{rin} E(c;\, a)$). It is easily verified that the set $E = E(c;\, a)$ is convex, is radially rotund in the sense that $\operatorname{rbd} E$ contains no segment, and is radially smooth in the sense that, relative to the affine hull of E, no point of $\operatorname{rbd} E$ lies on more than one hyperplane that misses $\operatorname{rin} E$. The set $E(c;\, a)$ has nonempty interior with respect to the l^2 topology for X if and only if $\inf\{a(i): i \in O_n\} > 0$, and when $X = l^2(n)$ the same is true with respect to the finest locally convex topology for X. However, when $X = l_f(n)$ and $a(i) > 0$ for all $i \in O_n$, the set $\operatorname{rin} E(c; a)$ is open with respect to the finest locally convex topology, and hence for this topology $E(c; a)$ is genuinely smooth and rotund in the sense defined earlier.

When $a\colon O_n \to]0, 1]$, the set $E(c; a)$ will be called an *n-canonical ellipsoid*, or simply an *ellipsoid* if there is no danger of confusion.

2.4. THEOREM. *If m is an infinite cardinal for which $m^{\aleph_0} = m$ then each of the spaces $l^2(m)$ and $l_f(m)$ admits a disjoint covering by m-canonical ellipsoids. For $l_f(m)$ this is a smooth rotund fc-tiling with respect to the finest locally convex topology.*

Proof. Let $n = m^+$, and let S_n and the S_αs be as in Corollary 2.2. For each $\alpha \leqslant n$ let $\boldsymbol{K}_\alpha$ denote the family of all disjoint collections of α-canonical ellipsoids in S_α, so that condition (iv) is certainly satisfied. To apply Corollary 2.2 we want to prove that condition (vi) is satisfied if the I_αs are defined by (v) and condition (viii) is satisfied if the J_λs are defined by (vii).

Suppose that $\mathscr{K}'$ is as in (vi), so that for each $K \in \mathscr{K}'$ with $|K| > 1$ there exist $c_K \in S_\alpha$ and $a_K\colon O_\alpha \to]0, 1]$ such that

$$K = \left\{x \in S_\alpha\colon \sum_{i \in O_\alpha} \left(\frac{x(i) - c_K(i)}{a_K(i)}\right)^2 \leqslant 1\right\}.$$

If $\mathscr{K}' = \mathscr{K}$, then for each $K \in \mathscr{K}$ let

$$L_K = \left\{x \in S_{\alpha+1}\colon \sum_{i \in O_{\alpha+1}} \left(\frac{x(i) - c'_K(i)}{a'_K(i)}\right)^2 \leqslant 1\right\},$$

where $c'_K = (c_K, 0) \in S_{\alpha+1}$ and $a'_K = (a_K, 1)\colon O_{\alpha+1} \to]0, 1]$; set $\mathscr{L} = \{L_K\colon K \in \mathscr{K}\}$. If $\mathscr{K}' = \mathscr{K} \cup \{\{p\}\}$, take in $S_{\alpha+1}$ (equipped with the l^2 norm) the ball B of unit radius and center $p' = (p, 1)$. It follows from B's uniform rotundity that each $K \in \mathscr{K}$ is at positive distance from B; hence

$$\inf\{y\colon (x, y) \in B \text{ for some } x \in K\} = d_K > 0.$$

Set $\mathscr{L} = \{L_K\colon K \in \mathscr{K}\} \cup \{B\}$, where the L_Ks are defined as above with $c'_K = (c_K, 0)$ and $a'_K = (a_K, \min\{1, d_K/2\})$. In both cases the collection $\mathscr{L}$ belongs to $\boldsymbol{K}_{\alpha+1}$ and the restriction of I_α to $\mathscr{L}$ maps $\mathscr{L}$ onto $\mathscr{K}'$. This establishes (vi).

Now consider a limit ordinal $\lambda \leqslant n$ and a collection $\mathscr{K}$ of subsets of $S_\lambda^<$ as described in (viii). We want to produce a collection $\mathscr{L} \in \boldsymbol{K}_\lambda$ such that the restriction of J_λ to $\mathscr{L}$ is a one-to-one mapping of $\mathscr{L}$ onto $\mathscr{K}$. Note that for each $K \in \mathscr{K}$ it is true that $|K| > 1$ and hence there exists $\alpha_K < n$ such that $K \cap S_\alpha$ is an α-canonical ellipsoid whenever $\alpha_K < \alpha < \lambda$; also,

$$(K \cap S_\beta) \cap S_\alpha = K \cap S_\alpha \quad \text{for } \alpha_K < \alpha < \beta < \lambda.$$

From this it follows that if L_K is the closure of K in S_λ (with respect to the l^2 topology) then L_K is a λ-canonical ellipsoid. (Note that $L_K = K$ in the case of $l_f(n)$, and also in the case of $l^2(n)$ when λ is of cofinality $> \aleph_0$.) The collection $\mathscr{L} = \{L_K\colon K \in \mathscr{K}\}$ is mapped one-to-one by J_λ onto $\mathscr{K}$. Also, $\mathscr{L}$ is disjoint, for if $G, K \in \mathscr{K}$, $p \in L_G \cap L_K$, and

$$\max\{\alpha_G, \alpha_K\} < \beta < \lambda$$

then the natural projection of p onto S_β belongs to both $G \cap S_\beta$ and $K \cap S_\beta$, an impossibility since these intersections belong to the same member of $\boldsymbol{K}_\beta$. Since (vi) and (viii) have been verified, the first part of 2.2 yields the existence of $\mathscr{K} \in \boldsymbol{K}_n$ with $\bigcup \mathscr{K}$ equal to $l^2(n)$ or $l_f(n)$. To pass from $\mathscr{K}$ to the desired covering of $l^2(m)$ or $l_f(m)$, apply the second part of 2.2 with the choice-function that associates to any $K \in \mathscr{K}$ its center. Note that the subspace S_U is a replica of $l^2(m)$ or $l_f(m)$, and each intersection $K \cap S_U$ is a canonical ellipsoid in S_U. ∎

3. Finding points that are covered by many nonoverlapping translates. Various terms (e.g., packing, disjoint, tiling) are used to describe a collection $\mathscr{K}$ of rooted bodies if they apply properly to the collection of bodies associated with $\mathscr{K}$. The main results of this section concern packings of rooted bodies. They are applied in Section 4 to establish limitations on smoothness and rotundity of tilings.

A rooted body (K, r) is *homothetic* (resp. *positively homothetic*) to a rooted body (K', r') if there exists $\lambda \neq 0$ (resp. > 0) such that $K - r = \lambda(K' - r')$.

For each rooted body (K, r) in X, let $\mathscr{S}^d(K, r)$ denote the set of all $G \in \mathscr{G}^d$ such that $T(K \cap F) = G$ for some d-dimensional affine subspace (flat) F of X through r and some affine transformation T of F onto $\boldsymbol{R}^d$ with $T(r) = 0$. Thus $\mathscr{S}^d(K, r)$ represents the d-sections of K through r. For each collection $\mathscr{K}$ of rooted bodies, let

$$\mathscr{S}^d(\mathscr{K}) = \bigcup \{\mathscr{S}^d(K, r)\colon (K, r) \in \mathscr{K}\}$$

and let $S^d_-(\mathscr{K})$ denote the closure of $\mathscr{S}^d(\mathscr{K})$ in $\mathscr{G}^d$. The "nonoverlapping translates" of the section heading are in $\mathscr{S}^d_-(\mathscr{K})$ and they appear in the following theorem.

3.1. THEOREM. *Suppose that $\mathscr{K}$ is a packing of rooted bodies in a normed linear space X, and that there exist functions $\xi\colon \mathscr{K} \to]0, \infty[$ and $\eta\colon \mathscr{K} \to]0, \infty[$ such that*

(a) *for each* $(K, r) \in \mathscr{K}$, $r + B_{\xi(K,r)} \subset K \subset r + B_{\eta(K,r)}$,

(b) $\sup_{(K,r) \in \mathscr{K}} \eta(K, r)/\xi(K, r) < \infty$,

(c) $\inf_{(K,r) \in \mathscr{K}} \xi(K, r) > 0$.

Suppose also that there exist a subset V of X and a function $(K(\cdot), r(\cdot))\colon V \to \mathscr{K}$ such that V has an accumulation point in X and for each $v \in V$, $K(v) \cap V = \{v\}$. Then for each positive integer $d \leqslant \dim X$ there are members $C_1, \ldots, C_d$ of $\mathscr{S}^d_-(\mathscr{K})$ and points $t_1, \ldots, t_d$ of $\boldsymbol{R}^d$ such that the translates $C_1 + t_1, \ldots, C_d + t_d$ have a common boundary point but disjoint interiors. If the members of $\mathscr{K}$ are all positively homothetic (resp. homothetic) to a single rooted body, it can be arranged that each C_j is equal to C_1 (resp. $\pm C_1$).

Proof. We start by noting that the packing $\mathscr{K}$ can be assumed to be normal, i.e. to satisfy condition (a) with constant functions $\xi(K, r)$ and $\eta(K, r)$. To prove this claim, consider the subset of $\mathscr{K}$ whose elements are the rooted bodies $(K(v), r(v))$, for $v \in V$. For notational simplicity set

$$\xi(v) = \xi\big(K(v), r(v)\big), \qquad \eta(v) = \eta\big(K(v), r(v)\big),$$
$$\xi = \inf_{v\in V} \xi(v), \qquad \sigma = \sup_{v\in V} \eta(v)/\xi(v).$$

Define, for each $v \in V$, the (affine) function $f_v \colon X \to X$

$$f_v(x) = \big(\xi/\xi(v)\big)x + \big(1-(\xi/\xi(v))\big)v$$

and set

$$K'(v) = f_v\big(K(v)\big), \qquad r'(v) = f_v\big(r(v)\big).$$

The collection $\mathscr{K}' = \{(K'(v), r'(v))\}$ is a packing, because $K'(v) \subset K(v)$ for each v, and, for the same reason, $K'(v) \cap V = \{v\}$. Moreover, for a suitable $\delta > 0$, and for each $v \in V$,

$$r'(v) + B_{\delta^{-1}} \subset r'(v) + B_\xi \subset K'(v) \subset r'(v) + B_{\xi\sigma} \subset r'(v) + B_\delta .$$

Since $\mathscr{S}^d_-(\mathscr{K}') \subset \mathscr{S}^d_-(\mathscr{K})$, we can replace in the proof conditions (a), (b) and (c) by the assumption that, for some positive δ.

(a′) $r + B_{\delta^{-1}} \subset K \subset r + B_\delta$.

We assume also, without loss of generality, that the origin 0 is an accumulation point of V. For each positive integer i let $v_i^1, \ldots, v_i^d$ be d distinct points of the set V at distance $\leqslant 1/i$ from 0. Set $K_i^j = K(v_i^j)$, $r_i^j = r(v_i^j)$, and for each i let A_i be a d-dimensional subspace of X such that

$$\{r_i^1, \ldots, r_i^d\} \subset A_i .$$

Set

$$M_i^j = (K_i^j - r_i^j) \cap A_i, \qquad M_i = \operatorname{co} \bigcup_{j=1}^{d} M_i^j .$$

Then for each i

$$B_{\delta^{-1}} \cap A_i \subset M_i \subset B_\delta \cap A_i$$

because of (a′). It follows that

$$M_i \subset \delta^2(M_i \cap -M_i),$$

whence, by the theorem of Mahler [10] and Macbeath [9] mentioned earlier, there is a linear transformation T_i of A_i onto $\boldsymbol{R}^d$ that carries M_i onto a body

$$T_i(M_i) \in G_{\varrho_d\delta^4} .$$

The transformations T_i and T_i^{-1} are equibounded, for

$$\|T_i\| = \sup_{x\in B_1\cap A_i} \|T_i x\| \leqslant \delta \sup_{x\in M_i} \|T_i x\| \leqslant \delta \sup_{y\in B^d_{\varrho_d\delta^2}} \|y\| = \varrho_{p_d\delta^3},$$

$$\|T_i^{-1}\| = \sup_{y\in B_1^d} \|T_i^{-1} y\| \leqslant \sup_{y\in T_i(M_i)} \|T_i^{-1} y\| \leqslant \delta .$$

The bound on $\|T_i^{-1}\|$ implies that, for all i and j,

$$B^d_{\delta^{-2}} \subset T_i(M_i^j) \subset B^d_{\varrho_d\delta^3}$$

because if $y \in B^d_{\delta^{-2}}$ then $\|T^{-1}y\| \leqslant \delta^{-1}$ and hence

$$T_i^{-1} y \in B_{\delta^{-1}} \cap A_i \subset M_i^j .$$

In particular, $T_i(M_i^j) \in \mathscr{S}^d(\mathscr{K})$ for all i and j, and since the set

$$\{G \in G^d\colon\ B^d_{\delta^{-2}} \subset G \subset B^d_{\varrho_d\delta^3}\}$$

is compact, we may assume (passing to a subsequence if necessary) that for each j,

$$T_i(M_i^j) \to C_j \in \mathscr{S}^d_-(\mathscr{K}) \qquad \text{as } i \to \infty .$$

At this point, it may be helpful to outline the remainder of the proof.

The next step is to show that

(i) points $z_i^j \in K_i^j \cap A_i$ can be chosen, for $j = 1, \ldots, d$ and $i = 1, 2, \ldots$ in such a way that for each j,

$$z_i^j \to 0 \qquad \text{as } i \to \infty .$$

We then define

(ii) $$t_i^j = T_i(z_i^j - r_i^j) \in B^d_{\varrho_d\delta^2}$$

and may assume (passing to further subsequences if necessary) that for each j,

(iii) $$t_i^j \to t_j \in C_j \qquad \text{as } i \to \infty$$

where the convergence is a consequence of compactness of $B^d_{\varrho_d\delta^2}$ and the membership of t_j in C_j follows from the relevant definitions. We then prove that the sets

(iv) $$C_j - t_j$$

have the stated properties.

A final paragraph considers the case in which the members of $\mathscr{K}$ are all homothetic to a single rooted body.

Now suppose that (i) fails. Then an $\varepsilon > 0$ and an infinite set I_0 of i's exists such that, for at least one j,

$$d(K_i^j \cap A_i, 0) > \varepsilon \qquad \text{for all } i \in I_0 .$$

We may also assume $\|v_i^j\| < \varepsilon$ for $i \in I_0$, whence $v_i^j \notin A_i$. For such j and $i \in I_0$, in the 2-dimensional subspace spanned by v_i^j and r_i^j, set

$$u_i^j = \varepsilon r_i^j / \|r_i^j\|$$

and consider the line through the points v_i^j and u_i^j and the line through r_i^j parallel to the segment $[0, v_i^j]$. Let w_i^j be the unique point of intersection of these lines. For each $i \in I_0$, the two triangles $\mathrm{co}(0, u_i^j, v_i^j)$ and $\mathrm{co}(r_i^j, u_i^j, w_i^j)$ are homothetic, and eventually we have

$$\|r_i^j\| < 4\delta/3$$

because $d(0, K_i^j) < 1/i$ and $K_i^j \subset r_i^j + B_\delta$. Therefore, as $i \to \infty$ and $\|v_i^j\| \to 0$, also

$$\|w_i^j - r_i^j\| \to 0.$$

But w_i^j does not belong to K_i^j, because u_i^j, that is a convex combination of v_i^j $(\in K_i^j)$ and w_i^j, is not in K_i^j; hence, in consequence of (a′),

$$\|w_i^j - r_i^j\| > \delta^{-1},$$

a contradiction.

Now we want to prove that the sets $C_j - t_j$ introduced in (iv) have disjoint interiors but all have a common boundary point. It is obvious from the relevant definitions that 0 belongs to all the $(C_j - t_j)$s. Suppose now that two of them have overlapping interiors, i.e. suppose that $a \in \mathbf{R}^d$, $\varepsilon > 0$, and two indices j and $k \neq j$, exist such that

$$a + B_\varepsilon^d \subset (C_j - t_j) \cap (C_k - t_k).$$

Then

$$a + t_j + B_\varepsilon^d \subset C_j$$

and, in view of the definitions of t_j and C_j, taking into account the fact that if B is a ball contained in the (Hausdorff) limit C of a sequence of convex sets, with $\mathrm{dist}(B, \partial C) > 0$, then B is eventually contained in any set of the sequence (this fact can be proved as an easy consequence of the separation theorem), we obtain

$$a + t_i^j + B_{\varepsilon/2}^d \subset T_i(M_i^j) \quad \text{for } i > i_0.$$

Since each T_i is one-to-one and the set $\{\|T_i\|\}$ is bounded, for a suitable $\theta > 0$

$$T_i^{-1}(a + t_i^j) + B_\theta \cap A_i \subset M_i^j.$$

Then, by linearity of T_i^{-1} and the relevant definitions

$$T_i^{-1}(a) + z_i^j - r_i^j + B_\theta \cap A_i \subset (K_i^j - r_i^j) \cap A_i.$$

Since $T_i^{-1}(a)$ and r_i^j belong to A_i, and $z_i^j \to 0$, eventually

$$T_i^{-1}(a) + B_{\theta/2} \cap A_i \subset K_i^j.$$

Analogously, for k,

$$T_i^{-1}(a) + B_{\theta/2} \cap A_i \subset K_i^k.$$

The hypothesis that $\mathcal{K}$ is a packing implies that $T_i^{-1}(a) + B_{\theta/2} \cap A_i$, being contained in $K_i^j \cap K_i^k$, is contained in the boundary of each of the two sets. But this is contradicted by the fact that it contains points of the open segment $]r_i^j, T_i^{-1}(a)[$, which is contained in A_i because both r_i^j and $T_i^{-1}(a)$ are, and which is interior to K_i^j.

To complete the proof of Theorem 3.1, we consider the case in which the members of $\mathcal{K}$ are all positively homothetic (homothetic), with respect to their roots, to a single rooted body $(K, 0)$. In this case

$$K_i^j = r_i^j + \lambda_i^j K, \quad \lambda_i^j > 0 \ (\lambda_i^j \in \mathbf{R}).$$

Under this assumption

$$M_i^j = \lambda_i^j (K \cap A_i) \quad \text{and} \quad T_i(M_i^j) = \lambda_i^j T_i(K \cap A_i).$$

Since

$$B_{\delta^{-1}} \cap A_i \subset M_i^j \subset B_\delta \cap A_i$$

and we may obviously assume (passing to a bigger δ if necessary)

$$B_{\delta^{-1}} \subset K \subset B_\delta$$

we obtain for any i, j

$$\delta^{-2} < |\lambda_i^j| < \delta^2.$$

From this inequality, and from the fact (proved in the preceding pages), that for any i, j

$$B_{\delta^{-2}}^d \subset T_i(M_i^j) \subset B_{\varrho_a \delta^2}^d$$

we can deduce (passing to a subsequence if necessary) that, as $i \to \infty$

$$T_i(K \cap A_i) \to C$$

and, for any j,

$$\lambda_i^j \to \lambda_j > 0 \quad (\neq 0)$$

It is easy to see that

$$T_i(M_i^j) = \lambda_i^j T_i(K \cap A_i) \to \lambda_j C.$$

Hence the C_js can be chosen of the form $C_j = \lambda_j C$. To produce sets C_j' and points t_j' which satisfy the required conditions and with the C_j's all equal to the same set C_1' $\langle \pm C_1' \rangle$, assume, without loss of generality,

$$|\lambda_1| = \min\{|\lambda_j| : j = 1, \ldots, n\}.$$

Set, for each j,

$$C_j' = |\lambda_1/\lambda_j| C_j = \lambda_1 C, \quad t_j' = |\lambda_1/\lambda_j| t_j.$$

Obviously

$$C_j' - t_j' = |\lambda_1/\lambda_j| (C_j - t_j) \subset C_j - t_j,$$

hence the sets $(C_j' - t_j')$s have pairwise disjoint interiors, but all contain the point 0. This completes the proof. ∎

In the situation described in the next result, the conditions of Theorem 3.1 are satisfied in a strengthened form.

3.2. PROPOSITION. *Suppose that $\mathscr{L}$ is a disjoint collection of rooted bodies in a normed linear space, and L is a line covered by the bodies. Then there exist $\mathscr{K} \subset \mathscr{L}$, $\delta > 0$, an uncountable subset V of L, and a function $(K(\cdot), r(\cdot))$: $V \to \mathscr{K}$ such that for each $(K, r) \in \mathscr{K}$,*

$$r + B_\delta \subset K \subset r + B_{1/\delta},$$

and for each $v \in V$, $K(v) \cap L = \{v\}$.

Proof. For each $x \in L$ let the rooted body $(K(x), r(x)) \in \mathscr{K}$ be defined by the condition that $x \in K(x)$ and let $\varrho(x) > 0$ be such that

$$r(x) + B_{\varrho(x)} \subset K(x) \subset r(x) + B_{1/\varrho(x)}.$$

For each $x \in L$ the intersection $K(x) \cap L$ is the singleton $\{x\}$ or a closed proper subsegment of L. The former must occur for uncountably many $x \in L$, for otherwise (since $\mathscr{C}$ is disjoint) L would be the union of a countable disjoint collection of singletons and segments, contradicting Sierpiński's theorem. In the uncountable set of $x \in L$ for which $K(x) \cap L = \{x\}$, there is an uncountable subset V for which $\inf_{v \in V} \varrho(v) = \delta > 0$. ■

A collection $\mathscr{C}$ of sets is *point-finite* if no point belongs to infinitely many members of $\mathscr{C}$. The following result is an aid in applying Theorem 3.1.

3.3. PROPOSITION. *Suppose that $\mathscr{C}$ is a bc-tiling of a normed linear space X that has an infinite-dimensional reflexive subspace. Then there exist a convergent sequence $\{v_n\}_1^\infty = V$ in X and a function $C(\cdot)$: $V \to \mathscr{C}$ such that for each $v \in V$, $C(v) \cap V = \{v\}$.*

Proof. By Corson's theorem [1], $\mathscr{C}$ is not locally finite. We may assume that $\mathscr{C}$ is not locally finite at 0, whence for any positive integer n there exists an infinite subcollection $\mathscr{C}_n$ of $\mathscr{C}$ such that any member of $\mathscr{C}_n$ intersects the open ball $B_{1/n}$. Choose $C_1 \in \mathscr{C}_1$, $v_1 \in \operatorname{int} C_1 \cap B_1$ and let $C(v_1) = C_1$, then choose $C_2 \in \mathscr{C}_2 \sim \{C_1\}$, $v_2 \in \operatorname{int} C_2 \cap B_{1/2}$ and let $C(v_2) = C_2$. Continuing in this manner leads to the desired conclusion. ■

4. Some limitations on tile-shape. Recall that a normed linear space X is said to be *smooth* or *rotund* if its unit ball has these properties, and that uniform smoothness and uniform rotundity have also been much studied (see Day [2], for example). Intuitively, X is uniformly smooth (resp. uniformly rotund) if and only if its unit sphere does not "come arbitrarily close to" containing a segment (resp. having more than one supporting hyperplane at some point). In terms of the definitions of Section 3, involving collections $\mathscr{K}$ of rooted bodies, this is made precise in the following result from [7], whose proof depends on the theorem of Mahler [10] and Macbeath [9] mentioned earlier.

4.1. THEOREM. *Suppose that d is an integer $\geqslant 2$, X is a normed linear space of dimension $\geqslant d$, B_1 is the unit ball of X, and $\mathscr{K}$ is the collection $\{(B_1, 0)\}$ of rooted bodies. Then X is uniformly smooth (resp. uniformly rotund) if and only if each member of $\mathscr{S}^d_-(\mathscr{K})$ is smooth (resp. rotund).*

Because of 4.1, an arbitrary collection $\mathscr{K}$ of rooted bodies is said to be *uniformly d-smooth* (resp. *uniformly d-rotund*) if each of the bodies is of dimension $\geqslant d$ and each body in the set $\mathscr{S}^d_-(\mathscr{K})$ is smooth (resp. rotund). It is shown in [7] that if $2 \leqslant d \leqslant e$ and each member of $\mathscr{K}$ is of dimension $\geqslant e$, then the conditions for d imply those for e, and when each rooted body in $\mathscr{K}$ is symmetric with respect to its root, the conditions for d are equivalent to those for e.

Theorems 4.6 and 4.7 below assert that under certain additional conditions, a tiling of an infinite-dimensional space cannot be uniformly smooth or uniformly rotund. The proofs are based on Theorem 3.1 and on the intersection properties of smooth and rotund bodies that are expressed in Corollary 4.5.

4.2. REMARK. *If A is a smooth convex subset of a topological vector space and B is a convex set that intersects A but misses the interior of A, then A and B admit a unique separating hyperplane.*

Proof. Use the basic separation theorem and the definition of smoothness. ■

We note in passing that as a property of balls, 4.2 characterizes smooth spaces:

4.3. PROPOSITION. *The following three conditions on a normed linear space X are equivalent:*

(i) *X is smooth;*

(ii) *Whenever two balls in X intersect but have disjoint interiors, there is a unique separating hyperplane;*

(iii) (ii) *restricted to balls of unit radius.*

Proof. In view of 4.2, it suffices to show that if (i) fails so does (iii). If G and H are hyperplanes that support the unit ball B_1 at a boundary point p, then $-G$ and $-H$ both support the set $-B_1 = B_1$ at the point $-p$. But $-G = G - 2p$ and $-H = H - 2p$, so the hyperplanes G and H support both B_1 and $B_1 + 2p$ at the point p, hence separate B_1 from $B_1 + 2p$. ■

4.4. PROPOSITION. *The following three conditions on a normed linear space X are equivalent:*

(i) *X is rotund;*

(ii) *whenever U and V are balls in X, with centers u and v, with disjoint interiors and a common boundary point p, then $p \in [u, v]$;*

(iii) (ii) *restricted to balls of unit radius.*

Proof. If (i) fails then the unit sphere S_1 contains a segment $[q, r]$. With $U = B_1$ and $V = B_1 + q + r$, U and V are balls of unit radius with disjoint interiors and common boundary points q and r. Of course q and r cannot both belong to the segment $[0, q + r]$, so (iii) fails. This shows (iii) implies (i).

In showing that (i) implies (ii), we may assume that $U = B_1$ and $V = \gamma B_1 + v$. Let $q = (1+\gamma)^{-1} v \in [0, v]$. Since V intersects U, $\|v\| \leqslant 1 + \gamma$ and hence $q \in U$. At the same time,

$$\|v - q\| = \gamma(1+\gamma)^{-1}\|v\| \leqslant \gamma$$

and hence $q \in V$. But if U and V are as in (ii) and U is rotund, then of course $U \cap V = \{p\}$ and hence $p = q \in [u, v]$. ■

4.5. COROLLARY. *If $\mathscr{P}$ is a packing of smooth convex sets or of rotund balls, then no point belongs to more than two members of $\mathscr{P}$.*

Proof. Suppose that some point p belongs to three distinct members U, V and W of $\mathscr{P}$. If U, V and W are smooth it follows from 4.2 that there is a single hyperplane through p which simultaneously separates U from V, V from W and W from U. That is obviously impossible. If U, V and W are rotund balls with centers u, v and w it follows from 4.4 that

$$p \in [u, v] \cap [v, w] \cap [w, u],$$

another impossibility. ■

It follows from Corollary 1.2 that there is no rotund tiling of a separable normed linear space of dimension $\geqslant 2$. We do not know whether some such spaces admit smooth *bc*- or *fc*-tilings. However, Theorem 4.7 below gives some results for uniformly d-smooth *bc*-tilings. For the nonseparable case, the following results may be compared with the examples in Section 2.

4.6. THEOREM. *Suppose that $\mathscr{C}$ is a collection of balls in a uniformly rotund normed linear space X. If*

(i) *$\mathscr{C}$ is disjoint or*

(ii) *X is complete and at least 2-dimensional, and the radii of $\mathscr{C}$'s members are bounded away from 0,*

then $\mathscr{C}$ is not a tiling of X.

4.7. THEOREM. *Suppose that X is a normed linear space and the collection $\mathscr{K}$ of rooted bodies in X is uniformly d-smooth for some $d \geqslant 2$. If*

(i) *$\mathscr{K}$ is disjoint or*

(ii) *$2 \leqslant \dim X < \infty$ or X contains an infinite-dimensional reflexive subspace, and there exist functions $\xi\colon \mathscr{K} \to]0, \infty[$ and $\eta\colon \mathscr{K} \to]0, \infty[$ satisfying conditions* (a), (b) *and* (c) *of Theorem* 3.1,

then $\mathscr{K}$ is not a tiling of X.

Proof of 4.6 and 4.7. When X is finite dimensional, 4.6 is subsumed by 1.2, and 4.7 (i) by [5]. To prove 4.7 (ii), assume that $\mathscr{K}$ is a tiling. We claim that it is locally finite which, by known results, contradicts the smoothness assumption. To prove the claim, suppose $\mathscr{K}$ is not locally finite at a point x. For a bounded neighborhood W of x, consider the infinite subcollection of $\mathscr{K}$

$$K_1 = \{K \in \mathscr{K}\colon K \cap W \neq \emptyset\}.$$

Working as in the first part of the proof of Theorem 3.1, produce a normal packing $\mathscr{P} = \{P\}$ such that any P meets W. By normality of $\mathscr{P}$, a bounded subset of the finite dimensional space X contains an infinite set with no accumulation point, a contradiction.

Now we may assume X is infinite-dimensional. For 4.6, form a collection $\mathscr{K}$ of rooted bodies by rooting each member of $\mathscr{C}$ at its center; then $\mathscr{K}$ is uniformly 2-rotund by Theorem 4.1. Under 4.6 (ii), X is reflexive (for it is a uniformly rotund Banach space) and conditions (a), (b) and (c) of 3.1 are satisfied by taking both $\xi\colon \mathscr{K} \to]0, \infty[$ and $\eta\colon \mathscr{K} \to]0, \infty[$ as the radius function. We can now deal simultaneously with 4.6 and 4.7. Suppose that the collection $\mathscr{K}$ is a tiling of X. Then it follows from Propositions 3.2 and 3.3 that Theorem 3.1 is applicable. By 3.1 there are members C_1, C_2 and C_3 of $S^3_-(\mathscr{K})$ that have a common boundary point but disjoint interiors. The C_is are smooth under 4.7, rotund and centrally symmetric under 4.6. And under 4.6 it can be arranged that C_2 and C_3 are translates of C_1. Thus 4.5 is contradicted and the proof is complete. ■

The first part of the following was announced in [5].

4.8. COROLLARY. *If a normed linear space X is uniformly smooth or uniformly rotund, then X does not admit a disjoint tiling by balls. If, in addition, X is complete and infinite-dimensional, then X does not admit a tiling by balls whose radii are bounded away from* 0.

Comments added in proof. The following beautiful generalization of Sierpiński's theorem is proved by Dijkstra in [17]: *If n is a nonnegative integer, X is a compact Hausdorff space, and $\{F_1, F_2, \ldots\}$ is a countable closed covering of X such that* $\dim(F_i \cap F_j) < n$ *whenever $i \neq j$, then every continuous mapping from F_1 into the n-sphere S^n is continuously extendable over X.* As Dijkstra observes, this implies that if $\mathscr{T}$ is a countable compact covering of R^d and $\mathscr{G}$ is the collection of all sets G such that $\dim G \geqslant d-1$ and G is the intersection of two distinct members of G, then $\mathscr{G}$ is nonempty. When $\mathscr{T}$ is actually a *bc*-tiling, this $\mathscr{G}$ is the same as the collection $\mathscr{F}$ of our Theorem 1.5, and the stronger conclusion of 1.5 applies.

In [18] Klee and Tricot define the notion of a *plump* convex body in such a way that every rotund body and every smooth body is plump. They prove the following: *For an arbitrary complete metric linear space, each locally countable tiling by plump bodies consists of the parallel strips given by a countable family of mutually parallel hyperplanes.* This implies, in particular, that no complete metric linear space of dimension $\geqslant 2$ admits a smooth *fc*-tiling.

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